ASYMPTOTICS IN ALL REGIMES FOR THE SCHRÖDINGER EQUATION WITH TIME-INDEPENDENT COEFFICIENTS

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Abstract. Using the recent analysis of the output of the low-energy resolvent of Schrödinger operators on asymptotically conic manifolds (including Euclidean space) when the potential is short-range, we produce asymptotic expansions for the solutions of the initial-value problem for the Schrödinger equation, assuming Schwartz initial data. Asymptotics are calculated in all joint large-radii large-time regimes, which correspond to the boundary hypersurfaces of a particular compactification of spacetime.

Contents

1. Introduction

In recent works [Hin22; LS], wave propagation on stationary, asymptotically flat spacetimes has been analyzed using Vasy's spectral methods [Vas21a; Vas21b], which involve low-energy resolvent estimates for Schrödinger operators (i.e. for the "time-independent" Schrödinger equation, in physicists' preferred terminology) and are closely related to earlier work of Guillarmou–Hassell–Sikora [GH08; GH09; GHS13], with numerous precursors in the wider literature. These tools apply to Schrödinger operators with short range potentials, which in [Hin22; LS] means decaying cubically or faster. The quadratic case is similar, requiring only minor modifications. Long range potentials (e.g. any Coulomb-like potential, decaying like $\sim r^{-1}$) require serious modifications which we do not discuss here.

We focus on the 3-dimensional case of most physical interest, for which the spectral methods are most developed. Let (X, g) denote an asymptotically conic manifold [Mel94; Mel95] (see below), and let ρ denote a boundary-defining-function on X. Let $\mathcal{S}(X) = \bigcap_{k\in\mathbb{N}} \rho^k C^\infty(X)$ denote the Fréchet space of Schwartz functions on X. The reader is invited to consider the case of exact Euclidean space. Then,

$$X = \overline{\mathbb{R}^3} = \mathbb{R}^3 \cup \infty\mathbb{S}^2, \tag{1}$$

which is the compactification of $\mathbb{R}^3$ constructed by adding on the 2-sphere $\infty\mathbb{S}^2$ at infinity, and g is the exact Euclidean metric. The results below are novel even in this case. Here, a convenient choice of bdf is $\rho = 1/\langle r\rangle$, where r is the Euclidean radial coordinate and $\langle r\rangle = (1 + r^2)^{1/2}$ is the Japanese bracket. Also, $\mathcal{S}(X) = \mathcal{S}(\mathbb{R}^3)$ is just the usual set of Schwartz functions on 3-dimensional Euclidean space. Because of our familiarity with the Euclidean case, even when working with general X

it is usually easier to use the coordinate $r(x) = \rho(x)^{-1} \in C^\infty(X^\circ)$ in place of the bdf $\rho(x)$. For example, g is given to leading order (with respect to some collar neighborhood of the boundary) by $\mathrm{d}r^2 + r^{-2} g_{\partial X}$ for $g_{\partial X}$ a Riemannian metric on X. This is the metric of an exact cone. However, it should be kept in mind that, in the exact Euclidean case, $r(x) = \langle r \rangle^{-1}$, where r is the Euclidean radial coordinate. This slightly overloaded notation should not cause confusion. Whenever we use 'r' below, we mean $\rho(x)^{-1}$.

In this paper, we apply Vasy–Hintz–Looi's low-energy toolkit to the Schrödinger initial value problem

$$\begin{cases} -i\partial_t u = \Delta_g u + iA\cdot\nabla_g u + (V + 2^{-1} i \nabla_g \cdot A)u, \\ u(0,x) = f(x), \end{cases} \tag{IVP}$$

posed on the manifold $\mathbb{R}_t \times X^\circ$, where $f \in \mathcal{S}(X)$,

- Δ_g is the positive semidefinite Laplace–Beltrami operator,
- $\nabla_g : C^\infty(X^\circ) \to \mathcal{V}(X^\circ)$ is the gradient operator which is anti-self-adjoint with respect to the $L^2(X,g)$-inner product, $A \cdot \nabla_g u(x) = g(A(x), \nabla_g u(x))$, and $\nabla_g \cdot$ is the corresponding divergence operator, and
- $A \in r^{-2}\mathcal{V}(X;\mathbb{R})$, and $V \in r^{-3} C^\infty(X;\mathbb{R})$.

Note that the coefficients of the PDE are static, i.e. constant in the time coordinate t. Moreover, the differential operator

$$P = \Delta_g + iA \cdot \nabla_g + 2^{-1} i \nabla_g \cdot A + V \in \mathrm{Diff}^2(X^\circ) \tag{2}$$

is formally symmetric with respect to the $L^2(X,g)$-inner product, $\langle \phi, \psi \rangle_{L^2(X,g)} = \int_X \phi^* \psi \,\mathrm{dVol}_g$. This enables the application of spectral-theoretic tools. We also make the following assumptions, as in Hintz's work:

(I) (No zero energy resonance or bound state.) The operator P has trivial nullspace acting on $r^{-1} C^\infty(X)$.
(II) The high energy estimates stated in [Hin22, Def. 2.9] apply.

In [Hin22, Def. 2.8], (I) is stated using the conormal space $\mathcal{A}^1(X)$ instead. In the present context, these formulations are equivalent. The high energy estimates apply whenever the metric g is non-trapping or exhibits only normally hyperbolic trapping. In particular, (II) holds in the exact Euclidean case or in any sufficiently small perturbation thereof.

Theorem A. *The solution of the initial-value problem eq. (IVP) is of exponential-polyhomogeneous type on the compactification $M \hookleftarrow (0,\infty)_t \times X$ given by the iterated blowup*

$$M = [[[0,\infty]_t \times X; \{\infty\} \times \partial X]; \beta^{-1}(\{\infty\} \times \partial X) \cap \mathrm{cl}\, \beta^{-1}(\{\infty\} \times X)], \tag{3}$$

where $\beta : [[0,\infty]_t \times X; \{\infty\} \times \partial X] \to [0,\infty]_t \times X$ is the blowdown map.

We will explain the construction of M in more detail below. For now, see Figure 1, which describes M near $\beta^{-1}([0,\infty] \times \partial X)$ via an atlas. Also, see the discussion below for the definition of the notion of exponential-polyhomogeneity appearing in the theorem. It is a term used to state that asymptotic expansions hold without specifying the forms of those asymptotic expansions. More precise theorems (in particular, Theorem B) appear later.

As far as we are aware, Theorem A should hold in any number of dimensions, not just $d = 3$, and for any $A \in r^{-1}\mathcal{V}(X)$ and $V \in r^{-2} C^\infty(X)$, regardless of whether or not there is a zero energy resonance or bound state. Our techniques are quite general, but – as described below – the spectral side has yet to be developed sufficiently for our techniques to apply. E.g. we cite [Hin22; LS] as a black box, but these works are restricted to the $d = 3$ case. Extending them provides an avenue for further work.

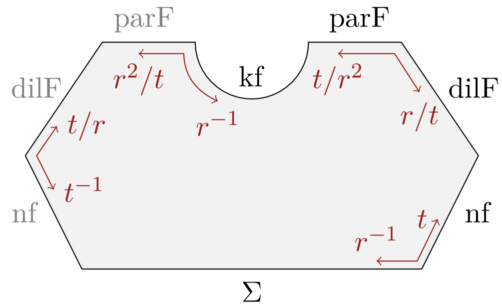

FIGURE 1. The mwc $M \supset \mathbb{R}_t \times X$, with an atlas of coordinate charts near $\mathrm{cl}_M\{\rho = 0\}$ depicted. Here, $r = 1/\rho$. The Cauchy hypersurface $\mathrm{cl}_M\{t = 0\}$ is Σ. Note that some of the faces appear disconnected only because the drawing is in 1+1D.

1.1. **More precise form of the main theorem.** For a function $v \in C^\infty(M^\circ)$ to be *of exponential-polyhomogeneous type* on the manifold-with-corners (mwc) M means that v can be written as a finite sum $v = \sum_{n=1}^N e^{i\theta_n} v_n$ for $\theta_n, v_n \in C^\infty(M^\circ)$ polyhomogeneous functions on M. Polyhomogeneity is a generalization of smoothness due to Melrose [Mel92; Mel93] – and used widely since – which allows bounded powers of logarithms to appear in Taylor series. Such "generalized Taylor series" are called *polyhomogeneous expansions*. The specific combinations of logarithms and powers allowed are specified by an *index set* $\mathcal{E}_{\mathrm{f}} \subset \mathbb{C} \times \mathbb{N}$ at each boundary hypersurface $\mathrm{f} \subset M$ of M. (An index set consists of a pair of numbers that are used to characterize the singularities of a distribution or a solution to a PDE.) Schematically, a polyhomogeneous function $v : M^\circ \to \mathbb{C}$ with index set $\mathcal{E}_{\mathrm{f}}$ at f admits the polyhomogeneous expansion

$$v \sim \sum_{(j,k)\in\mathcal{E}_{\mathrm{f}}} v_{\mathrm{f};j,k} \varrho_{\mathrm{f}}^j \log^k(\varrho_{\mathrm{f}}) \tag{4}$$

at f, where $\varrho_{\mathrm{f}} \in C^\infty(M)$ is a boundary-defining-function of f and the $v_{\mathrm{f};j,k} \in C^\infty(\mathrm{f}^\circ)$ are polyhomogeneous functions on f, which itself is a mwc of one lower dimension. Polyhomogeneity at corners guarantees the existence of *joint* asymptotic expansions there. This is equivalent to saying that the polyhomogeneous expansions at adjacent boundary hypersurfaces are compatible. Concretely, this means that $(v_{\mathrm{f};j,k})_{\mathrm{f}\cap\mathrm{F};J,K} = (v_{\mathrm{F};J,K})_{\mathrm{f}\cap\mathrm{F};j,k}$ for any adjacent boundary hypersurfaces f, F and $(j,k) \in \mathcal{E}_{\mathrm{f}}, (J,K) \in \mathcal{E}_{\mathrm{F}}$. So, the notion of exponential-polyhomogeneous type is a formalization of the notion of admitting a full atlas of (term-by-term differentiable, in all directions) asymptotic expansions in terms of elementary functions. If M is compact, then, in some sense, exponential-polyhomogeneity means that our set of asymptotic expansions is complete. (The opposite extreme is when M has no boundary, in which case exponential-polyhomogeneity just means smoothness and therefore says nothing regarding asymptotics.)

We refer to the cited works [Mel93; Mel92][Gri01][Hin22][She22] for further discussion of polyhomogeneity and the function spaces capturing it, as well as for the related notion of conormality, which e.g. features in the assumptions of the theorem. Our notational conventions mostly follow [Hin22; LS] and are explained as needed. In particular, "$\mathcal{A}^{(\mathcal{E},\alpha)}$" is used to refer to partially polyhomogeneous behavior with index set $\mathcal{E}$ and a conormal error of order $\alpha \in \mathbb{R}$, and "$\mathcal{A}^{\mathcal{E}}$" means purely polyhomogeneous behavior. One abbreviation used throughout is that, for $j \in \mathbb{R}$ and $k \in \mathbb{N}$, "(j,k)" means the index set $\{(j+n,\kappa) \in \mathbb{C} \times \mathbb{N} : n \in \mathbb{N}, \kappa \leq k\}$. Also, '$\infty$' means empty index set, i.e. Schwartz behavior. For instance, $\mathcal{A}^\infty(X) = \mathcal{S}(X)$, and $\mathcal{A}^{(j,0)}(X) = r^{-j} C^\infty(X)$.

So, Theorem A states that solutions of the Schrödinger equation (with Schwartz initial data) are governed by four asymptotic regimes (five if we include the Cauchy hypersurface $\Sigma = \{t = 0\} \subset M$) one regime for each of the four boundary hypersurfaces $\mathrm{nf}, \mathrm{dilF}, \mathrm{parF}, \mathrm{kf} \subset M$ of the mwc M

appearing in the theorem. A more precise version of the theorem – which, when combined with Proposition 1.1, which studies the sum over eigenfunctions in eq. (5), yields Theorem A – is:

Theorem B. *Let $\phi_1,\cdots,\phi_N\in\mathcal{S}(X)$ denote the (automatically Schwartz) eigenfunctions of P, so that $P\phi_n=-E_n\phi_n$ for some $E_n>0$. Let $f\in\mathcal{S}(X)$. If $u\in C^\infty([0,\infty)_t\times X^\circ)$ solves the initial-value problem eq. (IVP), and if $\chi\in C^\infty_{\mathrm{c}}(\mathbb{R})$ satisfies $0\notin\operatorname{supp}(1-\chi)$, then*

$$u(t,x)=\exp\Big[-\frac{i(1-\chi(t))}{4t\rho(x)^2}\Big]\frac{u_{\mathrm{phg}}(t,x)}{(t+i\epsilon)^{3/2}}+\sum_{n=1}^N e^{-iE_nt}\phi_n(x)\langle\phi_n,f\rangle_{L^2(X,g)} \tag{5}$$

for $u_{\mathrm{phg}}=u_{\mathrm{phg}}[\chi]$ polyhomogeneous on M, with $u_{\mathrm{phg}}\in\mathcal{A}^{(0,0)\cup(1/2,0)\cup\mathcal{E},(0,0)\cup\mathcal{F},(0,0),\infty,(0,0)}(M)$ for some index sets $\mathcal{E}\subset(2^{-1}\mathbb{N}^{\geq2})\times\mathbb{N}$, $\mathcal{F}\subset\mathbb{N}^{\geq1}\times\mathbb{N}$, where the index sets are specified at kf, parF, dilF, nf, and Σ, respectively. So, the index set at kf is $(0,0)\cup\mathcal{E}$, the index set at parF is $(0,0)\cup\mathcal{F}$, the index set at nf is empty, and the remaining index sets are $(0,0)$.

Moreover, the leading order behavior $\mathrm{kf}\cup\mathrm{parF}$ has the following form: for some $w\in C^\infty(X)$ and polyhomogeneous $v\in\mathcal{A}^{1-}(X)$.

$$u_{\mathrm{phg}}(t,x)-\chi(r/t)w(x)-\chi(r^2/t)v(x)\in\mathcal{A}^{(1/2,0)\cup\mathcal{E},\mathcal{F},(0,0),\infty,(0,0)}(M), \tag{6}$$

The restriction $w|_{\partial X}$ is constant, being of the form $\Lambda(f)$ for some linear functional $\Lambda:\mathcal{S}(X)\to\mathbb{C}$. In fact, $w(x)=-\sqrt{\pi}irP^{-1}f$, where $P^{-1}f\in r^{-1}C^\infty(X)$ is the unique solution to $Pw=f$ in $\mathcal{A}^1(X)$, and letting

$$L=-i\frac{\mathrm{d}}{\mathrm{d}\sigma}M_{\exp(-i\sigma r)}PM_{\exp(i\sigma r)}\Big|_{\sigma=0}\in\mathrm{Diff}^1(X^\circ), \tag{7}$$

in which $M_\bullet:w\mapsto\bullet w$ denotes a multiplication operator, the function v is given by $v(x)=\sqrt{\pi}iP^{-1}(-r+LP^{-1})f$. The functional Λ is given by $\Lambda(f)=\langle f,u^{(0)}\rangle_{L^2(X,g)}$ for some $u^{(0)}\in C^\infty(X)$ constant at ∂X.

Remark. Under the stated assumptions, it is the case [Hin22, §2] that we have a well-defined one-sided inverse $P^{-1}:\mathcal{A}^{2+\alpha}(X)\to\mathcal{A}^{\alpha-}(X)$ for any $\alpha\in(0,1)$, where $\mathcal{A}^{\alpha-}(X)=\bigcap_{\epsilon>0}\mathcal{A}^{\alpha-\epsilon}(X)$. So, $P^{-1}f\in\mathcal{A}^{1-}(X)$. A standard argument lets us upgrade this to $P^{-1}f\in\rho C^\infty(X)$. Because L, which is given by

$$L=-2r^{-1}(r\partial_r+1)\bmod r^{-2}\,\mathrm{Diff}^1_{\mathrm{b}}(X),$$

(cf. [Hin22, below eq. 1.11], our L being related to $L(\sigma)$ there by $L=-iL'(\sigma)|_{\sigma=0}$) satisfies $L\rho\in\rho^3C^\infty(X)$, it is the case that $LP^{-1}f\in\rho^3C^\infty(X)$ and therefore that $P^{-1}LP^{-1}f\in\mathcal{A}^{1-}(X)$. This will not in general be smooth – but it can be shown that $P^{-1}LP^{-1}f$ is still polyhomogeneous. This justifies the description of the profiles v,w in Theorem B. We refer to [Hin22, §3] for the details, which also include the large r asymptotics of w.

For the reader uncomfortable with the notion of polyhomogeneity, the following L^∞-based corollary follows immediately from the theorem:

Corollary. *For any $K\in\mathbb{N}$,*

$$\begin{aligned}u(t,x)=\exp\Big[-\frac{i(1-\chi(t))}{4t\rho(x)^2}\Big]\frac{\chi(r/t)w(x)+\chi(r^2/t)v(x)}{(t+i\epsilon)^{3/2}}+\sum_{n=1}^N e^{-iE_nt}\phi_n(x)\langle\phi_n,f\rangle_{L^2(X,g)}\\+O\Big(\frac{1}{\langle t\rangle^{3/2}}\Big\langle\frac{r}{t}\Big\rangle^{-K}\Big\langle\frac{t}{r}\Big\rangle^{-1/2}\Big).\end{aligned} \tag{8}$$

The big-O term in eq. (8), which is bounded above by $O(r^{1/2}/t^2)$, is suppressed relative to the other terms as $t\to\infty$ in $t\gg r$. That is, for any $c,\epsilon>0$ the big-O term is $o(t^{-3/2})$ if $r=o(t)$. Moreover, for any $\varepsilon>0$,

$$u(t,x)=\exp\Big[-\frac{i(1-\chi(t))}{4t\rho(x)^2}\Big]\frac{\Lambda(f)}{(t+i\epsilon)^{3/2}}+O\Big(\Big\langle\frac{rt}{t+r^2}\Big\rangle^{-4+\varepsilon}\Big), \tag{9}$$

The big-O term in eq. (9) suppressed relative to the other terms for $t \sim r^2$. That is, if $cr^2 < t < Cr^2$ for some $0 < c < C$, then the big-O term is $O(t^{-2+\varepsilon/2}) = O(r^{-4+\varepsilon})$. ■□

Remark. One minor improvement of Theorem B is that the index sets $\mathcal{E}, \mathcal{F}$ can be related to the polynomial decay rate of the coefficients of $P - \Delta_{g_0}$ for g_0 the exactly conic metric on which g is modeled. The larger the degree, the smaller these index sets can be taken. This improvement follows from Theorem C below and a corresponding improvement of [LS].

The proof below is essentially constructive, in the sense that it yields an algorithm for computing the asymptotic expansions of u_{phg} in all possible regimes, not just $\mathrm{kf} \cup \mathrm{parF}$, and not just leading order. The algorithm can be extracted from the proof below. It produces expansions on M in terms of the coefficients of the expansions in [LS]. Insofar as these coefficients are explicit, so too are the asymptotics on M.

Remark. Since [LS] is as-of-yet unpublished, it is worth mentioning that if [Hin22, Thm. 3.1] is used instead of [LS], then one gets Theorem B except with only conormal estimates of the remainder in eq. (6). In particular, the L^∞-based corollary above can still be deduced. Moreover, this applies even if V, A merely satisfy *symbolic* estimates (so, do not necessarily extend smoothly to ∂X), except in this case w, v are only known to be partially polyhomogeneous.

1.2. **Geometric setup and spacetime compactification.** Concretely, that (X, g) be an asymptotically conic manifold means that X is a smooth manifold-with-boundary and g is a Riemannian metric on X° satisfying the following: for some $\bar{\rho} > 0$ and embedding $\iota : [0, \bar{\rho}]_\rho \times \partial X \to X$ satisfying $\iota(0, -) = \mathrm{id}_{\partial X}$ (that is, a collar neighborhood of the boundary), and for some Riemannian metric $g_{\partial X}$ on ∂X, the pullback $\iota^* g$ has the form

$$\iota^* g - \rho^{-4} \mathrm{d}\rho^2 - \rho^{-2} g_{\partial X} \in \rho C^\infty(\mathrm{Sym}\, {}^{\mathrm{sc}}T^*([0, \bar{\rho})_\rho \times \partial X)), \tag{10}$$

where ${}^{\mathrm{sc}}T^*X$ is the vector bundle over X whose smooth sections are given by $C^\infty(X)\rho^{-2}\mathrm{d}\rho$, $C^\infty(X)\rho^{-1}\omega$ for $\omega \in \Omega^1(\partial X)$. That is, g differs from the exactly conic metric $\rho^{-4}\mathrm{d}\rho^2 + \rho^{-2} g_{\partial X}$ by suitably decaying terms. In the exact Euclidean case, $g_{\partial X}$ is the standard metric (or any scalar multiple thereof) on the 2-sphere at infinity. The first component of ι^{-1} serves as a boundary-defining-function (bdf). That is, there exists a bdf $\rho \in C^\infty(X; [0, \infty))$ such that $\rho(\iota(\varrho, \theta)) = \varrho$ for all $\varrho \in [0, \bar{\rho}]$. That this is a bdf means that $\rho^{-1}(\{0\}) = \partial X$ and that $d\rho$ is nonvanishing on ∂X. Going forwards, we will identify $[0, \bar{\rho}]_\rho \times \partial X$ with its image under ι. We will use the notation $\dot{X}[R] = [0, R^{-1}]_\rho \times \partial X_\theta$, and this can be considered as a subset of X as long as $R > \bar{\rho}^{-1}$. The subscripts here signal preferred variable names used to parametrize each factor, and similar notation is used throughout below.

We now describe the construction of M in a bit more detail. As a starting point, let C denote the "cylinder" $C = [0, \infty]_t \times X$. Consider the mwc

$$M/\mathrm{parF} = [C; \{\infty\} \times \partial X] = C^\circ \cup \Sigma \cup \mathrm{nf} \cup \mathrm{dilF}_0 \cup \mathrm{kf}_0 \tag{11}$$

resulting from performing a polar blowup of the corner $\{\infty\} \times \partial X \subset C$ of C. Here, $\Sigma = \{t = 0\}$, and the remaining three boundary hypersurfaces $\mathrm{nf}, \mathrm{dilF}_0, \mathrm{kf}_0$ are the lift of $[0, \infty]_t \times \partial X$, the front face of the blowup, and the lift of $\{\infty\} \times X$, respectively. Then, M can be constructed in terms of M/parF as $M = [M/\mathrm{parF}; \mathrm{dilF}_0 \cap \mathrm{kf}_0]$, which is the result of performing a polar blowup of the corner $\mathrm{dilF}_0 \cap \mathrm{kf}_0$ of $[C; \{\infty\} \times \partial X]$. The construction of M is depicted in Figure 2. The notation M/parF indicates that this mwc is, as a topological space, the quotient resulting from collapsing parF.

There are a number of other compactifications of $\mathbb{R}^+_t \times X^\circ$ via mwcs used in this paper. In the next subsection, we use $C_1 = [0, \infty)_{t/r^2} \times X$. We refer to the boundary hypersurfaces $\{t/r^2 = \infty\}$, $\{r = \infty\}$, and $\{t = 0\}$ of this mwc as kf, parF_1, Σ_1 respectively. As the notation suggests, a small neighborhood of kf in C_1 is identifiable with a neighborhood of kf in M, and the interiors of $\mathrm{parF}_1, \Sigma_1$

are identifiable with their counterparts in parF, Σ, respectively. An alternative construction of M involves blowing up the lower corner of C_1 and then blowing up the new lower corner of the resultant mwc, which we call M/nf.

Another compactification of note is M/dilF, which is the result of blowing down dilF in M. In §B, we address the question as to whether Theorem B holds on M/f for $\mathrm{f} \in \{\mathrm{nf}, \mathrm{dilF}\}$. The upshot (which holds also for parF, but requires a different argument) is that Theorem B fails on both. (A similar analysis also applies to Theorem A, but we do not present it.) In the Euclidean case, we can also define M/kf using the coordinates $1/t \in [0, \infty)$, $x_j/t^{1/2} \in \mathbb{R}$ near the blown down locus, but if bound states are present then it can be immediately concluded from Theorem B that exponential-polyhomogeneity does not hold on this compactification. So, in some sense, the mwc M is the simplest mwc on which solutions of the Schrödinger equation with Schwartz initial data have the desired form. However, it is expected that, when g is a Schwartz perturbation of an exactly conic metric and A, V are Schwartz, then the theorems above hold, mutatis mutandis, with M/parF in place of M. This is expected to follow from the collapsibility, in this case, of the transitional asymptotic regime in [Hin22; LS] vis-à-vis exponential-polyhomogeneity.

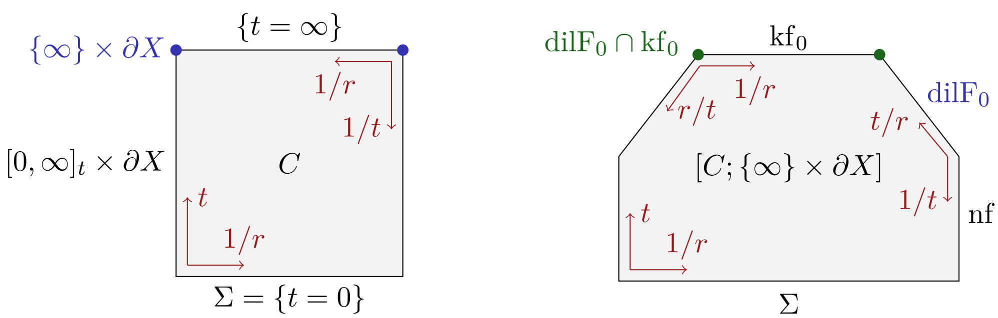

FIGURE 2. The cylinder $C = [0, \infty]_t \times X$ and the blowup $M/\mathrm{parF} = [C; \{\infty\} \times \partial X]$ constructed in the process of constructing M. The submanifolds to be blown up are depicted in blue and green.

1.3. **Outline of proof.**

Consider the differential operator $P = \Delta_g + iA \cdot \nabla + 2^{-1} i \nabla_g \cdot A + V$. The hypotheses are such that $P : C_c^\infty(X^\circ) \to L^2(X, g)$ defines an essentially self-adjoint operator. The closure is the map $H^2(X, g) \to L^2(X, g)$ given by restricting $P : \mathcal{D}'(X) \to \mathcal{D}'(X)$, defined in the sense of distributions, to the L^2-based Sobolev space $H^2(X, g)$. The spectrum $\sigma(P) = \sigma_{\mathrm{pp}}(P) \cup \sigma_{\mathrm{ac}}(P)$ of P consists of finitely many negative eigenvalues and a continuous spectrum on the whole nonnegative real axis, so $\sigma_{\mathrm{pp}}(P) = \{-E_1, \cdots, -E_N\}$ for some $0 < E_N \leq \cdots \leq E_1$, and $\sigma_{\mathrm{ac}}(P) = [0, \infty)$. Note the absence of embedded eigenvalues (recall that we are assuming the nonexistence of a bound state at zero energy) or of singular continuous spectrum. That $E_N \neq 0$ is the assumption that no bound state exists at zero energy. Here $N \in \mathbb{N}$, with $N = 0$ corresponding to the absence of pure-point spectrum. Let $\Pi : \mathrm{Borel}(\mathbb{R}) \to \mathcal{L}(L^2(X, g))$ denote the spectral measure of P. So, for each Borel set $S \subset \mathbb{R}$, $\Pi(S)$ is a projection operator on $L^2(X)$.

Via the functional calculus, there exists a 1-parameter subgroup $U : t \mapsto U(t) \in \mathrm{U}(L^2(X, g))$ such that the solution $u(t, x) : \mathbb{R}_t \times X_x \to \mathbb{C}$ to eq. (IVP) is given by $u(t, x) = (U(t)f)(x)$, and $U(t)$ is given by

$$U(t) = \int_{-\infty}^{\infty} e^{iEt} \, \mathrm{d}\Pi(E), \tag{12}$$

this integral being well-defined e.g. when applied to an element of $\mathcal{S}(X)$.

For each $n \in \{0, \ldots, N\}$, let ϕ_n be an $L^2(X, g)$-normalized bound state with $P\phi_n = E_n \phi_n$, such that $\phi_0, \cdots, \phi_N$ are orthogonal. Via a standard elliptic estimate – e.g. ellipticity in Melrose's

$\mathrm{Diff}_{\mathrm{sc}}(X)$ [Mel94] – we have $\phi_n \in \mathcal{S}(X)$ for each n. Let

$$\Pi_{\mathrm{pp}}(E) = \sum_{n=1}^{N} \delta(E - E_n)\phi_n \langle \phi_n, - \rangle \tag{13}$$

be the spectral projection onto the pure-point spectrum. Stone's theorem says that the spectral projection onto the continuous spectrum, $\Pi_{\mathrm{ac}}(E) = \Pi(E) - \Pi_{\mathrm{pp}}(E)$, which is supported on $[0,\infty)_E$, has Radon–Nikodym derivative given by

$$\mathrm{d}\Pi_{\mathrm{ac}}(E) = \frac{1}{2\pi i}(R(E + i0) - R(E - i0)), \tag{14}$$

where, for each $E \notin \sigma(P)$, $R(E) : L^2(X, g) \to H^2(X, g)$ denotes the resolvent $R(E) = (P - E)^{-1}$, and where, for each $E > 0$, $R(E \pm i0) = \lim_{\epsilon\to 0^+} R(E \pm i\epsilon)$, these limits existing in the strong operator topology.

Combining eq. (14) and eq. (13),

$$U(t) = \sum_{n=0}^{N} e^{-iE_n t}\phi_n\langle\phi_n, -\rangle + \frac{1}{2\pi i}\int_0^\infty e^{iEt}(R(E + i0) - R(E - i0))\,\mathrm{d}E, \tag{15}$$

the integral being absolute convergent in the strong sense. In other words, for any $f \in \mathcal{S}(X)$,

$$(U(t)f)(x) = \sum_{n=0}^{N} e^{-iE_n t}\phi_n(x)\langle\phi_n, f\rangle + \frac{1}{2\pi i}\int_0^\infty e^{iEt}(R(E + i0) - R(E - i0))f(x)\,\mathrm{d}E, \tag{16}$$

where, for each $x \in X^\circ$, the integral on the second line is absolutely convergent.

The terms on the first sum in eq. (16) are easily analyzed — see Proposition 1.1. The crux of our problem is to analyze the oscillatory integral $I(t, x) = I_+(t, x) - I_-(t, x)$, where

$$I_\pm(t, x) = \int_0^\infty e^{iEt}R(E \pm i0)f(x)\,\mathrm{d}E = 2\int_0^\infty e^{i\sigma^2 t}R(\sigma^2 \pm i0)f(x)\sigma\,\mathrm{d}\sigma. \tag{17}$$

The key input, coming from [Vas21a; Vas21b][Hin22; LS] is a detailed analysis of the output $e^{\mp i\sigma r}R(\sigma^2 \pm i0)e^{\pm i\sigma r}f(x) : \mathbb{R}^+_\sigma \times X_x \to \mathbb{C}$ of the "conjugated (limiting) resolvent" $e^{\mp i\sigma r}R(\sigma^2 \pm i0)e^{\pm i\sigma r}$ for $f \in \mathcal{S}(X)$. Indeed, we have the following result. Let $X^{\mathrm{sp}}_{\mathrm{res}} = [[0,\infty]_\sigma \times X; \{\infty\} \times \partial X] \hookleftarrow \mathbb{R}^+_\sigma \times X$. Label its faces $\mathrm{zf}, \mathrm{tf}, \mathrm{bf}, \infty\mathrm{f}$, as in Figure 3. Then:

Theorem ([Hin22; LS])**.** *For any $f(x) \in C^\infty([0,\infty)_\sigma; \mathcal{S}(X))$, there exist $u_0 \in \rho C^\infty(X)$ and $u_1 \in \mathcal{A}^{(1,1)}(X)$ such that*

$$e^{\mp i\sigma r}R(\sigma^2 \pm i0)e^{\pm i\sigma r}f(\sigma, x) = u_0(x) \pm i\sigma u_1(x) + \phi_\pm(\sigma, x) = \varphi_\pm(\sigma, x) \tag{18}$$

for $\phi_\pm(\sigma, x) \in \mathcal{A}^{(2,1)\cup\mathcal{E}_0,\mathcal{F}_0,(1,0)}_{\mathrm{loc}}(X^{\mathrm{sp}}_{\mathrm{res}}\backslash\infty\mathrm{f})$ and $\varphi_\pm(\sigma, x) \in \mathcal{A}^{(0,0)\cup(2,1)\cup\mathcal{E}_0,(1,0)\cup\mathcal{F}_0,(0,0)}_{\mathrm{loc}}(X^{\mathrm{sp}}_{\mathrm{res}}\backslash\infty\mathrm{f})$ for some index sets $\mathcal{E}_0 \subseteq \mathbb{N}^{\geq 3} \times \mathbb{N}$ and $\mathcal{F}_0 \subseteq \mathbb{N}^{\geq 2} \times \mathbb{N}$. Here, the index sets are specified at the faces $\mathrm{zf}, \mathrm{tf}, \mathrm{bf}$.

In fact, $u_0(x) = P^{-1}f(0, x)$ and $u_1 = -P^{-1}LP^{-1}f(0, x) \mp iP^{-1}f'(0, x)$, where $f'(\sigma, x) = \partial_\sigma f'(\sigma, x)$ and L is as above.

We supplement this with the usual high-energy bounds, namely that if $f(x) \in \mathcal{S}(X)$, then

$$e^{\mp i\sigma r}R(\sigma^2 \pm i0)f(x) \in \mathcal{A}^{(1,0),\infty}_{\mathrm{loc}}((0,\infty]_\sigma \times X_x), \tag{19}$$

where the '∞' denotes Schwartz behavior as $\sigma \to \infty$. Indeed, if we are given $f \in \mathcal{S}(X)$, then we write $e^{\mp i\sigma r}R(\sigma^2 \pm i0)f(x) = e^{\mp i\sigma r}R(\sigma^2 \pm i0)e^{\pm i\sigma r}\tilde{f}(\sigma, x)$ for $\tilde{f}(\sigma, x) \in C^\infty([0,\infty)_\sigma; \mathcal{S}(X))$ given by $\tilde{f}(\sigma, x) = e^{\mp i\sigma r}f(x)$.

In order to understand the cancellations between $I_\pm(t,x)$ in $I(t,x) = I_+(t,x) - I_-(t,x)$, it is useful to note that the expansion of $R(\sigma^2 \pm i0)f(x)$ has the form

$$R(\sigma^2 \pm i0)f(x) \sim \sum_{(j,k)\in(0,0)\cup\mathcal{E}_0} (\pm i\sigma)^j \log^k(\pm i\sigma)\phi_{j,k}(x) \tag{20}$$

for some $\phi_{j,k} \in C^\infty(\mathrm{zf}^\circ)$, assuming that $(0,0) \in \mathcal{E}_0$, where the key point is that $\phi_{j,k}$ does not depend on the choice of sign. So, in the expansion of the spectral projection $(R(\sigma^2 + i0) - R(\sigma^2 - i0))f(x)$, all of the terms with even j up to the first logarithmic term cancel. In particular,

$$(R(\sigma^2 + i0) - R(\sigma^2 - i0))f(x) \sim 2i\sigma\phi_{1,0}(x) - \pi i\sigma^2\phi_{2,1} + \sum_{(j,k)\in\mathcal{E}_0} \sigma^j \log^k(\sigma)\tilde{\phi}_{j,k}(x) \tag{21}$$

for some $\tilde{\phi}_{j,k} \in C^\infty(\mathrm{zf}^\circ)$. So, combining the theorem of Hintz–Looi with eq. (19), we have:

Corollary. *Let $f \in \mathcal{S}(X)$. For any $\epsilon > 0$, there exists $\phi_{\mathrm{even}}(\sigma,x) \in \mathcal{A}^{(2,1)\cup\mathcal{E}_0,\mathcal{F}_0,\infty,\infty}_{\mathrm{loc}}(X^{\mathrm{sp}}_{\mathrm{res}})$ such that*

$$R(\sigma^2 \pm i0)f(x) = e^{\pm i\sigma r - \epsilon\sigma^2}(u_0(x) \pm i\sigma u_1(x)) + \phi_{\mathrm{even}}(\sigma,x) + e^{\pm i\sigma r}\phi_\pm(\sigma,x) \tag{22}$$

for some $\phi_\pm = \phi_\pm[\epsilon] \in \mathcal{A}^{(2,0)\cup\mathcal{E}_0,\mathcal{F}_0,(1,1),\infty}_{\mathrm{loc}}(X^{\mathrm{sp}}_{\mathrm{res}})$ (differing from the $\phi_\pm$ in eq. (18)), where the final '∞' means Schwartz behavior as $\sigma \to \infty$. Moreover, $u_0(x) = P^{-1}\tilde{f}(0,x) = P^{-1}f(x)$ and $u_1(x) = -P^{-1}LP^{-1}\tilde{f}(0,x) \mp iP^{-1}\tilde{f}'(0,x) = -P^{-1}LP^{-1}f(x) - P^{-1}(rf(x))$. ■□

Another version of our main theorem, which we will also call our "main lemma," is:

Theorem C (Main lemma). *Let $\mathcal{E},\mathcal{F},\mathcal{G}$ denote index sets and $\alpha,\beta,\gamma \in \mathbb{R}\cup\{\infty\}$, and suppose that $\min\{\alpha, \Re j : (j,k) \in \mathcal{E}\} > -2$. Let $\phi \in \mathcal{A}^{(\mathcal{E},\alpha),(\mathcal{F},\beta),(\mathcal{G},\gamma),\infty}(X^{\mathrm{sp}}_{\mathrm{res}})$. Then, letting*

$$I_\pm[\phi](t,x) = 2\int_0^\infty e^{i\sigma^2 t \pm i\sigma r}\phi(\sigma,x)\sigma \,\mathrm{d}\sigma : \mathbb{R}_t \times X_x \to \mathbb{C}, \tag{23}$$

we have $I_+[\phi] \in \mathcal{A}^{(\mathcal{E}/2+1,\alpha/2+1),(\mathcal{F}+2,\beta+2),(0,0)}(C_1)$, and for any $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R})$ such that $0 \notin \mathrm{supp}(1-\chi)$, a decomposition of $I_-[\phi]$ of the form $I_-[\phi] = \exp(-i(1-\chi(t))/4t\rho^2)I_{\mathrm{osc}}[\phi] + I_{-,\mathrm{phg}}[\phi]$ for some functions

$$I_{-,\mathrm{phg}}[\phi] \in \mathcal{A}^{(\mathcal{E}/2+1,\alpha/2+1),(\mathcal{F}+2,\beta+2),(0,0)}(C_1) \tag{24}$$

and $I_{\mathrm{osc}}[\phi] \in \mathcal{A}^{(\mathcal{E}/2+1,\alpha+1),(\mathcal{F}/2+2,\beta+2),(\mathcal{G}+1/2,\gamma+1/2),\infty,(0,0)}(M)$. Here, the index sets on C_1 are specified in the order at kf, parF_1, and Σ_1, respectively.

Together with eq. (16) and eq. (22), Theorem C yields the theorems above. Indeed, the integral I above is given by $I(t,x) = I_+[\varphi_+] - I_-[\varphi_-]$ where

$$\varphi_\pm \in \mathcal{A}^{(1,0)\cup(2,1)\cup\mathcal{E}_0,(1,0)\cup\mathcal{F}_0,(0,0),\infty}(X^{\mathrm{sp}}_{\mathrm{res}}) \tag{25}$$

are as in eq. (18). Each $I_\pm[\varphi_\pm]$ has the form described by our main lemma, Theorem C, and it turns out that the combination $I_{\mathrm{phg}} = I_+[\varphi] - I_{-,\mathrm{phg}}[\varphi]$ must be Schwartz at $\mathrm{dilF}\cup\mathrm{nf}$. We provide a proof via microlocal analysis in §A, but the simplest way to see this is that each term in the Taylor series of I_{phg} at Σ differs from that of the solution $u(t,x)$ by a Schwartz function (coming from I_{osc} and the sum over bound states in eq. (5)). But, the initial data is Schwartz, eq. (IVP) implies that each term in the Taylor series of $u(t,x)$ at Σ is Schwartz. So, each term in the expansion of I_{phg} at Σ is Schwartz. Since I_{phg} is polyhomogeneous already on C_1, this implies Schwartzness at $\mathrm{dilF}\cup\mathrm{nf}$ when viewed as a function on M. So, in fact

$$\begin{aligned} e^{ir^2(1-\chi(t))/4t}I(t,x) &\in \mathcal{A}^{(1,0)\cup(1+\mathcal{E}_0/2),(3,0)\cup(\mathcal{F}_0+2),(3/2,0),\infty,(0,0)}(M) \\ &\subseteq (t+i\epsilon)^{-3/2}\mathcal{A}^{(-1/2,0)\cup\mathcal{E},(0,0)\cup\mathcal{F},(0,0),\infty,(0,0)}(M) \end{aligned} \tag{26}$$

where $\mathcal{E} = (1,0) \cup (-1/2 + \mathcal{E}_0/2) \subset (2^{-1}\mathbb{N}^{\geq 2}) \times \mathbb{N}$ and $\mathcal{F} = \mathcal{F}_0 - 1 \subset \mathbb{N}^{\geq 1} \times \mathbb{N}$. So, combining eq. (16) and eq. (26), we get the desired eq. (5) for

$$u_{\mathrm{phg}} \in \mathcal{A}^{(-1/2,0)\cup\mathcal{E},(0,0)\cup\mathcal{F},(0,0),\infty,(0,0)}(M) \tag{27}$$

defined by $u_{\mathrm{phg}} = (t+i\epsilon)^{3/2} e^{ir^2(1-\chi(t))/4t} I(t,x)$. Theorem B requires a slightly more refined analysis of u_{phg}. To this end, write

$$I(t,x) = 2\int_{-\infty}^{\infty} e^{i\sigma^2 t + i\sigma r - \epsilon\sigma^2}(u_0(x) + i\sigma u_1(x))\sigma \,\mathrm{d}\sigma + I_+[\phi_+] - I_-[\phi_-], \tag{28}$$

where $\phi_\pm \in \mathcal{A}^{(2,0)\cup\mathcal{E}_0,\mathcal{F}_0,(1,1),\infty}(X^{\mathrm{sp}}_{\mathrm{res}})$ are as in eq. (22). Note that the ϕ_{even} term in eq. (22) does not contribute. The first term in eq. (28) is explicitly computable:

$$\exp\Big(\frac{ir^2}{4(t+i\epsilon)}\Big)\int_{-\infty}^{\infty} e^{i\sigma^2 t + i\sigma r - \epsilon\sigma^2}(u_0(x) + i\sigma u_1(x))\sigma\,\mathrm{d}\sigma = -\sqrt{\pi i}\Big[\frac{r u_0(x)}{2(t+i\epsilon)^{3/2}} + \frac{i(r^2+2it-2\epsilon)u_1(x)}{4(t+i\epsilon)^{5/2}}\Big] \in (t+i\epsilon)^{-3/2}\mathcal{A}^{(0,0),(0,0)\cup(1,1),(0,1),(-1,1),(0,0)}(M). \tag{29}$$

Applying Theorem C to $I[\phi] = I_+[\phi_+] - I_-[\phi_-]$, the conclusion is that $I[\phi] = \exp(-i(1-\chi(t))/4t\rho^2) I_{\mathrm{osc}}[\phi] + I_{\mathrm{phg}}[\phi]$ for

$$I_{\mathrm{osc}}[\phi] \in \mathcal{A}^{(2,0)\cup(1+\mathcal{E}_0/2),\mathcal{F}_0+2,(3/2,1),\infty,(0,0)}(M) = (t+i\epsilon)^{-3/2}\mathcal{A}^{(1/2,0)\cup\mathcal{E},\mathcal{F},(0,1),\infty,(0,0)}(M) \tag{30}$$

and some $I_{\mathrm{phg}}[\phi] \in \mathcal{A}^{\mathcal{E},\mathcal{F},(0,0)}(C_1)$. Combining eq. (28), eq. (29), and eq. (30), the result is, assuming without loss of generality that $(1,1) \in \mathcal{F}$, that $u_{\mathrm{phg}} \in \mathcal{A}^{(0,0)\cup(1/2,0)\cup\mathcal{E},(0,0)\cup\mathcal{F},(0,1),(-1,1),(0,0)}(M)$. Combining this with eq. (27), we get

$$u_{\mathrm{phg}} \in \mathcal{A}^{(0,0)\cup(1/2,0)\cup\mathcal{E},(0,0)\cup\mathcal{F},(0,0),\infty,(0,0)}(M), \tag{31}$$

which was what was claimed in Theorem B. In order to complete the deduction of that theorem, we need to verify that u_{phg} has the claimed behavior at $\mathrm{parF}\cup\mathrm{kf}$. Combining eq. (28), eq. (29), and eq. (30),

$$u_{\mathrm{phg}} = -\sqrt{\pi i}\exp\Big(\frac{i(1-\chi(t))r^2}{4t} - \frac{ir^2}{4(t+i\epsilon)}\Big)\Big[r u_0(x) + \frac{i(r^2+2it-2\epsilon)u_1(x)}{2(t+i\epsilon)}\Big] + \mathcal{A}^{(1/2,0)\cup\mathcal{E},\mathcal{F},(0,1),(-1,1),(0,0)}(M). \tag{32}$$

This can be simplified using that (I)

$$r^2(t+i\epsilon)^{-1}u_1(x) \in \mathcal{A}^{(-1,0),(-1,1)}(C) \subseteq \mathcal{A}^{\mathcal{E},\mathcal{F},(0,1),(-1,1),(0,0)}(M), \tag{33}$$

(II) $(1-\chi(r/t)) r u_0(x) \in \mathcal{A}^{\infty,\infty,(0,0),(0,0),(0,0)}(M)$ and $(1-\chi(r^2/t))u_1(x) \in \mathcal{A}^{\infty,(1,1),(1,1),(1,1),(0,0)}(M)$, and (III) the exponential differs in eq. (32) from 1 only in a neighborhood of Σ disjoint from all boundary hypersurfaces of M besides nf, so

$$u_{\mathrm{phg}} = -\sqrt{\pi i}(\chi(r/t) r u_0 - \chi(r^2/t)u_1(x)) + \mathcal{A}^{(1/2,0)\cup\mathcal{E},\mathcal{F},(0,1),(-1,1),(0,0)}(M). \tag{34}$$

Since $\chi(r/t) r u_0(x) \in \mathcal{A}^{(0,0),(0,0),(0,0),\infty,\infty}(M)$ and $\chi(r^2/t)u_1(x) \in \mathcal{A}^{(0,0),(1,1),\infty,\infty,\infty}(M)$, combining this with eq. (31) shows that the error term in eq. (34) lies in

$$\mathcal{A}^{(1/2,0)\cup\mathcal{E},\mathcal{F},(0,1),(-1,1),(0,0)}(M) \cap (\mathcal{A}^{(0,0)\cup(1/2,0)\cup\mathcal{E},(0,0)\cup\mathcal{F},(0,0),\infty,(0,0)}(M) \cup \mathcal{A}^{(0,0),(0,0),(0,0),\infty,\infty}(M) \cup \mathcal{A}^{(0,0),(1,1),\infty,\infty,\infty}(M)) = \mathcal{A}^{(1/2,0)\cup\mathcal{E},\mathcal{F},(0,0),\infty,(0,0)}(M); \tag{35}$$

for each boundary hypersurface. In summary, we have improved eq. (34) to eq. (6), which completes the deduction of Theorem B from Theorem C.

So, we can regard the main theorems in this paper as corollaries of the "main lemma" Theorem C, when the latter is combined with the Hintz–Looi theorem cited above and a bit of computation

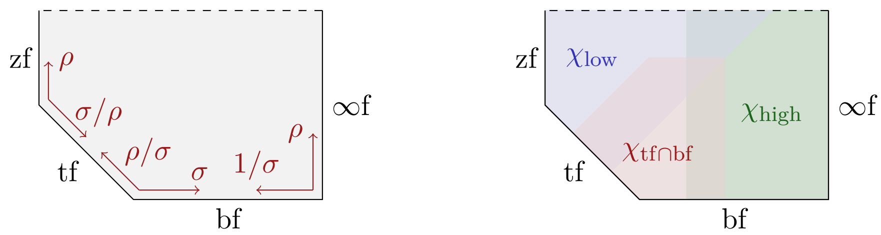

FIGURE 3. The mwc $X^{\mathrm{sp}}_{\mathrm{res}}$, with an atlas of coordinate charts (left), and the supports of the cutoffs $\chi_{\mathrm{low}}, \chi_{\mathrm{tf}\cap\mathrm{bf}}, \chi_{\mathrm{high}}$ (right). Since $E = \sigma^2$, "high" means high energy, and "low" means low energy.

(which, it should be mentioned, is not even necessary if a less sharp theorem is desired). We will prove Theorem C over the course of three sections, §2, §4, §3. The idea is to write

$$I_\pm[\phi] = I_\pm[\chi_{\mathrm{low}}\phi] + I_\pm[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi] + I_\pm[\chi_{\mathrm{high}}\phi] \tag{36}$$

for $\chi_{\mathrm{low}}, \chi_{\mathrm{tf}\cap\mathrm{bf}}, \chi_{\mathrm{high}} \in C^\infty(X^{\mathrm{sp}}_{\mathrm{res}})$ a partition of unity on $X^{\mathrm{sp}}_{\mathrm{res}}$ such that the support of χ_{low} is disjoint from $\mathrm{bf} \cup \infty\mathrm{f}$, hence supported at low energies and radii, the support of $\chi_{\mathrm{tf}\cap\mathrm{bf}}$ is disjoint from $\mathrm{zf} \cup \infty\mathrm{f}$, and the support of χ_{high} is disjoint from $\mathrm{zf} \cup \mathrm{tf}$, hence supported at high energy. The low energy contribution $I_\pm[\chi_{\mathrm{low}}\phi]$ is analyzed in §2, the high energy contribution $I_\pm[\chi_{\mathrm{high}}\phi]$ is analyzed in §3, and the final contribution $I_\pm[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]$ is analyzed in §4.

Below, we compute, for all $\phi_\pm$ as in Theorem C, full asymptotic expansions of $I_{\mathrm{phg}}[\phi]$ at $\mathrm{dilF} \cup \mathrm{nf}$. It is not always the case that $I_{\mathrm{phg}}[\phi]$ is Schwartz there. It may therefore seem a bit miraculous that, as stated above, Schwartzness does hold when $\phi_\pm(\sigma, x) = R(\sigma^2 \pm i0)f(x)$ for $f \in \mathcal{S}(X)$. In principle, it should be possible to prove this fact by verifying that all of the terms in the expansions below vanish at $\mathrm{dilF} \cup \mathrm{nf}$. We present an alternative argument in §A based on microlocal tools. These tools are based on spacetime Fourier transforms, whereas we only work with the Fourier transform in time elsewhere in this paper. One takeaway: that which is transparent when working with spacetime methods may be hidden when working with spectral methods, and vice versa.

1.4. **Bound states.** The contribution from bound states is a finite sum of functions $v : \mathbb{R}_t \times X_x \to \mathbb{C}$ of the form $w(t, x) = e^{-iEt}\varphi(x)$ for some $E \in \mathbb{R}$ and Schwartz $\varphi \in \mathcal{S}(X)$.

Proposition 1.1. *If $v(t, x) = e^{-iEt}\varphi(x)$ for some $E > 0$ and Schwartz $\varphi \in \mathcal{S}(X)$, then v is of exponential-polyhomogeneous type on M and Schwartz at $\mathrm{nf} \cup \mathrm{dilF} \cup \mathrm{parF}$.* ■

Proof. On the cylinder $C = [0, \infty]_t \times X$, each such v is already of exponential-polyhomogeneous type, with

$$v \in e^{-iEt} \bigcap_{k \in \mathbb{N}} \rho(x)^k C^\infty(C) = e^{-iEt} \bigcap_{k \in \mathbb{N}} \varrho_{\mathrm{nf}}^k \varrho_{\mathrm{dilF}}^k \varrho_{\mathrm{parF}}^k C^\infty(M). \tag{37}$$

We can choose $\varrho_{\mathrm{kf}} = t^{-1}\rho(x)^{-1}(\rho(x) + 1/t\rho(x))^{-1}$, $\varrho_{\mathrm{parF}} = \rho(x) + 1/t\rho(x)$, and $\varrho_{\mathrm{dilF}} = \rho(x)(\rho(x) + 1/t\rho(x))^{-1}$, as follows from the construction of M from M/parF. So, $t = \varrho_{\mathrm{dilF}}^{-1}\varrho_{\mathrm{parF}}^{-2}\varrho_{\mathrm{kf}}^{-1}$ is polyhomogeneous on M. □

2. Low energy contribution

We now analyze $I_\pm[\phi](t, x) = 2\int_0^\infty e^{i\sigma^2 t \pm i\sigma/\rho(x)}\phi(\sigma, x)\sigma \,\mathrm{d}\sigma$ for ϕ polyhomogeneous on X^+_{res} and supported away from $\mathrm{bf} \cap \infty\mathrm{f}$, in other words within $\{\sigma < \Sigma\}$ for some $\Sigma > 0$. This therefore constitutes the *low energy* contribution to our overall integrals. We begin with a few geometric preliminaries. Let $\lambda = \sigma/\rho(x)$, so that the map $\mathbb{R}^+ \times X^\circ \ni (\sigma, x) \mapsto (\lambda, x) \in \mathbb{R}^+ \times X^\circ$ extends to a diffeomorphism

$$\iota : X^{\mathrm{sp}}_{\mathrm{res}} \backslash (\mathrm{bf} \cup \infty\mathrm{f}) \to [0, \infty)_\lambda \times X \tag{38}$$

Let $\varphi = \phi \circ \iota^{-1} \in C^\infty((0,\infty)_\lambda \times X_x^\circ)$. Then, ϕ being supported away from $\mathrm{bf} \cap \infty\mathrm{f}$ is equivalent to $\operatorname{supp} \varphi \Subset [0,\infty)_\lambda \times X$. That is, $\varphi(\lambda, -)$ vanishes identically if λ is sufficiently large. In terms of φ,

$$2^{-1} I_\pm[\phi](t,x) = \int_0^\infty e^{iEt \pm i\sigma/\rho(x)} \varphi(\sigma/\rho(x), x)\sigma \,\mathrm{d}\sigma = \rho(x)^2 \int_0^\infty e^{i\lambda^2 t\rho(x)^2 \pm i\lambda} \varphi(\lambda, x)\lambda \,\mathrm{d}\lambda. \tag{39}$$

In order to express the spacetime asymptotics of $I_\pm[\phi]$, it is convenient to work with the compactification

$$C_1 = [0,\infty]_\tau \times X_x \hookleftarrow \mathbb{R}^+_t \times X^\circ_x \tag{40}$$

defined using $\tau = t\rho(x)^2$. As mwcs, $C_1 \cong C$, but these differ as compactifications of spacetime (except over spatially bounded regions). The three boundary hypersurfaces of C_1 are $\overline{\mathbb{R}}_\tau \times \partial X$, $\{\infty\} \times X_x$, and $\{0\} \times X_x$. Note that

$$M/\mathrm{nf} \cong [C_1; \{0\} \times \partial X], \tag{41}$$

which is a precise way of saying that C_1 results from M by blowing down both nf and dilF. This blowdown identifies kf with $\{\infty\}_\tau \times X$, parF\dilF with $(0,\infty]_\tau \times \partial X$, Σ\nf with $\{0\} \times X^\circ$, and maps $\mathrm{nf} \cup \mathrm{dilF}$ to the corner $\{0\}_\tau \times \partial X$.

So, in order to specify asymptotics of $I_\pm[\phi]$ on M, it suffices to specify them on C_1. The main proposition of this section, most of the details of the proof of which are relegated to Proposition 2.3, below, reads:

Proposition 2.1. *Suppose that $\varphi(\lambda, x) \in \mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha)}([0,\infty)_\lambda; \mathcal{A}^{(\mathcal{F},\beta)}(X_x))$ for some index set $\mathcal{E} \subseteq \{z \in \mathbb{C} : \Re z > -2\} \times \mathbb{N}$, $\alpha \in \mathbb{R}^{>-2} \cup \{\infty\}$, index set $\mathcal{F} \subset \mathbb{C} \times \mathbb{N}$, and $\beta \in \mathbb{R} \cup \{\infty\}$. Then,*

$$I_\pm[\phi](t,x) \in \mathcal{A}^{(\mathcal{E}/2+1,\alpha/2+1),(\mathcal{F}+2,\beta+2),(0,0)}(C_1), \tag{42}$$

where $\mathcal{E}/2+1$ is the index set at $\{\infty\}_\tau \times X$, $\mathcal{F}+2$ is the index set at $[0,\infty]_\tau \times \partial X$, and $(0,0)$ is the index set at $\{0\} \times X$. ■

See below for notational conventions regarding the Fourier transform.

Remark 2.2. The proof shows that the expansion of $I_\pm[\phi]$ at $[0,\infty]_\tau \times \partial X$, i.e. as $r \to \infty$, is just

$$I_\pm[\phi](t,x) \sim \sum_{(j,k)\in\mathcal{F}, \Re j \leq \beta} \rho(x)^{j+2} \log^k \rho(x) \mathcal{F}_{\xi\to\tau}(e^{\pm i\xi^{1/2}} \varphi_{j,k}(\xi^{1/2}, \theta))(\tau), \tag{43}$$

where $\varphi_{j,k}(\lambda) \in \mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha)}([0,\infty)_\lambda \times \partial X_\theta)$ are the coefficients in the polyhomogeneous expansion of $\varphi(\lambda,x)$ at $[0,\infty)_\lambda \times \partial X$, i.e. as $x \to \partial X$.

Similarly, if we let $\varphi^{j,k}(x) \in \mathcal{A}^{(\mathcal{F},\beta)}(X_x)$ denote the coefficients in the $\lambda \to 0^+$ expansion of $\varphi(\lambda, x)$, then the $\tau \to \infty$ expansion of $I_\pm[\phi]$ is

$$I_\pm[\phi](t,x) \sim \rho(x)^2 \sum_{(j,k)\in\mathcal{E}, \Re j \leq \alpha} \Big[\sum_{j_0=0}^\infty \sum_{K \geq k \text{ s.t. } (j,K)\in\mathcal{E}} \frac{(\pm i)^{j_0}}{j_0! 2^K} \varphi^{j-j_0,K}(x) c_{j/2,K,k} \Big] |\tau|^{-j/2-1} \log^k |\tau|. \tag{44}$$

If $\lambda^j \log^k(\lambda) \varphi^{j,k}(x)$ denotes the leading term in the $\lambda \to 0^+$ expansion of $\varphi(\lambda, x)$, then the leading term in the $\tau \to \infty$ expansion of $I_\pm[\phi]$ is given by $I_\pm[\phi] \sim \rho(x)^2 \varphi^{j,k} i^{1+j/2} (-1)^k \Gamma(1 + j/2)\tau^{-j/2-1} \log^k(\tau)$.

Proof. Rewriting the integral in terms of $\xi = \lambda^2\sigma^2/\rho(x)^2$, we have $I_\pm(t,x) = \tilde{I}_\pm(t\rho(x)^2, x)$ for

$$\tilde{I}_\pm(\tau, x) = \rho(x)^2 \int_0^\infty e^{i\xi\tau} \tilde{\varphi}_\pm(\xi, x) \,\mathrm{d}\xi = \rho(x)^2 \mathcal{F}_{\xi\to\tau}(\Theta(\xi)\tilde{\varphi}_\pm(\xi, x))(\tau), \tag{45}$$

where $\tilde{\varphi}_\pm(\xi, x) = e^{\pm i\xi^{1/2}} \varphi(\xi^{1/2}, x)$. Because $\varphi(\lambda, x) \in \mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha)}([0,\infty)_\lambda; \mathcal{A}^{(\mathcal{F},\beta)}(X_x))$, we have

$$\tilde{\varphi}_\pm(\xi, x) \in \mathcal{A}_{\mathrm{c}}^{(\mathcal{E}/2,\alpha/2)}([0,\infty)_\xi; \mathcal{A}^{(\mathcal{F},\beta)}(X_x)). \tag{46}$$

So, via Proposition 2.3, the Fourier transform on the right-hand side of eq. (45) lies in the function space $\mathcal{A}^{(\mathcal{E}/2+1,\alpha/2+1)}(\overline{\mathbb{R}}_\tau;\mathcal{A}^{(\mathcal{F},\beta)}(X_x))$.

The form of the expansions given follow from Proposition 2.3. Indeed, the large-τ expansions are stated as part of that proposition: letting $\tilde{\varphi}_{\pm;j,k}\in\mathcal{A}^{(\mathcal{F},\beta)}(X)$ denote the coefficients in the expansion of $e^{\pm i\xi^{1/2}}\varphi(\xi^{1/2},x)$ as $\xi\to 0^+$, then the expansion of $\rho(x)^{-2}I_\pm[\phi]$ at $\{\infty\}_\tau\times X$ is given by

$$\varphi(x)^{-2}I_\pm[\phi](t,x)\sim\sum_{(j,k)\in\mathcal{E},\Re j\leq\alpha}|\tau|^{-j/2-1}\log^k|\tau|\Big[\sum_{(j,K)\in\mathcal{E},K\geq k}\tilde{\varphi}_{j/2,K}(x)c_{j/2,K,k}\Big],\tag{47}$$

where the $c_\bullet=c_{\bullet;+}$'s are given by eq. (61) below. Since $\tilde{\varphi}_{j/2,k}(x)=\sum_{j_0=0}^\infty(\pm i)^{j_0}(j_0!2^k)^{-1}\varphi_{j-j_0,k}(x)$, eq. (44) follows. Note that this sum is finite, since $\varphi_{j-j_0,k}$ vanishes if j_0 is too large.

Also, letting $\varphi_{j,k}(\lambda)\in\mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha)}([0,\infty)_\lambda\times\partial X_\theta)$ be the coefficients in the polyhomogeneous expansion of $\varphi(\lambda,x)$ at $[0,\infty)_\lambda\times\partial X$, and defining φ_γ by

$$\varphi(\lambda,x)=\varphi_\gamma(\lambda,x)+\sum_{(j,k)\in\mathcal{F},\Re j\leq\gamma}\rho(x)^j\log^k\rho(x)\varphi_{j,k}(\lambda,\theta),\tag{48}$$

we have $\varphi_\gamma(\lambda,x)\in\mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha)}([0,\infty)_\lambda;\mathcal{A}^\gamma(X_x))$. Proposition 2.3 then says that the error in truncating the expansion in eq. (43) to order γ lies in $\mathcal{A}^{(\mathcal{E}/2+1,\alpha/2+1),\gamma+2,(0,0)}(C_1)$, where the $\gamma+2$ is the order at $[0,\infty]_\tau\times\partial X$. Since γ can be any real number $\leq\beta$, we conclude that eq. (43) holds. □

2.1. Fourier transforms of polyhomogeneous functions on the half-line.

Our convention for the Fourier transform $\mathcal{F}:\mathcal{S}'(\mathbb{R})\to\mathcal{S}'(\mathbb{R})$ is

$$\mathcal{F}\phi(\tau)=\int_{-\infty}^{+\infty}e^{i\xi\tau}\phi(\xi)\,\mathrm{d}\xi.\tag{49}$$

We will also write $\mathcal{F}\phi(\tau)$ as $\mathcal{F}_{\xi\to\tau}(\phi(\xi))(\tau)$ when it is useful to name the dual variable, 'ξ' in this case.

The main proposition of this subsection is:

Proposition 2.3. *Suppose that $\mathcal{X}$ is a Fréchet space over $\mathbb{C}$. Fix $\alpha\in(-1,\infty)\cup\{\infty\}$ and an index set $\mathcal{E}\subset\{z\in\mathbb{C}:\Re z>-1\}\times\mathbb{N}$, so that*

$$\mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha)}([0,\infty);\mathcal{X})\subseteq L^1(\mathbb{R};\mathcal{X}).\tag{50}$$

Then, if $\phi\in\mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha)}([0,\infty);\mathcal{X})$, the Fourier transform $\mathcal{F}\phi$ satisfies $\mathcal{F}\phi(\tau)\in\mathcal{A}^{(\mathcal{E}+1,\alpha+1)}(\overline{\mathbb{R}}_\tau;\mathcal{X})$. Moreover, if $\phi_{j,k}\in\mathcal{X}$ are the coefficients in the polyhomogeneous expansion

$$\phi(\xi)\sim\sum_{(j,k)\in\mathcal{E},\Re j\leq\alpha}\phi_{j,k}\xi^j\log^k\xi\tag{51}$$

of $\phi(\xi)$ as $\xi\to 0^+$, then

$$\mathcal{F}\phi(\tau)\sim\sum_{(j,k)\in\mathcal{E},\Re j\leq\alpha}\Big[\sum_{K\geq k\ \ s.t\ (j,K)\in\mathcal{E}}\phi_{j,K}c_{j,K,k;\pm}\Big]|\tau|^{-j-1}\log^k|\tau|\tag{52}$$

is the polyhomogeneous expansion of $\mathcal{F}\phi$ as $\tau\to\pm\infty$, where the $c_\bullet$'s are given by eq. (61). ■

This proposition links the regularity results before and after Fourier transform, giving an explicit way to 'transform' between an expansion into its dual (Fourier) expansion.

Recall that the 'c' subscript means that $\phi\in\mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha)}([0,\infty);\mathcal{X})$ implies that there exists some $\xi_0>0$ such that $\phi(\xi)=0$ for all $\xi\geq\xi_0$.

For $\phi\in\mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha)}[0,\infty)$, we let $\phi(-\xi)=0$ for $\xi>0$, this being implicit in eq. (50).

Proof. For simplicity, we prove the claim when $\mathcal{X} = \mathbb{C}$. The general case is completely analogous.

Let $\beta \in (-1, \infty)$ satisfy $\beta \le \alpha$. (If $\alpha < \infty$, then there is no reason not to take $\beta = \alpha$.) We can write

$$\phi(\lambda) = \phi^{(\beta)}(\xi) + \sum_{(j,k)\in\mathcal{E},\, \Re j \le \beta} \phi_{j,k} \xi^j \log^k \xi \tag{53}$$

for $\phi_{j,k} \in \mathbb{C}$ which do not depend on β, where $\phi^{(\beta)} \in \mathcal{A}^\beta([0,\infty))$. Because $\mathcal{E}$ is an index set, the sum here is finite. (In the future, we will simply use that sums of this form are finite without stating so explicitly.)

Let $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R})$ equal 1 identically on a neighborhood of $\{0\} \cup \operatorname{supp} \phi$. Then,

$$\mathcal{F}\phi(\tau) = \int_0^\infty e^{i\xi\tau} \chi(\xi) \phi^{(\beta)}(\xi) \,\mathrm{d}\xi + \sum_{(j,k)\in\mathcal{E},\, \Re j \le \beta} \phi_{j,k} \int_0^\infty e^{i\xi\tau} \chi(\xi) \xi^j \log^k \xi \,\mathrm{d}\xi. \tag{54}$$

Let $E_\beta(\tau) = \int_0^\infty e^{i\xi\tau}\chi(\xi)\phi^{(\beta)}(\xi)\,\mathrm{d}\xi$, and, for each $(j,k) \in \mathbb{C}\times\mathbb{N}$, let

$$I_{j,k}[\chi](\tau) = \int_0^\infty e^{i\xi\tau} \chi(\xi) \xi^j \log^k \xi \,\mathrm{d}\xi, \tag{55}$$

so that

$$\mathcal{F}\phi(\tau) = E_\beta(\tau) + \sum_{(j,k)\in\mathcal{E},\, \Re j \le \beta} \phi_{j,k} I_{j,k}[\chi](\tau). \tag{56}$$

It follows immediately from [Hin22, Lemma 3.6] that $E_\beta \in \mathcal{A}^{\beta+1}(\overline{\mathbb{R}}_\tau)$. On the other hand, $I_{j,k}[\chi]$ can be written as

$$I_{j,k}[\chi] = \mathcal{F}\chi(\tau) * \mathcal{F}_{\xi\to\tau}(\Theta(\xi)\xi^j \log^k \xi). \tag{57}$$

By Proposition 2.4,

$$\mathcal{F}_{\xi\to\tau}(\Theta(\xi)\xi^j \log^k \xi) \in \mathcal{S}'(\mathbb{R}_\tau) \cap \mathcal{A}^{(j+1,k)}(\overline{\mathbb{R}}_\tau \backslash \{0\}) \subseteq \mathcal{S}'(\mathbb{R}_\tau) \cap \mathcal{A}^{\mathcal{E}+1}(\overline{\mathbb{R}}_\tau \backslash \{0\}). \tag{58}$$

So, by Lemma 2.6, $I_{j,k}[\chi](\tau) \in \mathcal{A}^{\mathcal{E}+1}(\overline{\mathbb{R}}_\tau)$.

So, $\mathcal{F}\phi(\tau) \in \mathcal{A}^{(\mathcal{E}+1,\beta+1)}(\overline{\mathbb{R}}_\tau)$. Given the arbitrariness of $\beta \le \alpha$, this implies the first clause of the proposition. The argument shows that the polyhomogeneous expansion of $\mathcal{F}_\phi$ is given, at the level of formal series, by

$$\phi \sim \sum_{(j,k)\in\mathcal{E}} \phi_{j,k} I_{j,k}[\chi](\tau), \tag{59}$$

where by $I_{j,k}[\chi](\tau)$ we mean the polyhomogeneous expansion of each $I_{j,k}[\chi](\tau)$ as $\tau \to \pm\infty$. By Lemma 2.6, the polyhomogeneous expansion of $I_{j,k}[\chi](\tau)$ in this limit is the same as that of $\mathcal{F}_{\xi\to\tau}(\Theta(\xi)\xi^j \log^k \xi)(\tau)$, which we compute in Proposition 2.4. Substituting this into eq. (59) yields eq. (52). □

Proposition 2.4. *For any $j \in \{z \in \mathbb{C} : \Re z > -1\}$ and $k \in \mathbb{N}$, $\mathcal{F}_{\xi\to\tau}(\Theta(\xi)\xi^j \log^k \xi)$ is smooth away from the origin, and, for $\tau > 0$,*

$$\mathcal{F}_{\xi\to\tau}(\Theta(\xi)\xi^j \log^k \xi)(\tau) = \tau^{-j-1} \sum_{\kappa=0}^{k} c_{j,k,\kappa} \log^\kappa \tau \tag{60}$$

for some $c_{j,k,\kappa} \in \mathbb{C}$. In fact, $c_{j,k,k} = i^{j+1}(-1)^k \Gamma(j+1)$, where $\Gamma : \mathbb{C}\backslash\mathbb{Z}^{\le 0} \to \mathbb{C}$ denotes Euler's gamma function and $(\pm i)^z = \exp(\pm \pi i z/2)$ for $z \in \mathbb{C}$. ■

Remark 2.5. The proof shows that $c_{j,k,\kappa}$ is given by

$$c_{j,k,\kappa} = i^{j+1}(-1)^\kappa \binom{k}{\kappa} \sum_{\varkappa=0}^{k-\kappa} \Big(\pm \frac{\pi i}{2}\Big)^{k-\kappa-\varkappa} \binom{k-\kappa}{\varkappa} \frac{\mathrm{d}^\varkappa \Gamma(j+1)}{\mathrm{d}j^\varkappa} \tag{61}$$

for all $j \in \{z \in \mathbb{C} : \Re z > -1\}$, $k \in \mathbb{N}$, and $\kappa \in \{0, \dots, k\}$.

Proof. For $\tau\neq 0$, the Fourier transform $\mathcal{F}_{\xi\to\tau}(\Theta(\xi)\xi^j\log^k\xi)$ is given by

$$\mathcal{F}_{\xi\to\tau}(\Theta(\xi)\xi^j\log^k\xi)=\lim_{\epsilon\to 0^+}\int_0^\infty e^{i\xi\tau-\epsilon\xi|\tau|}\xi^j\log^k(\xi)\,\mathrm{d}\xi. \tag{62}$$

Letting $\xi=|\tau|\xi$, the integral on the right-hand side can be written

$$\int_0^\infty e^{i\xi\tau-\epsilon\xi|\tau|}\xi^j\log^k(\xi)\,\mathrm{d}\xi=|\tau|^{-j-1}\sum_{\kappa=0}^k(-1)^\kappa\binom{k}{\kappa}\log^\kappa|\tau|\int_0^\infty e^{\pm i\xi-\epsilon\xi}\xi^j\log^{k-\kappa}(\xi)\,\mathrm{d}\xi, \tag{63}$$

where the $\pm$ is the sign of τ. All we need to do is compute the $\epsilon\to 0^+$ limit of the integrals on the right-hand side.

By Cauchy's integral theorem,

$$\begin{aligned}\int_0^\infty e^{\pm i\xi-\epsilon\xi}\xi^j\log^k(\xi)\,\mathrm{d}\xi&=\int_0^{\pm i\infty}e^{\pm iz-\epsilon z}z^j\log^k(z)\,\mathrm{d}z\\&=\pm i\int_0^\infty e^{-\xi\mp i\epsilon\xi}(\pm i\xi)^j\log^k(\pm i\xi)\,\mathrm{d}\xi,\end{aligned} \tag{64}$$

where we are using the principal branch of the logarithm in order to fix the phase of $(\pm i\xi)^j=e^{\pm j\pi i}\xi^j$ and $\log(\pm i\xi)=\pm\pi i/2+\log\xi$. So, taking $\epsilon\to 0^+$,

$$\lim_{\epsilon\to 0^+}\int_0^\infty e^{\pm i\xi-\epsilon\xi}\xi^j\log^k(\xi)\,\mathrm{d}\xi=\pm i\int_0^\infty e^{-\xi}(\pm i\xi)^j\log^k(\pm i\xi)\,\mathrm{d}\xi. \tag{65}$$

The integral on the right-hand side can be written, after justifying differentiating under the integral sign, as

$$\frac{\mathrm{d}^k}{\mathrm{d}j^k}\int_0^\infty e^{-\xi}(\pm i\xi)^j\,\mathrm{d}\xi=\frac{\mathrm{d}^k}{\mathrm{d}j^k}((\pm i)^j\Gamma(j+1))=(\pm i)^j\sum_{\kappa=0}^k\binom{k}{\kappa}\Big(\pm\frac{\pi i}{2}\Big)^{k-\kappa}\frac{\mathrm{d}^\kappa\Gamma(j+1)}{\mathrm{d}j^\kappa}. \tag{66}$$

Chaining together these equalities yields the proposition. □

Lemma 2.6. *Suppose that* $\chi\in\mathcal{S}(\mathbb{R})$ *is identically* 1 *near the origin, and suppose that* $f\in\mathcal{S}'(\mathbb{R})\cap\mathcal{A}^{\mathcal{E}}_{\mathrm{loc}}(\overline{\mathbb{R}}_\tau\backslash\{0\})$ *for some index set* $\mathcal{E}$. *Then,* $\mathcal{F}\chi*f(\tau)-(1-\psi(\tau))f(\tau)\in\mathcal{S}(\mathbb{R}_\tau)$ *for any* $\psi\in C^\infty_{\mathrm{c}}(\mathbb{R}_\tau)$ *identically* 1 *near the origin.* ■

Proof. Write $f(\tau)=E(\tau)+F(\tau)$ for $E\in\mathcal{E}(\mathbb{R})$ a compactly supported distribution and $F\in\mathcal{A}^{\mathcal{E}}(\overline{\mathbb{R}}_t)$. Then,

$$\mathcal{F}\chi*f(\tau)-(1-\psi(\tau))f(\tau)=(\mathcal{F}\chi*F(\tau)-F(\tau))+\mathcal{F}\chi*E(\tau)-(1-\psi(\tau))E+\psi F(\tau). \tag{67}$$

The last two terms are Schwartz. Indeed, $\psi F\in C^\infty_{\mathrm{c}}(\mathbb{R})$, and necessarily $\operatorname{singsupp}E\subseteq\{0\}$, so $(1-\psi)E\in C^\infty_{\mathrm{c}}(\mathbb{R})$ as well. Now consider $\mathcal{F}\chi*E(\tau)$. We prove that this is Schwartz, which is equivalent to proving that $\chi\mathcal{F}^{-1}E$ is Schwartz. Since

$$E(\tau)\in\bigcup_{m\in\mathbb{R}}\bigcap_{s\in\mathbb{R}}\langle\tau\rangle^sH^m(\mathbb{R}_\tau), \tag{68}$$

applying $\mathcal{F}^{-1}$ yields $\mathcal{F}^{-1}E(\xi)\in\bigcup_{m\in\mathbb{R}}\bigcap_{s\in\mathbb{R}}\langle D\rangle^s\langle\xi\rangle^{-m}L^2(\mathbb{R}_\xi)$. So, $\chi(\xi)\mathcal{F}^{-1}E(\xi)\in\mathcal{S}(\mathbb{R}_\xi)$.

In order to conclude that the left-hand side of eq. (67) is Schwartz, we prove that the remaining term on the right-hand side, $\mathcal{F}\chi*F-F$, is as well. This is equivalent to $(1-\chi(\xi))\mathcal{F}^{-1}F(\xi)\in\mathcal{S}(\mathbb{R}_\xi)$, which follows if $\mathcal{F}^{-1}F$ is Schwartz except at the origin, i.e. smooth except at the origin and Schwartz outside of some compact subset.

We can write $F(\tau)=(1+\tau^2)^jF_0(\tau)$ for some $F_0\in\mathcal{A}^1(\overline{\mathbb{R}})$ and $j\in\mathbb{N}$. Because

$$1+\Delta_\xi=\mathcal{F}^{-1}_{\tau\to\xi}\circ M_{1+\tau^2}\circ\mathcal{F}_{\xi\to\tau} \tag{69}$$

preserves the space of tempered distributions on the real line that are Schwartz except at the origin, it suffices to prove that the claim holds for F_0. Equivalently, it suffices to consider the case $F=F_0$.

So, assume that $F \in \mathcal{A}^1(\overline{\mathbb{R}})$. Then, $\partial_\tau^{k+\ell}(\tau^k F(\tau)) \in \mathcal{A}^1(\overline{\mathbb{R}}_\tau) \subseteq L^2(\mathbb{R}_\tau)$ for all $k, \ell \in \mathbb{N}$. Taking the inverse Fourier transform yields

$$\xi^{k+\ell}\partial_\xi^k \mathcal{F}^{-1}F(\xi) \in L^2(\mathbb{R}_\xi) \tag{70}$$

for all $k, \ell \in \mathbb{N}$, which implies that $\mathcal{F}^{-1}F$ is Schwartz except at the origin, as desired. □

3. Radiation-field analogue and the high energy contribution

Fix an index set $\mathcal{E} \subset \mathbb{C} \times \mathbb{N}$ and $\alpha \in \mathbb{R} \cup \{\infty\}$. Suppose that $\phi \in \mathcal{S}(\mathbb{R}_\sigma; \mathcal{A}^{(\mathcal{E},\alpha)}_{\mathrm{loc}}(X))$ vanishes identically in $\{\sigma < \sigma_0\} \subset \mathbb{R}_\sigma \times \dot{X}$ for some $\sigma_0 > 0$. We denote the set of such functions as

$$\dot{\mathcal{S}}([\sigma_0, \infty)_\sigma; \mathcal{A}^{(\mathcal{E},\alpha)}_{\mathrm{loc}}(X)). \tag{71}$$

In this section, we analyze

$$I_\pm[\phi](t, x) = \int_0^\infty e^{i\sigma^2 t \pm i\sigma r(x)} \phi(\sigma, x) \,\mathrm{d}\sigma, \tag{72}$$

where $r(x) = \rho(x)^{-1}$. The argument is a straightforward application of the method of stationary phase. The phase appearing in the oscillatory integral is $\theta_\pm(t, x; \sigma) = \sigma^2 t \pm \sigma r$, which has derivative $\partial_\sigma \theta_\pm(t, x; \sigma) = 2\sigma t \pm r$. Remembering that $\rho > 0$ and $\sigma, t \geq 0$, $\partial_\sigma \theta_+$ is nonvanishing, while $\partial_\sigma \theta_-$ vanishes at the "critical" frequency $\sigma_{\mathrm{crit}} = r/2t$. Thus, following the oscillatory integral $I_\pm[\phi]$ along level sets of $r/t \in C^\infty(\mathrm{dilF}^\circ)$, an observer either sees rapid decay or else asymptotics in accordance with the stationary phase expansion.

For $I_+[\phi]$, the method of nonstationary phase also yields rapid decay at nf. The reason why $I_-[\phi]$ decays rapidly at nf is that, in this asymptotic regime, $r \to \infty$ and $t/r \to 0$, which means that $\sigma_{\mathrm{crit}} \to \infty$. As $\phi(\sigma, -)$ decays rapidly as $\sigma \to \infty$, the data in the stationary phase approximation decays rapidly as well. A less careful version of this reasoning (valid only in nf°) is that, in nf°, only r is a large parameter, so the relevant portion of the phase is $\theta_{\pm,0} = \pm r/2\sigma$, whose gradient $\partial_\sigma \theta_{\pm,0}$ is nonvanishing, so the method of nonstationary phase applies, regardless of the sign.

3.1. **Nonstationary case of sign.**

We first turn to the nonstationary case. Actually, in addition to discussing $I_+[\phi]$, we discuss the contribution $2I_{-,\mathrm{non}}[\sigma\phi(\sigma,-), \psi]$ to $I_-[\phi]$, where

$$I_{-,\mathrm{non}}[\phi, \psi](t, x) = \int_0^\infty e^{i\sigma^2 t - i\sigma r(x)} \Big[1 - \psi\Big(\sigma - \frac{r}{2t}\Big)\Big] \phi(\sigma, x) \,\mathrm{d}\sigma \in C^\infty(\mathbb{R}_t^+ \times X_x^\circ), \tag{73}$$

where $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ is identically 1 in some neighborhood of the origin and, for convenience, $\operatorname{supp}\psi \Subset (-\sigma_0, \sigma_0)$, where recall that σ_0 is chosen such that $\phi(\sigma, -) = 0$ whenever $\sigma \leq \sigma_0$.

In the next proposition, let $\mathcal{A}^{\infty,\infty,(0,0)}(C)$ denote the set of smooth functions on $C = [0, \infty]_t \times X$ that are Schwartz at $\partial C \backslash \{t = 0\}$. Such functions are smooth on M and Schwartz at all faces except $\Sigma = \mathrm{cl}_M\{t = 0\}$.

Proposition 3.1. *If $\phi \in \dot{\mathcal{S}}([\sigma_0, \infty)_\sigma; \mathcal{A}^{(\mathcal{E},\alpha)}(X))$, then $I_+[\phi], I_{-,\mathrm{non}}[\phi] \in \mathcal{A}^{\infty,\infty,(0,0)}(C)$.* ■

Proof. For any $R > 0$, we have $e^{\pm iE^{1/2} r(x)} \phi(E^{1/2}, x) \in \dot{\mathcal{S}}([\sigma_0^2, \infty); C^\infty(\{r(x) < R\}))$. So, since the Fourier transform has the mapping property

$$\mathcal{F}_{E\to t} : \mathcal{S}(\mathbb{R}_E; C^\infty(\{r(x) < R\})) \to \mathcal{S}(\mathbb{R}_t; C^\infty(\{r(x) < R\})), \tag{74}$$

we deduce, since $I_+[\phi](t, x) = \mathcal{F}_{E\to t}(e^{iE^{1/2} r(x)} \phi(E^{1/2}, x))(t)$, that $I_+[\phi](t, x) \in \mathcal{S}(\mathbb{R}_t; C^\infty(X_x^\circ))$. Similarly, if t is sufficiently large so that an $R/2T$-neighborhood of $\operatorname{supp}\psi$ is still a subset of $(-\sigma_0, \sigma_0)$, then if $r(x) < R$,

$$I_{-,\mathrm{non}}[\phi](t, x) = \mathcal{F}_{E\to t}(e^{-iE^{1/2} r(x)} \phi(E^{1/2}, x))(t). \tag{75}$$

So, $I_{-,\mathrm{non}}[\phi](t,x) \in \mathcal{S}(\mathbb{R}_t; C^\infty(X^\circ_x))$. So, in order to prove the proposition, it suffices to restrict attention to any neighborhood in C of $[0,\infty]_t \times \partial X$, at least one of which is identifiable with $[0,\infty]_t \times \dot{X}[R]$ for $\dot{X}[R] = (R,\infty]_r \times \partial X_\theta$.

Let $\mathcal{A}^{\infty,\infty,(0,0)}_{\mathrm{loc}}(\dot{C}[R])$ denote the set of smooth functions on $\dot{C}[R] = [0,\infty]_t \times \dot{X}[R]$ Schwartz at $\partial\dot{C}[R]\backslash\{t=0\}$. It suffices to prove that

$$I_+[\phi] \in \mathcal{A}^{\infty,\infty,(0,0)}_{\mathrm{loc}}(\dot{C}[R]). \tag{76}$$

Let $L \in \mathrm{Diff}_{\mathrm{b}}(\dot{X})$, i.e. L is a differential operator in the $C^\infty(\dot{X})$-algebra generated by vector fields tangent to ∂X. For any $j \in \mathbb{N}$, we can write $\partial_t^j L I_+[\phi] = I_+[\phi_{j,L}]$ for some

$$\phi_{j,L} \in \mathcal{S}(\mathbb{R}_E; \mathcal{A}^{(\mathcal{E},\alpha)}_{\mathrm{loc}}(\dot{X})) \tag{77}$$

vanishing identically in $\{E < E_0\}$. In order to prove that $I_+[\phi] \in \dot{\mathcal{S}}$, it suffices to prove that

$$I_+[\phi_{j,L}] \in \langle t+r\rangle^{-K} L^\infty_{\mathrm{loc}}([0,\infty]_t \times \dot{X}) \tag{78}$$

for every $K \in \mathbb{Z}$. Here, $L^\infty_{\mathrm{loc}}([0,\infty]_t \times \dot{X})$ is the set of functions $f(t,r,\theta)$ on $\mathbb{R}^+_t \times \dot{X}_{r,\theta}$, such that, for each $r_0 > 0$, there exists some $C[r_0] > 0$ such that $|f(t,r,\theta)| \leq C[r_0]$ whenever $r \geq r_0$.

More generally, we show that the bound eq. (78) holds for $I_+[\psi]$ whenever $\psi(t,E,r,\theta) \in C^\infty(\mathbb{R}^+_t \times \mathbb{R}_E \times \dot{X}_{r,\theta})$ is vanishing identically on $\{E < E_0\}$ and satisfies the following bounds: there exists some $J \in \mathbb{R}$ such that, for all $k, K' \in \mathbb{N}$, and for all $Q \in \mathrm{Diff}_{\mathrm{sc}}(\dot{X})$,

$$\frac{\partial^k}{\partial E^k} Q\psi(t,E,r,\theta) \in \langle E\rangle^{-K'} \langle t+r\rangle^J L^\infty(\mathbb{R}^+_t \times \mathbb{R}_E \times (\dot{X}_{r,\theta} \cap \{r \geq r_0\})) \tag{79}$$

for all $r_0 > 0$. For the $\phi_{j,L}$ above, this holds with $J > 0$ sufficiently large such that $\mathcal{A}^{(\mathcal{E},\alpha)}_{\mathrm{loc}}(\dot{X}) \subset \langle r\rangle^J L^\infty_{\mathrm{loc}}(\dot{X})$, but it will be useful to consider other values of J.

Applying eq. (79) with $k = 0$ and $K' = 2$ yields

$$|I_+[\psi](t,r,\theta)| \leq \int_{E_0}^\infty |\psi(t,E,r,\theta)| \,\mathrm{d}E \in \langle t+r\rangle^J L^\infty_{\mathrm{loc}}([0,\infty]_t \times \dot{X}), \tag{80}$$

so eq. (78) holds with $K = -J$. This is the base case of the inductive argument.

Let $\hat{K} \in \mathbb{N}$. Suppose we have shown that eq. (78) holds for $K = -J + \tilde{K}$ for all $\tilde{K} \in \{0,\ldots,\hat{K}\}$, whenever ψ satisfies eq. (79). The inductive step, which once handled completes the argument, is to show that the bound holds also for

$$K = -J + \hat{K} + 1, \tag{81}$$

i.e. that $I_+[\psi] \in \langle t+r\rangle^{J-\hat{K}-1} L^\infty_{\mathrm{loc}}([0,\infty]_t \times \dot{X})$. Writing

$$I_+[\psi] = \int_{E_0}^\infty \Big(\frac{\partial\theta_+}{\partial E}\Big)^{-1} \Big(\frac{\partial}{\partial E} e^{iEt + i\sigma(E)r}\Big) \psi(t,E,-) \,\mathrm{d}E \tag{82}$$

and integrating by parts, the result is $I_+[\psi] = I_+[\psi_1] + I_+[\psi_2]$ for

$$\psi_1 = -\Big(\frac{\partial\theta_+}{\partial E}\Big)^{-1} \frac{\partial\psi}{\partial E}, \qquad \psi_2 = \Big(\frac{\partial\theta_+}{\partial E}\Big)^{-2} \frac{\partial^2\theta_+}{\partial E^2}\psi. \tag{83}$$

Note that $\partial_E^2\theta_+ = -1/4E^{3/2}$. Thus, Lemma 3.2 implies that for each $\nu \in \{1,2\}$, all $k, K' \in \mathbb{N}$, and for all $Q \in \mathrm{Diff}_{\mathrm{sc}}(\dot{X})$,

$$\frac{\partial^k}{\partial E^k} Q\psi_\nu(t,E,r,\theta) \in \langle E\rangle^{-K'} \langle t+r\rangle^{J-\nu} L^\infty(\mathbb{R}^+_t \times \mathbb{R}_E \times (\dot{X}_{r,\theta} \cap \{r \geq r_0\})) \tag{84}$$

for all $r_0 > 0$. In other words, each of ψ_1, ψ_2 also satisfies eq. (79), except with a lower value of J. Thus, by the phrasing of the inductive hypothesis,

$$I[\psi_\nu] \in \langle t+r\rangle^{J-\hat{K}-\nu} L^\infty_{\mathrm{loc}}([0,\infty]_t \times \dot{X}) \subseteq \langle t+r\rangle^{J-\hat{K}-1} L^\infty_{\mathrm{loc}}([0,\infty]_t \times \dot{X}), \tag{85}$$

as desired. □

Lemma 3.2. *For any $j,k\in\mathbb{N}$, and for any $r_0,\sigma_0>0$, there exists a constant $C=C(j,k,\sigma_0,r_0)>0$ such that*

$$\Big|\frac{\partial^j\partial^k}{\partial\sigma^j\partial r^k}\frac{\sigma}{2\sigma t+r}\Big|\leq\frac{C\langle\sigma\rangle}{\langle t+r\rangle^{k+1}}\tag{86}$$

holds for all $t>0$, $r\geq r_0$, and $\sigma\geq\sigma_0$. ■

Proof. We have $\partial_\sigma^j\partial_r^k(\sigma/(2\sigma t+r))=(-1)^kk!\partial_\sigma^j(\sigma/(2\sigma t+r)^{k+1})$, and

$$\frac{\partial^j}{\partial\sigma^j}\frac{\sigma}{(2\sigma t+r)^{k+1}}=\frac{(k+j-1)!}{k!}\frac{j(-2t)^{j-1}}{(2\sigma t+r)^{k+j}}+\frac{(k+j)!}{k!}\frac{(-2t)^j\sigma}{(2\sigma t+r)^{k+j+1}}.\tag{87}$$

The first term on the right-hand side, which is only nonzero if $j\neq0$, satisfies the required estimate, as, for $j\geq1$,

$$0\leq t^{j-1}(2\sigma t+r)^{-k-j}\leq(2\sigma_0)^{-j+1}(2\sigma_0t+r)^{-k-1}\leq C_{j,k}\langle t+r\rangle^{-k-1}\tag{88}$$

for some $C_{j,k}>0$. The second term in eq. (87) is under control as well, as $0\leq t^j(2\sigma t+r)^{-k-j-1}\leq(2\sigma_0)^{-j}(2\sigma_0t+r)^{-k-1}\leq C_{j,k}\langle t+r\rangle^{-k-1}$, for a possibly different $C_{j,k}$. So, the bound eq. (86) follows. □

3.2. **Stationary remainder.** We now turn to the remaining contribution $2I_{-,\mathrm{stat}}[\sigma\phi(\sigma,-),\psi]$ to $I_-[\phi]$, where

$$I_{-,\mathrm{stat}}[\phi,\psi](t,x)=\int_0^\infty e^{i\sigma^2t-i\sigma r(x)}\psi\Big(\sigma-\frac{r}{2t}\Big)\phi(\sigma,x)\,\mathrm{d}\sigma\in C^\infty(\mathbb{R}_t^+\times X_x^\circ).\tag{89}$$

The main proposition of this section says:

Proposition 3.3. *Given $\phi\in\mathcal{S}_{\mathrm{c}}(\mathbb{R}^+_\sigma;\mathcal{A}^{(\mathcal{E},\alpha)}(X))$, $\psi\in C_{\mathrm{c}}^\infty(\mathbb{R})$ satisfying $\operatorname{supp}\psi\Subset(-\sigma_0,\sigma_0)$, and $\chi\in C_{\mathrm{c}}^\infty(\mathbb{R})$ identically 1 near the origin,*

$$I_{-,\mathrm{stat}}[\phi,\psi]\in e^{-i(1-\chi(t))r^2/4t}\mathcal{A}^{\infty,\infty,(\mathcal{E}+1/2,\alpha+1/2),\infty,\infty}(M),\tag{90}$$

with $I_{-,\mathrm{stat}}[\phi,\psi]$ vanishing identically $\mathrm{cl}_M\{r/2t\leq\epsilon\}$. The expansion at dilF *is given by eq. (103).* ■

Proof. Since $\phi(\sigma,x)=0$ for $\sigma\leq\sigma_0$, and since we chose ψ such that $\operatorname{supp}\psi\Subset(-\sigma_0,\sigma_0)$, and therefore $\operatorname{supp}\psi\subset(-\sigma_0+\epsilon,\sigma_0-\epsilon)$ for some $\epsilon>0$, the integral $I_{-,\mathrm{stat}}[\phi,\psi](t,x)$ is vanishing in $\{r/2t\leq\epsilon\}$. Thus, we work on the sub-mwc

$$M_{\epsilon,R}=M\cap\mathrm{cl}_M\{r/2t\geq\epsilon,r(x)>R\}\tag{91}$$

for $R>0$ sufficiently large such that we can identify $X\cap\mathrm{cl}_X\{r(x)>R\}$ with $\dot{X}[R]=[0,R^{-1})_\rho\times\partial X_\theta$ via a choice of boundary collar $\dot{X}[R]\hookrightarrow X$.

We can write $M_{\epsilon,R}=M^\circ_{\epsilon,R}\cup U_0\cup U$, where, for any $T>0$ satisfying $T\epsilon>R$ and $R_0>R$ satisfying $R_0/2T>\epsilon$,

$$U_0\cong[0,2T)_t\times\dot{X}[R_0],\qquad U\cong(T,\infty]_t\times(\epsilon,\infty]_{r/t}\times\partial X_\theta.\tag{92}$$

That is:

- the map $[0,2T)_t\times\dot{X}[R_0]\hookrightarrow M_{\epsilon,R}$, applying the boundary collar to the right factor, is a diffeomorphism onto U_0, and
- the composition

$$(T,\infty)_t\times(\epsilon,\infty)_{r/t}\times\partial X_\theta\hookrightarrow(T,\infty)_t\hookrightarrow\dot{X}[R]_{r,\theta}\hookrightarrow M^\circ_{\epsilon,R},\tag{93}$$

  where the first map sends $(t,\hat{r},\theta)\mapsto(t,(t\hat{r},\theta))$ and the second map applies the boundary collar, extends to a diffeomorphism $(T,\infty]_t\times(\epsilon,\infty]_{r/t}\times\partial X_\theta\to U$.

So, in order to conclude the proposition, it suffices to prove that $I_{-,\mathrm{stat}}[\phi,\psi](t,r,\theta)\in\mathcal{S}_{\mathrm{loc}}([0,\infty)_t\times\dot{X})$ and

$$I_{-,\mathrm{stat}}[\phi,\psi](t,\hat{r}t,\theta)\in e^{-ir^2/rt}\mathcal{A}^{(\mathcal{E}+1/2,\alpha+1/2),\infty}_{\mathrm{loc}}((T,\infty]_t\times(\epsilon,\infty]_{\hat{r}}\times\partial X_\theta), \tag{94}$$

where $\mathcal{E}+1/2$ is the index set at $t=\infty$ and the ∞ denotes Schwartz behavior at $\hat{r}=\infty$. These claims are proven below. The first is in Proposition 3.4, and the second is in Proposition 3.5. □

A modification of Equation (89),

$$I_{-,\mathrm{stat}}[\phi,\psi](t,r,\theta)=\int_0^\infty e^{i\sigma^2t-i\sigma r}\psi\Big(\sigma-\frac{r}{2t}\Big)\phi(\sigma,r,\theta)\,\mathrm{d}\sigma\in C^\infty(\mathbb{R}^+_t\times\dot{X}^\circ[R]_{r,\theta}) \tag{95}$$

defines a function $I_{-,\mathrm{stat}}[\phi,\psi]:\mathbb{R}_t\times\dot{X}[R]\to\mathbb{C}$ for any $\phi\in\mathcal{S}(\mathbb{R}_\sigma;\mathcal{A}^{(\mathcal{E},\alpha)}_{\mathrm{loc}}(\dot{X}))$, for any index set $\mathcal{E}$ and any $\alpha\in\mathbb{R}\cup\{\infty\}$, and for any $\psi\in C^\infty_{\mathrm{c}}(\mathbb{R})$.

Proposition 3.4. *For any $\phi\in\mathcal{S}(\mathbb{R}_\sigma;\mathcal{A}^{(\mathcal{E},\alpha)}_{\mathrm{loc}}(\dot{X}))$ and $\psi\in C^\infty_{\mathrm{c}}(\mathbb{R})$, the function $I_{-,\mathrm{stat}}[\phi,\psi]$ satisfies*

$$I_{-,\mathrm{stat}}[\phi,\psi]\in\mathcal{S}_{\mathrm{loc}}([0,\infty)_t\times\dot{X}), \tag{96}$$

i.e. is Schwartz at both boundary hypersurfaces $\{t=0\}$ and $[0,\infty)_t\times\partial\dot{X}$. ■

Proof. First, we prove that $I_{-,\mathrm{stat}}[\phi,\psi]\in L^\infty_{\mathrm{loc}}([0,\infty)_t\times\dot{X})$. Indeed, usying the rapid decay of $\phi(\sigma,-)$ as $\sigma\to\infty$ in some weighted L^∞-space $\langle r\rangle^J L^\infty_{\mathrm{loc}}(\dot{X})$, we have, for all $r_0>0$ and $r\geq r_0$,

$$|I_{-,\mathrm{stat}}[\phi,\psi](t,r,\theta)|\leq\|\psi\|_{L^1}\sup_{\sigma\geq r/2t-\sigma_0}|\phi(\sigma,r,\theta)|\preceq\Big\langle\frac{r}{2t}-\sigma_0\Big\rangle^{-K}\langle r\rangle^J, \tag{97}$$

which holds for some $J\geq 0$ and all $K\geq 0$, where the constant involved depends on r_0. Since $\langle r/2t-\sigma_0\rangle\preceq\langle r\rangle\langle t^{-1}\rangle$, we conclude that $I_{-,\mathrm{stat}}[\phi,\psi]\in\langle r\rangle^{-\infty}\langle t\rangle^{-\infty}L^\infty_{\mathrm{loc}}([0,\infty)_t\times\dot{X})$.

In order to control derivatives, we use the identities

$$\partial_t I_{-,\mathrm{stat}}[\phi,\psi]=iI_{-,\mathrm{stat}}[\sigma^2\phi,\psi]+(r/2t^2)I_{-,\mathrm{stat}}[\phi,\psi'] \tag{98}$$

$$\partial_r I_{-,\mathrm{stat}}[\phi,\psi]=-iI_{-,\mathrm{stat}}[\sigma\phi,\psi]-(1/2t)I_{-,\mathrm{stat}}[\phi,\psi']+I_{-,\mathrm{stat}}[\partial_r\phi,\psi]. \tag{99}$$

Applying these inductively, and applying the L^∞-bounds derived in the previous paragraph, it can be concluded that

$$\partial_t^j\partial_r^k I_{-,\mathrm{stat}}[\phi,\psi]\in\langle r\rangle^{-\infty}\langle t\rangle^{-\infty}L^\infty_{\mathrm{loc}}([0,\infty)_t\times\dot{X}) \tag{100}$$

for all $j,k\in\mathbb{N}$. So, $I_{-,\mathrm{stat}}[\phi,\psi]\in\mathcal{S}_{\mathrm{loc}}([0,\infty)_t\times\dot{X})$. □

Proposition 3.5. *For $\phi\in\mathcal{S}(\mathbb{R}_\sigma;\mathcal{A}^{(\mathcal{E},\alpha)}_{\mathrm{loc}}(\dot{X}))$,*

$$I_{-,\mathrm{stat}}[\phi,\psi](t,t\hat{r},\theta)\in e^{-ir^2/4t}\mathcal{A}^{(\mathcal{E}+1/2,\alpha+1/2),\infty}_{\mathrm{loc}}((0,\infty]_t\times(0,\infty]_{\hat{r}}\times\partial X_\theta). \tag{101}$$

The $t\to\infty$ expansion is given by

$$I_{-,\mathrm{stat}}[\phi,\psi](t,t\hat{r},\theta)\sim e^{-ir^2/4t}\sum_{j=0}^\infty\frac{\Gamma(j+1/2)}{(2j)!(-it)^{j+1/2}}\phi^{(2j)}(\hat{r}/2,t\hat{r},\theta), \tag{102}$$

$$\sim\frac{e^{-ir^2/4t}}{\sqrt{-it}}\sum_{(j,k)\in\mathcal{E}}\Big[\sum_{j_0=0}^\infty\hat{r}^{-j_0}\frac{\Gamma(j-j_0+1/2)}{(2(j-j_0))!(-i)^{j-j_0}}\times\sum_{k_0=0}^k\log^{k_0}(\hat{r})\binom{k+k_0}{k_0}\phi^{2(j-j_0)}_{j_0,k+k_0}\Big]t^{-j}\log^k(t), \tag{103}$$

where $\phi^{(k)}(\sigma,r,\theta)=\partial_\sigma^k\phi(\sigma,r,\theta)$ for $k\in\mathbb{N}$, and $\phi^{(k)}=0$ if $k<0$, and where

$$\phi^{(k)}(\sigma,r,\theta)\sim\sum_{(j,k)\in\mathcal{E}}\phi^{(k)}_{j,k}(\sigma,\theta)r^{-j}\log^k(r) \tag{104}$$

is the polyhomogeneous expansion of $\phi^{(k)}$ *at* $r = \infty$*, i.e. at* bf*. Here, each* $\phi^{(k)}_{j,k}(\sigma,\theta)$ *is in* $\dot{\mathcal{S}}([\sigma_0,\infty);C^\infty(\partial X_\theta))$*.* ■

Proof. Let $\tilde{I}_{-,\mathrm{stat}}[\phi,\psi](t,r,\theta) = e^{ir^2/4t} I_{-,\mathrm{stat}}[\phi,\psi](t,r,\theta)$. In terms of $\hat{r} = r/t$, this can be written

$$\tilde{I}_{-,\mathrm{stat}}[\phi,\psi](t,\hat{r}t,\theta) = \int_0^\infty e^{it(\sigma-\hat{r}/2)^2}\psi(\sigma-\hat{r}/2)\phi(\sigma,t\hat{r},\theta)\,\mathrm{d}\sigma, \tag{105}$$

Our goal is to prove that this lies in $\mathcal{A}^{(\mathcal{E}+1/2,\alpha+1/2),\infty}_{\mathrm{loc}}((0,\infty]_t\times(0,\infty]_{\hat{r}}\times\partial X_\theta)$.

For each $K\in\mathbb{N}$, Taylor's theorem says that

$$\phi(\sigma,r,\theta) = \sum_{k=0}^K \frac{1}{k!}\Big(\sigma-\frac{\hat{r}}{2}\Big)^k\phi^{(k)}\Big(\frac{\hat{r}}{2},r,\theta\Big) + \frac{1}{K!}\int_0^{\sigma-\hat{r}/2}\Big(\sigma-\frac{\hat{r}}{2}-\delta\Big)^K\phi^{(K+1)}\Big(\frac{\hat{r}}{2}+\delta,r,\theta\Big)\,\mathrm{d}\delta, \tag{106}$$

where the superscript on ϕ refers to differentiation in the first slot.

Thus, $\tilde{I}_{-,\mathrm{stat}}[\phi,\psi] = \sum_{k=0}^K\phi^{(k)}(\hat{r}/2,r,\theta)I_{-,\mathrm{stat},k}[\psi] + I_{-,\mathrm{stat},K,\mathrm{rem}}[\phi,\psi]$ for

$$\begin{aligned} I_{-,\mathrm{stat},k}[\psi] &= \frac{1}{k!}\int_{-\infty}^\infty e^{it(\sigma-\hat{r}/2)^2}\Big(\sigma-\frac{\hat{r}}{2}\Big)^k\psi\Big(\sigma-\frac{\hat{r}}{2}\Big)\,\mathrm{d}\sigma \\ &= \frac{1}{k!}\int_{-\infty}^\infty e^{it\delta^2}\delta^k\psi(\delta)\,\mathrm{d}\delta \end{aligned} \tag{107}$$

$$I_{-,\mathrm{stat},K,\mathrm{rem}}[\phi,\psi] = \frac{1}{K!}\int_{-\infty}^\infty e^{it\Delta^2}\psi(\Delta)\Big[\int_0^\Delta(\Delta-\delta)^K\phi^{(K+1)}\Big(\frac{\hat{r}}{2}+\delta,r,\theta\Big)\,\mathrm{d}\delta\Big]\,\mathrm{d}\Delta. \tag{108}$$

The stationary phase approximation suffices to show that $I_{-,\mathrm{stat},k}[\psi]\in t^{-(k+1)/2}C^\infty((0,\infty]_t)$. In fact, since $\psi=1$ identically near the origin, the difference

$$k!I_{-,\mathrm{stat},k}[\psi](t) - \begin{cases} 0 & (k\text{ odd}),\\ (-it)^{-(k+1)/2}\Gamma((k+1)/2) & (\text{otherwise}).\end{cases} \tag{109}$$

is, for large t, Schwartz.

Since $\phi^{(k)}(\hat{r}/2,r,\theta) = \phi^{(k)}(\hat{r}/2,\hat{r}t,\theta)$ lies in $\mathcal{S}(\mathbb{R}_{\hat{r}};\mathcal{A}^{(\mathcal{E},\alpha)}_{\mathrm{loc}}((0,\infty]_t))$, it follows that

$$\phi^{(k)}(\hat{r}/2,r,\theta)I_{-,\mathrm{stat},k}[\psi]\in\mathcal{S}(\mathbb{R}_{\hat{r}};t^{-1/2}\mathcal{A}^{(\mathcal{E},\alpha)}_{\mathrm{loc}}((0,\infty]_t)). \tag{110}$$

On the other hand, Lemma 3.6 shows that

$$|I_{-,\mathrm{stat},K,\mathrm{rem}}[\phi,\psi](t,\hat{r}t,\theta)|\in t^{-\lfloor(K+1)/2\rfloor}\hat{r}^{-\infty}\mathcal{A}^{0,0}_{\mathrm{loc}}((0,\infty]_t\times(0,\infty]_{\hat{r}}\times\partial X_\theta). \tag{111}$$

Combining everything, $I_{-,\mathrm{stat}}[\phi,\psi]\in\mathcal{S}(\mathbb{R}_{\hat{r}};\mathcal{A}^{(\mathcal{E}+1/2,\min\{\alpha+1/2,\lfloor(K+1)/2\rfloor\})}_{\mathrm{loc}}(0,\infty]_t))$. Since K can be taken arbitrarily large, the result follows. □

Lemma 3.6. *For each* $J,K\in\mathbb{N}$*,* $\phi\in\mathcal{S}(\mathbb{R}_\sigma;\mathcal{A}^0_{\mathrm{loc}}(\dot{X}))$*, and* $\psi\in C^\infty_{\mathrm{c}}(\mathbb{R})$*, consider the function* $\mathcal{I}_{J,K}[\phi,\psi]:\mathbb{R}^+_t\times\dot{X}^\circ_{r,\theta}\to\mathbb{C}$ *given by*

$$\mathcal{I}_{J,K}[\phi,\psi] = \int_{-\infty}^\infty e^{it\Delta^2}\Delta^J\psi(\Delta)\Big[\int_0^\Delta(\Delta-\delta)^K\phi\Big(\frac{r}{2t}+\delta,r,\theta\Big)\,\mathrm{d}\delta\Big]\,\mathrm{d}\Delta. \tag{112}$$

Then, $\mathcal{I}_{J,K}[\phi,\psi]\in t^{-\lfloor(J+K+1)/2\rfloor}(t/r)^\infty L^\infty_{\mathrm{loc}}((0,\infty]_t\times\dot{X}_{r,\theta})$*.*

In fact, $\tilde{\mathcal{I}}_{J,K}[\phi,\psi](t,\hat{r}t,\theta)\in t^{-\lfloor(J+K+1)/2\rfloor}\hat{r}^{-\infty}\mathcal{A}^{0,0}_{\mathrm{loc}}((0,\infty]_t\times[0,\infty)_{\hat{r}}\times\partial X_\theta)$*.* ■

Proof. We first prove the L^∞-bounds. For $J+K=0$, we have

$$|\mathcal{I}_{J,K}[\phi,\psi]|\le|\operatorname{supp}\psi|\|\psi\|_{L^1}\sup_{\delta\in\operatorname{supp}\psi}\Big|\phi\Big(\frac{r}{2t}+\delta,r,\theta\Big)\Big|\in(t/r)^\infty L^\infty_{\mathrm{loc}}((0,\infty]_t\times\dot{X}_{r,\theta}). \tag{113}$$

To handle the $J+K\ge1$ case, we integrate-by-parts, starting from

$$2it\mathcal{I}_{J,K}[\phi,\psi] = \int_{-\infty}^\infty\Big(\frac{\partial}{\partial\Delta}e^{it\Delta^2}\Big)\Delta^{J-1}\psi(\Delta)\Big[\int_0^\Delta(\Delta-\delta)^K\phi\Big(\frac{\hat{r}}{2}+\delta,r,\theta\Big)\,\mathrm{d}\delta\Big]\,\mathrm{d}\Delta. \tag{114}$$

Integrating-by-parts yields $-2it\mathcal{I}_{J,K}[\phi,\psi]+(J-1)\mathcal{I}_{J-2,K}[\phi,\psi]+\mathcal{I}_{J-1,K}[\phi,\psi']+K\mathcal{I}_{J,K-1}$ if $K\neq 0$ and

$$-2it\mathcal{I}_{J,0}[\phi,\psi]+(J-1)\mathcal{I}_{J-2,0}[\phi,\psi]+\mathcal{I}_{J-1,0}[\phi,\psi']+\tilde{\mathcal{I}}_{J-1}[\phi,\psi] \tag{115}$$

otherwise, where the last of these functions is defined by eq. (118).

So, $\mathcal{I}_{J,K}[\phi,\psi]\in t^{-\lfloor (J+K+1)/2\rfloor}(t/r)^\infty L^\infty_{\mathrm{loc}}((0,\infty]_t\times \dot{X}_{r,\theta})$ follows inductively.

In order to control derivatives, we use $L\mathcal{I}_{J,K}[\phi,\psi]=\mathcal{I}_{J,K}[L\phi,\psi]$, which holds for all $L\in \mathrm{Diff}(\partial X_\theta)$, and the identities

$$\partial_{\hat{r}}\mathcal{I}_{J,K}[\phi,\psi](t,\hat{r}t,\theta)=2^{-1}\mathcal{I}_{J,K}[\partial_\sigma\phi(\sigma,r,\theta),\psi](t,\hat{r}t,\theta)+\hat{r}^{-1}\mathcal{I}_{J,K}[r\partial_r\phi(\sigma,r,\theta),\psi](t,\hat{r}t,\theta) \tag{116}$$

$$\partial_t\mathcal{I}_{J,K}[\phi,\psi](t,\hat{r}t,\theta)=i\mathcal{I}_{J+2,K}[\phi,\psi](t,\hat{r}t,\theta)+t^{-1}\mathcal{I}_{J,K}[r\partial_r\phi(\sigma,r,\theta),\psi](t,\hat{r}t,\theta). \tag{117}$$

Using these inductively, and using the L^∞-bounds already proven, the final clause of the lemma follows. □

Lemma 3.7. *For each $J\in\mathbb{N}$, $\phi\in\mathcal{S}(\mathbb{R}_\sigma;\mathcal{A}^0_{\mathrm{loc}}(\dot{X}))$, and $\psi\in C^\infty_{\mathrm{c}}(\mathbb{R})$, consider the function $\mathcal{I}_J[\phi,\psi]:\mathbb{R}^+_t\times\dot{X}^\circ_{r,\theta}\to\mathbb{C}$ given by*

$$\tilde{\mathcal{I}}_J[\phi,\psi]=\int_{-\infty}^{\infty}e^{it\Delta^2}\Delta^J\psi(\Delta)\phi\Big(\frac{r}{2t}+\Delta,r,\theta\Big)\,\mathrm{d}\Delta. \tag{118}$$

Then, for each $K\in\mathbb{N}$, we have $\tilde{\mathcal{I}}_J[\phi,\psi](t,\hat{r}t,\theta)\in t^{-\lfloor J/2\rfloor}\hat{r}^{-\infty}\mathcal{A}^{0,0}_{\mathrm{loc}}((0,\infty]_t\times(0,\infty]_{\hat{r}}\times\partial X_\theta)$. ■

Proof. We first prove the L^∞-bounds. If $J=0$, then

$$|\tilde{\mathcal{I}}_J[\phi,\psi]|\leq\|\psi\|_{L^2}\sup_{\Delta\in\operatorname{supp}\psi}|\phi((r/2t)+\Delta,r,\theta)|\in(t/r)^\infty L^\infty_{\mathrm{loc}}((0,\infty]_t\times\dot{X}_{r,\theta}). \tag{119}$$

If $J\geq 1$, then integration-by-parts yields

$$-2it\tilde{\mathcal{I}}_J[\phi,\psi]=(J-1)\tilde{\mathcal{I}}_{J-2}[\phi,\psi]+\tilde{\mathcal{I}}_{J-1}[\phi,\psi']+\tilde{\mathcal{I}}_{J-1}[\phi',\psi], \tag{120}$$

where $\phi'(\sigma,r,\theta)=\partial_\sigma\phi(\sigma,r,\theta)$. Applying this inductively allows the deduction of $\tilde{\mathcal{I}}_J[\phi,\psi]\in t^{-\lfloor J/2\rfloor}(t/r)^K L^\infty_{\mathrm{loc}}((0,\infty]_t\times\dot{X}_{r,\theta})$ from the $J=0$ case.

To deduce the final clause of the lemma, we want to prove that the same L^∞-bounds apply to $(t\partial t)^j\partial^k_{\hat{r}}L\tilde{\mathcal{I}}_J[\phi,\psi](t,\hat{r}t,\theta)$ for every $j,k\in\mathbb{N}$ and $L\in\mathrm{Diff}(\partial X_\theta)$. Using the identities $L\tilde{\mathcal{I}}_J[\phi,\psi]=\tilde{\mathcal{I}}_J[L\phi,\psi]$,

$$\partial_{\hat{r}}\tilde{\mathcal{I}}_J[\phi',\psi](t,\hat{r}t,\theta)=2^{-1}\tilde{\mathcal{I}}_J[\partial_\sigma\phi(\sigma,r,\theta),\psi](t,\hat{r}t,\theta)+\hat{r}^{-1}\tilde{\mathcal{I}}_J[r\partial_r\phi(\sigma,r,\theta),\psi](t,\hat{r}t,\theta)], \tag{121}$$

$$\partial_t\tilde{\mathcal{I}}_J[\phi',\psi](t,\hat{r}t,\theta)=i\tilde{\mathcal{I}}_{J+1}[\phi,\psi](t,\hat{r}t,\theta)+t^{-1}\tilde{\mathcal{I}}_J[r\partial_r\phi(\sigma,r,\theta),\psi](t,\hat{r}t,\theta), \tag{122}$$

these bounds follow from those already proven. □

4. Remaining contribution

Finally, we examine $I_\pm[\phi](t,x)=2\int_0^\infty e^{i\sigma^2t\pm i\sigma r(x)}\phi(\sigma,r,\theta)\sigma\,\mathrm{d}\sigma$ for ϕ polyhomogeneous on $\dot{X}^{\mathrm{sp}}_{\mathrm{res}}$ and supported near $\mathrm{tf}\cap\mathrm{bf}$. Specifically, we consider the case $\phi(\sigma,r,\theta)=\varphi(\sigma,\sigma r,\theta)$ for

$$\varphi(\sigma,\lambda,\theta)\in\mathcal{A}^{(\mathcal{E},\alpha),(\mathcal{F},\beta)}_{\mathrm{c}}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times\partial X_\theta) \tag{123}$$

for some $\Sigma,\Lambda>0$, index sets $\mathcal{E},\mathcal{F}$, and $\alpha,\beta\in\mathbb{R}$. Here, $\mathcal{E}$ is the index set at $\sigma=0$, i.e. at tf, and $\mathcal{F}$ is the index set at $\lambda=\infty$, i.e. at bf. In order to formulate the asymptotics of $I_\pm$, it is useful to work with the manifold $\dot{M}/\mathrm{nf}$, which is defined analogously to M/nf with $\dot{X}$ in place of X, and whose faces we label correspondingly. Recall that $\dot{C}_1=[0,\infty]_\tau\times\dot{X}$. We summarize the results of this section in the following proposition:

Proposition 4.1. *Given the setup above, $I_+[\phi]\in\mathcal{A}^{\infty,(\mathcal{E}+2,\alpha+2),(0,0)}_{\mathrm{loc}}(\dot{C}_1)$, where the index sets are specified at* kf*,* parF*, and* dilF *respectively, and $I_-[\phi]=\exp(-ir^2/4t)\tilde{I}_-[\phi]+I_{-,\mathrm{phg}}[\phi]$ for some*

$$\tilde{I}_-[\phi]\in\mathcal{A}^{\infty,(\mathcal{E}+2,\alpha+2),(\mathcal{F}+1/2,\beta+1/2)}_{\mathrm{loc}}(\dot{M}/\mathrm{nf}) \tag{124}$$

and $I_{-,\mathrm{phg}}[\phi] \in \mathcal{A}^{\infty,(\mathcal{E}+2,\alpha+2),(0,0)}_{\mathrm{loc}}(\dot{C}_1)$. *Moreover, if* $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R})$ *satisfies* $\operatorname{supp}\chi \Subset (-1,1)$, *then* $\chi(2t\rho(x)\Sigma)\tilde{I}_-[\phi]$ *is Schwartz.* ■

Proof. The statment for $I_+[\phi]$ comes immediately from Proposition 4.2 and Proposition 4.4.

The partial compactifications $\mathbb{R}^+_t\times\dot{X}\hookrightarrow(\dot{M}/\mathrm{nf})\backslash(\mathrm{dilF}\cup\Sigma)$ and $\mathbb{R}^+_t\times\dot{X}\hookrightarrow\dot{C}_1\backslash\Sigma$ are equivalent, in the sense that the identity map on the interior extends to a diffeomorphism $(\dot{M}/\mathrm{nf})\backslash(\mathrm{dilF}\cup\Sigma)\cong\dot{C}_1\backslash\Sigma$. So, Proposition 4.2 tells us that

$$I_-[\phi] \in \mathcal{A}^{\infty,(\mathcal{E}+2,\alpha+2)}_{\mathrm{loc}}((\dot{M}/\mathrm{nf})\backslash\mathrm{dilF}), \tag{125}$$

where the index sets are specified at kf, parF, respectively On the other hand, the partial compactifications $\mathbb{R}^+_t\times\dot{X}\hookrightarrow(\dot{M}/\mathrm{nf})\backslash\mathrm{kf}$ and $\mathbb{R}^+_t\times\dot{X}\hookrightarrow[0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta$ given by $(t,r,\theta)\mapsto(t/r^2,t/r,\theta)$ are equivalent. So Proposition 4.7 tells us that

$$\tilde{I}_-[\phi] \in \mathcal{A}^{(\mathcal{E}+2,\alpha+2),(\mathcal{F}+1/2,\beta+1/2)}_{\mathrm{loc}}((\dot{M}/\mathrm{nf})\backslash\mathrm{kf}). \tag{126}$$

The last clause of this proposition follows from the last clause of Proposition 4.7. □

4.1. **Control for very large time.** The following establishes control of near kf:

Proposition 4.2. *For* $\varphi(\sigma,\lambda,\theta) \in \mathcal{A}^{(\mathcal{E},\alpha),(\mathcal{F},\beta)}_{\mathrm{c}}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times\partial X_\theta)$ *and* $\phi(\sigma,r,\theta)=\varphi(\sigma,\sigma r,\theta)$, *we have* $I_\pm[\phi](t,x)=\rho^2\bar{I}_\pm[\varphi](t\rho^2,x)$ *for some*

$$\bar{I}_\pm[\varphi](\tau,x) \in \mathcal{A}^{\infty,(\mathcal{E},\alpha)}_{\mathrm{loc}}(\dot{C}_1\backslash\Sigma), \tag{127}$$

where $\dot{C}_1\backslash\Sigma=(0,\infty]_\tau\times\dot{X}$, *where* $\mathcal{E}$ *is the index set at* $(0,\infty]_\tau\times\partial X_\theta$, *i.e. as* $r\to\infty$, *and the* ∞ *denotes Schwartz behavior as* $\tau\to\infty$. *Moreover, the expansion at* $(0,\infty]_\tau\times\partial X_\theta$ *is given by*

$$\bar{I}_\pm[\varphi](\tau,x) \sim \sum_{(j,k)\in\mathcal{E},\Re j\le\gamma}\frac{2}{r^j}\sum_{\kappa=0}^{k}(-1)^\kappa\binom{k}{\kappa}\log^\kappa(r)\int_\Lambda^\infty e^{i\lambda^2\tau\pm i\lambda}\varphi_{j,k}(\lambda,\theta)\lambda^{1+j}\log^k(\lambda)\,\mathrm{d}\lambda \tag{128}$$

where $\varphi(\sigma,\lambda,\theta)\sim\sum_{(j,k)\in\mathcal{E},\Re j\le\alpha}\varphi_{j,k}(\lambda,\theta)\sigma^j\log\sigma^k$ *is the polyhomogeneous expansion of* φ *as* $\sigma\to0^+$, *so that*

$$\varphi_{j,k}(\lambda,\theta)\in\mathcal{A}^{(\mathcal{F},\beta)}_{\mathrm{c}}((\Lambda,\infty]_\lambda\times\partial X_\theta). \tag{129}$$

The integrals on the right-hand side of eq. (128) are well-defined oscillatory integrals (though not necessarily absolutely convergent), e.g. via formal integration-by-parts. ■

Proof. We have $I_\pm[\phi](t,x)=\rho^2\bar{I}_\pm[\varphi](t\rho^2,x)$ for $\bar{I}_\pm[\varphi](\tau,x)$ defined by

$$\bar{I}_\pm[\varphi](\tau,x)=2\int_\Lambda^\infty e^{i\lambda^2\tau\pm i\lambda}\varphi(\lambda/r,\lambda,\theta)\lambda\,\mathrm{d}\lambda. \tag{130}$$

Defining $\varphi_\gamma(\sigma,\lambda,\theta)=\sigma^{-\gamma}(\varphi(\sigma,\lambda,\theta)-\sum_{(j,k)\in\mathcal{E},\Re j\le\gamma}\varphi_{j,k}(\lambda,\theta)\sigma^j\log\sigma^k)$ for $\gamma\in\mathbb{R}$ with $\gamma\le\alpha$, we have

$$\varphi_\gamma\in\mathcal{A}^{0,(\mathcal{F},\beta)}_{\mathrm{c}}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times\partial X_\theta). \tag{131}$$

Let $\chi\in C^\infty_{\mathrm{c}}(\mathbb{R})$ be identically 1 on $[-1,1]$. Then, we can write $\varphi(\sigma,\lambda,\theta)=\chi(\sigma\Sigma^{-1})\varphi(\sigma,\lambda,\theta)$ for all $\sigma,\lambda>0$ and $\theta\in\partial X$, so

$$\bar{I}_\pm[\varphi](\tau,x)=\sum_{(j,k)\in\mathcal{E},\Re j\le\gamma}\frac{2}{r^j}\sum_{\kappa=0}^{k}(-1)^\kappa\binom{k}{\kappa}\log^\kappa(r)\bar{I}_{\pm,j,k-\kappa}[\varphi](\tau,x)+r^{-\gamma}\bar{I}_{\pm,\gamma}[\varphi](\tau,x) \tag{132}$$

for

$$\bar{I}_{\pm,j,k}[\varphi](\tau,x)=\int_\Lambda^\infty e^{i\lambda^2\tau\pm i\lambda}\chi\Big(\frac{\lambda}{r\Sigma}\Big)\varphi_{j,k}(\lambda,\theta)\lambda^{1+j}\log^k(\lambda)\,\mathrm{d}\lambda,$$

$$\bar{I}_{\pm,\gamma}[\varphi](\tau,x)=\int_\Lambda^\infty e^{i\lambda^2\tau\pm i\lambda}\chi\Big(\frac{\lambda}{r\Sigma}\Big)\varphi_\gamma\Big(\frac{\lambda}{r},\lambda,\theta\Big)\lambda^{1+\gamma}\,\mathrm{d}\lambda. \tag{133}$$

By Lemma 4.3, $\bar{I}_{\pm,j,k}(\tau,x) \in \mathcal{A}^{\infty,(0,0)}_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x)$ and $\bar{I}_{\pm,\gamma}(\tau,x) \in \mathcal{A}^{\infty,0}_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x)$, where the $(0,0)$ denotes the index set at $(0,\infty]_\tau \times \dot{X}_x$ and ∞ denotes Schwartz behavior as $\tau \to \infty$.

So, $\bar{I}_\pm[\varphi](\tau,x) \in \mathcal{A}^{\infty,(\mathcal{E},\gamma)}_{\mathrm{loc}}(\dot{C}_1 \backslash \Sigma)$, and since $\gamma \le \alpha$ was arbitrary we conclude eq. (127). The explicit expansion follows from the argument above and the second half of Lemma 4.3. □

Lemma 4.3. *For $\varphi \in \bigcup_{K\in\mathbb{N}} \mathcal{A}^{0,-K}_{\mathrm{c}}([0,\Sigma)_\sigma \times (\Lambda,\infty]_\lambda \times \partial X_\theta)$, for $j \in \mathbb{C}$ and $k \in \mathbb{N}$, and for $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R})$, consider*

$$\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,r,\theta) = \int_\Lambda^\infty e^{i\lambda^2\tau \pm i\lambda} \chi\Big(\frac{\lambda}{r\Sigma}\Big)\varphi\Big(\frac{\lambda}{r},\lambda,\theta\Big)\lambda^j \log^k(\lambda)\,\mathrm{d}\lambda. \tag{134}$$

Then, $\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,x) \in \mathcal{A}^{\infty,0}_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x)$, where 0 is the order at $(0,\infty]_\tau \times \partial X$, i.e. as $r \to \infty$, and the ∞ denotes Schwartz behavior as $\tau \to \infty$. If $\varphi(\sigma,\lambda,\theta) = \varphi(\lambda,\theta)$ does not depend on σ, then

$$\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,x) \in \mathcal{A}^{\infty,(0,0)}_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x), \tag{135}$$

and moreover

$$\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,x) - \int_\Lambda^\infty e^{i\lambda^2\tau \pm i\lambda}\varphi(\lambda,\theta)\lambda^j\log^k(\lambda)\,\mathrm{d}\lambda \in \mathcal{A}^{\infty,\infty}_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x). \tag{136}$$

The second term on the right-hand side is a well-defined oscillatory integral, even though it may not be absolutely convergent. ■

Proof. It suffices to consider the case $\varphi \in \mathcal{A}^{0,0}_{\mathrm{c}}([0,\Sigma)_\sigma \times (\Lambda,\infty]_\lambda \times \partial X_\theta)$, as we can write $\mathcal{I}_{\pm,j,k}[\varphi,\chi] = \mathcal{I}_{\pm,j+K,k}[\lambda^{-K}\varphi,\chi]$.

We first prove that $\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,x) \in \tau^{-\infty}L^\infty_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x)$. First of all, if $\Re j < -1$, then $\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,r,\theta) \in L^\infty_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x)$, as follows immediately from an ML-bound. Using

$$2i\tau\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,r,\theta) = \int_\Lambda^\infty \Big[\frac{\partial}{\partial\lambda}e^{i\lambda^2\tau}\Big]e^{\pm i\lambda}\chi\Big(\frac{\lambda}{r\Sigma}\Big)\varphi\Big(\frac{\lambda}{r},\lambda,\theta\Big)\lambda^{j-1}\log^k(\lambda)\,\mathrm{d}\lambda, \tag{137}$$

integrating-by-parts yields

$$\begin{aligned} -2i\tau\mathcal{I}_{\pm,j,k}[\varphi,\chi] &= \pm i\mathcal{I}_{\pm,j-1,k}[\varphi,\chi] + r^{-1}\Sigma^{-1}\mathcal{I}_{\pm,j-1,k}[\varphi,\chi'] + \mathcal{I}_{\pm,j-2,k}[\sigma\partial_\sigma\varphi(\sigma,\lambda,\theta),\chi] \\ &\quad + \mathcal{I}_{\pm,j-2,k}[\lambda\partial_\lambda\varphi(\sigma,\lambda,\theta),\chi] + (j-1)\mathcal{I}_{\pm,j-2,k}[\varphi,\chi] + k\mathcal{I}_{\pm,j-2,k}[\varphi,\chi]. \end{aligned} \tag{138}$$

Each term on the right-hand side has the same form as the original integral (possibly times an extra L^∞_{loc} factor), but with j with smaller real part. Since the left-hand side of eq. (138) has one extra factor of τ, this sets up an inductive argument to conclude $O(\tau^{-\infty})$ decay from the L^∞_{loc} estimate already proven in the $\Re j < -1$ case.

Now suppose that $0 \notin \operatorname{supp}\chi$. Then, an ML-bound yields immediately that, if $\Re j < -1$, then

$$\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,x) \in r^{j+1+\epsilon}L^\infty_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x) \tag{139}$$

for any $\epsilon > 0$. So, in this case the inductive argument above yields additionally rapid decay as $r \to \infty$, i.e. that $\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,r,\theta) \in \tau^{-\infty}r^{-\infty}L^\infty_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x)$.

We now prove two sets of estimates on derivatives of $\mathcal{I}_{\pm,j,k}[\varphi,\chi]$. First of all, if $n,m \in \mathbb{N}$ and $L \in \operatorname{Diff}(\partial X_\theta)$, then

$$\partial_\tau^n(r\partial_r)^m L\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,r,\theta) \in \tau^{-\infty}L^\infty_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x). \tag{140}$$

Secondly, if $0 \notin \operatorname{supp}(1-\chi)$ and if $\varphi(\sigma,\lambda,\theta) = \varphi(\lambda,\theta)$ does not depend on σ, then, for $m \in \mathbb{N}^+$, $\partial_\tau^n\partial_r^m L\mathcal{I}_{\pm,j,k}[\varphi,\chi](\tau,r,\theta) \in \tau^{-\infty}r^{-\infty}L^\infty_{\mathrm{loc}}((0,\infty]_\tau \times \dot{X}_x)$. These follow from applying repeatedly the identities

$$L\mathcal{I}_{\pm,j,k}[\varphi,\chi] = \mathcal{I}_{\pm,j,k}[L\varphi,\chi], \qquad \partial_\tau\mathcal{I}_{\pm,j,k}[\varphi,\chi] = i\mathcal{I}_{\pm,j+2,k}[\varphi,\chi] \tag{141}$$

$$r\partial_r\mathcal{I}_{\pm,j,k}[\varphi,\chi] = -\mathcal{I}_{\pm,j+1,k}[\sigma\partial_\sigma\varphi(\sigma,\lambda,\theta),\chi] - \Sigma^{-1}r^{-1}\mathcal{I}_{\pm,j+1,k}[\varphi,\chi']. \tag{142}$$

For example, if $\varphi(\sigma,\lambda,\theta) = \varphi(\lambda,\theta)$ does not depend on σ, then the first term on the right-hand side of eq. (142) is zero, so $\partial_r\mathcal{I}_{\pm,j,k}[\varphi,\chi] = -\Sigma^{-1}r^{-2}\mathcal{I}_{\pm,j+1,k}[\varphi,\chi']$, and if $\chi = 1$ identically near the origin, then $0 \notin \operatorname{supp}\chi$, so we can apply the improved L^∞-estimates that apply to $\mathcal{I}_{\pm,j+1,k}[\varphi,\chi']$ in this case.

□

4.2. **Asymptotics elsewhere, I_+.** To complete our discussion of I_+, we prove:

Proposition 4.4. *For $\varphi(\sigma,\lambda,\theta) \in \mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha),(\mathcal{F},\beta)}([0,\Sigma)_\sigma \times (\Lambda,\infty]_\lambda \times \partial X_\theta)$ and $\phi(\sigma,r,\theta) = \varphi(\sigma,\sigma r,\theta)$, we have $I_+[\phi](t,r,\theta) = 2\rho^2 \bar{I}_+[\varphi](t/r^2,r,\theta)$ for some*

$$\bar{I}_+[\varphi](\tau,r,\theta) \in \mathcal{A}_{\mathrm{loc}}^{(\mathcal{E},\alpha),(0,0)}(\dot{C}_1\backslash\mathrm{kf}). \tag{143}$$

■

Proof. We have $I_+[\phi](t,r,\theta) = 2\rho^2\bar{I}_+[\varphi](t/r^2,r,\theta)$ for $\bar{I}_+[\varphi](\tau,r,\theta)$ defined by

$$\bar{I}_+[\varphi](\tau,r,\theta) = \int_\Lambda^\infty e^{i\lambda^2\tau + i\lambda}\varphi\Big(\frac{\lambda}{r},\lambda,\theta\Big)\lambda\,\mathrm{d}\lambda. \tag{144}$$

Now let $\varphi(\sigma,\lambda,\theta) \sim \sum_{(j,k)\in\mathcal{E},\Re j\leq\alpha}\varphi_{j,k}(\lambda,\theta)\sigma^j\log^k\sigma$ denote the polyhomogeneous expansion of $\varphi(\sigma,\lambda,\theta)$ at $\sigma=0$, so

$$\varphi_{j,k} \in \mathcal{A}_{\mathrm{c}}^{(\mathcal{F},\beta)}((\Lambda,\infty]_\lambda \times \partial X_\theta) \tag{145}$$

and, letting $\varphi_\gamma = \sigma^{-\gamma}(\varphi - \sum_{(j,k)\in\mathcal{E},\Re j\leq\gamma}\varphi_{j,k}(\lambda,\theta)\sigma^j\log^k\sigma)$, we have $\varphi \in \mathcal{A}_{\mathrm{c}}^{0,(\mathcal{F},\beta)}([0,\Sigma)_\sigma \times (\Lambda,\infty]_\lambda \times \partial X_\theta)$. Let $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ be identically 1 on $[-1,+1]$. Then, $\varphi(\sigma,\lambda,\theta) = \chi(\sigma\Sigma^{-1})\varphi(\sigma,\lambda,\theta)$, so we can write

$$\bar{I}_+[\varphi](\tau,r,\theta) = \sum_{(j,k)\in\mathcal{E},\Re j\leq\gamma}\Big(\frac{1}{r}\Big)^j\sum_{\kappa=0}^k\binom{k}{\kappa}\log^\kappa\Big(\frac{1}{r}\Big)\bar{I}_{+,j,k-\kappa}(\tau,s,\theta) + \Big(\frac{1}{r}\Big)^\gamma\bar{I}_{+,\gamma}(\tau,x) \tag{146}$$

for

$$\bar{I}_{+,j,k}[\varphi](\tau,r,\theta) = \int_\Lambda^\infty e^{i\lambda^2\tau+i\lambda}\chi\Big(\frac{\lambda}{r\Sigma}\Big)\varphi_{j,k}(\lambda,\theta)\lambda^{1+j}\log^k(\lambda)\,\mathrm{d}\lambda,$$
$$\bar{I}_{+,\gamma}[\varphi](\tau,r,\theta) = \int_\Lambda^\infty e^{i\lambda^2\tau+i\lambda}\chi\Big(\frac{\lambda}{r\Sigma}\Big)\varphi_\gamma\Big(\frac{\lambda}{r},\lambda,\theta\Big)\lambda^{1+\gamma}\,\mathrm{d}\lambda. \tag{147}$$

We now appeal to Lemma 4.6 to conclude that $\bar{I}_+[\varphi](\tau,r,\theta) \in \mathcal{A}_{\mathrm{loc}}^{(\mathcal{E},\gamma),(0,0)}(\dot{C}_1\backslash\mathrm{kf})$. Since $\gamma\leq\alpha$ was arbitrary, we can conclude eq. (143). □

Remark 4.5. Using the explicit expansions in Lemma 4.6, the proof of Proposition 4.4 shows that the expansion of $\bar{I}_+[\varphi](\tau,r,\theta)$ as $r\to\infty$ is given by

$$\bar{I}_+[\varphi] \sim \sum_{(j,k)\in\mathcal{E},\Re j\leq\alpha}\Big(\frac{1}{r}\Big)^j\log^k\Big(\frac{1}{r}\Big)\sum_{K\geq k,(j,K)\in\mathcal{E}}\binom{K}{k}\int_\Lambda^\infty e^{i\lambda^2\tau+i\lambda}\varphi_{j,K}(\lambda,\theta)\lambda^{1+j}\log^{K-k}(\lambda)\,\mathrm{d}\lambda. \tag{148}$$

Lemma 4.6. *Let $\Sigma>0$ and $\chi\in C_{\mathrm{c}}^\infty([0,\infty))$. Suppose that $\gamma\in\mathbb{C}$, $k\in\mathbb{N}$, and that $\phi\in\mathcal{A}_{\mathrm{c}}^{0,0,0,0}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times(0,\infty]_r\times[0,\infty)_\tau\times\partial X_\theta)$. Consider the integral*

$$\mathcal{I}_{+,\gamma,k}[\phi,\chi](\tau,r,\theta) = \int_\Lambda^\infty e^{i\lambda^2\tau+i\lambda}\phi\Big(\frac{\lambda}{r},\lambda,r,\tau,\theta\Big)\chi\Big(\frac{\lambda}{r\Sigma}\Big)\lambda^\gamma\log^k(\lambda)\,\mathrm{d}\lambda. \tag{149}$$

Then, $\mathcal{I}_{+,\gamma,k}[\phi,\chi](\tau,r,\theta)\in\mathcal{A}_{\mathrm{loc}}^{0,0}(\dot{C}_1\backslash\mathrm{kf})$. If $\phi(\sigma,\lambda,r,\tau,\theta)=\phi(\lambda,\theta)$ does not depend on any of σ,τ,r, then

$$\mathcal{I}_{+,\gamma,k}[\phi,\chi](\tau,r,\theta) - \int_\Lambda^\infty e^{i\lambda^2\tau+i\lambda}\phi(\lambda,\theta)\lambda^\gamma\log^k(\lambda)\,\mathrm{d}\lambda \in \mathcal{A}_{\mathrm{loc}}^{\infty,(0,0)}(\dot{C}_1\backslash\mathrm{kf}), \tag{150}$$

the integral on the left-hand side being a well-defined oscillatory integral. Here, the ∞ denotes Schwartz behavior as $r\to\infty$. ■

Proof. We first prove that

$$\mathcal{I}_{+,\gamma,k}[\phi,\chi](\tau,r,\theta)\in L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta). \tag{151}$$

If $\Re\gamma<-1$, then $\|\mathcal{I}_{+,\gamma,k}[\phi,\chi]\|_{L^\infty}\leq\|\phi\|_{L^\infty}\|\chi\|_{L^\infty}\int_\Lambda^\infty\lambda^\gamma\,\mathrm{d}\lambda<\infty$. In order to prove the claim for $\Re\gamma\geq-1$, we use an inductive argument. The key is the identity $\exp(i\lambda^2\tau+i\lambda)=-i(2\lambda+1)^{-1}\partial_\lambda\exp(i\lambda^2\tau+i\lambda)$, hence

$$\mathcal{I}_{+,\gamma,k}[\phi,\chi](\tau,r,\theta)=\int_\Lambda^\infty\Big[\frac{-i}{2\lambda\tau+1}\frac{\partial}{\partial\lambda}e^{i\lambda^2\tau+i\lambda}\Big]\phi\Big(\frac{\lambda}{r},\lambda,r,\tau,\theta\Big)\chi\Big(\frac{\lambda}{r\Sigma}\Big)\lambda^\gamma\log^k(\lambda)\,\mathrm{d}\lambda. \tag{152}$$

Integrate-by-parts, noting that no boundary terms arise:

$$\begin{split}\mathcal{I}_{+,\gamma,k}[\phi,\chi]=\mathcal{I}_{+,\gamma-1,k}[\varphi,\chi]+\mathcal{I}_{+,\gamma-1,k}[\phi_0,\chi]+\mathcal{I}_{+,\gamma-1,k}[\psi,\sigma\chi'(\sigma)]\\+\gamma\mathcal{I}_{+,\gamma-1,k}[\psi,\chi]+k\mathcal{I}_{+,\gamma-1,k-1}[\psi,\chi],\end{split} \tag{153}$$

where

$$\begin{split}\varphi(\sigma,\lambda,r,\tau,\theta)=\frac{i}{2\lambda\tau+1}\Big(\sigma\frac{\partial}{\partial\sigma}\phi(\sigma,r,\tau,\lambda,\theta)+\lambda\frac{\partial}{\partial\lambda}\phi(\sigma,r,\tau,\lambda,\theta)\Big)\\\in\mathcal{A}_{\mathrm{c}}^{0,0,0,0}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times(0,\infty]_r\times[0,\infty)_\tau\times\partial X_\theta)\end{split} \tag{154}$$

$$\phi_0(\sigma,\lambda,r,\tau,\theta)=\frac{-2i\lambda\tau\phi(\sigma,\lambda,r,\tau,\theta)}{(2\lambda\tau+1)^2}\in\mathcal{A}_{\mathrm{c}}^{0,0,0,0}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times(0,\infty]_r\times[0,\infty)_\tau\times\partial X_\theta), \tag{155}$$

and $\psi=i(2\lambda\tau+1)^{-1}\phi\in\mathcal{A}_{\mathrm{c}}^{0,0,0,0}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times(0,\infty]_r\times[0,\infty)_\tau\times\partial X_\theta)$. Since the right-hand side of eq. (153) involves only $\gamma-1$ in place of γ, this identity can be used repeatedly to reduce the to-be-proven claim eq. (151) to the $\Re\gamma<-1$ case.

A modification of this argument shows that if $0\notin\operatorname{supp}\chi$, then

$$\mathcal{I}_{+,\gamma,k}[\phi,\chi](\tau,r,\theta)\in(1/r)^\infty L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta). \tag{156}$$

Indeed, in this case, the integrand in eq. (152) is supported for $\lambda\sim r$. Thus, the initial ML-bound yields $\mathcal{I}_{+,\gamma,k}[\phi,\chi](\tau,r,\theta)\in r^{\Re\gamma+1}L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta)$. The inductive argument then shows that the same bound holds with $\gamma+K$ in place of γ, for all $K\in\mathbb{N}$, which then implies eq. (156).

In order to deduce that $\mathcal{I}_{+,\gamma,k}[\phi,\chi]\in\mathcal{A}^{0,0}_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta)$, we want to show that

$$(r\partial_r)^j(\tau\partial_\tau)^\kappa L\mathcal{I}_{+,\gamma,k}[\phi,\chi](\tau,r,\theta)\in L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta) \tag{157}$$

for all $j,\kappa\in\mathbb{N}$ and $L\in\mathrm{Diff}(\partial X)$. As elsewhere, we use $L\mathcal{I}_{+,\gamma,k}[\phi,\chi]=\mathcal{I}_{+,\gamma,k}[L\phi,\chi]$, and now we have $\tau\partial_\tau\mathcal{I}_{+,\gamma,k}[\phi,\chi]=i\tau\mathcal{I}_{+,\gamma+2,k}[\phi,\chi]+\mathcal{I}_{+,\gamma,k}[\vartheta,\chi]$ for $\vartheta(\sigma,\lambda,r,\tau,\theta)=\tau\partial_\tau\phi(\sigma,\lambda,r,\tau,\theta)$ and $r\partial_r\mathcal{I}_{+,\gamma,k}[\phi,\chi]=\mathcal{I}_{+,\gamma,k}[\psi,\chi]-\mathcal{I}_{+,\gamma,k}[\phi,\sigma\chi'(\sigma)]$ for $\psi(\sigma,\lambda,r,\tau,\theta)=(-\sigma\partial_\sigma+r\partial_r)\phi(\sigma,\lambda,r,\tau,\theta)$. So, the desired bounds in eq. (157) follow from the L^∞-bounds proven above via the usual inductive argument.

Suppose now that $\phi(\sigma,\lambda,r,\tau,\theta)=\phi(\lambda,\theta)$ does not depend on any of σ,τ,r. Then, $\partial_\tau\mathcal{I}_{+,\gamma,k}[\phi,\chi]=i\mathcal{I}_{+,\gamma+2,k}[\phi,\chi]+r^{-1}\Sigma^{-1}\mathcal{I}_{+,\gamma+1,k}[\varphi,\chi']$ and $\partial_r\mathcal{I}_{+,\gamma,k}[\phi,\chi]=-r^{-1}\mathcal{I}_{+,\gamma,k}[\phi,\sigma\chi'(\sigma)]$. So, a similar inductive argument to the above shows that

$$\partial_\tau^\kappa L\mathcal{I}_{+,\gamma,k}[\phi,\chi]\in L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta) \tag{158}$$

$$\partial_\tau^\kappa\partial_r^{j+1}L\mathcal{I}_{+,\gamma,k}[\phi,\chi]\in r^{-\infty}L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta) \tag{159}$$

for all $j,\kappa\in\mathbb{N}$ and $L\in\mathrm{Diff}(\partial X_\theta)$. These estimates suffice to show that

$$\mathcal{I}_{+,\gamma,k}[\phi,\chi](\tau,r,\theta)\in\mathcal{A}^{(0,0),(0,0)}_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta) \tag{160}$$

and that only the $O(1)$ term in the $r\to\infty$ expansion is nontrivial. It can be checked, e.g. via the integration-by-parts argument above, that this leading term is that specified by eq. (150). □

4.3. **Asymptotics elsewhere, remaining case.** Finally, to complete our discussion of I_-:

Proposition 4.7. *For $\varphi(\sigma,\lambda,\theta) \in \mathcal{A}_{\mathrm{c}}^{(\mathcal{E},\alpha),(\mathcal{F},\beta)}([0,\Sigma)_\sigma \times (\Lambda,\infty]_\lambda \times \partial X_\theta)$ and $\phi(\sigma,r,\theta) = \varphi(\sigma,\sigma r,\theta)$, we have*

$$I_-[\phi](t,x) = I_{-,\mathrm{phg}}[\varphi](t/r^2,r,\theta) + I_{-,\mathrm{osc}}[\varphi](t/r^2,t/r,\theta) \tag{161}$$

for

$$I_{-,\mathrm{phg}}[\varphi](\tau,r,\theta) \in r^{-2}\mathcal{A}_{\mathrm{loc}}^{(\mathcal{E},\alpha),(0,0)}(\dot{C}_1\backslash\mathrm{kf}) \tag{162}$$

and $I_{-,\mathrm{osc}}[\varphi](\tau,s,\theta) \in e^{-i/4\tau}\tau^{1/2}s^{-2}\mathcal{A}_{\mathrm{loc}}^{(\mathcal{E},\alpha),(\mathcal{F},\beta)}([0,\infty)_\tau \times (0,\infty]_s \times \partial X_\theta)$, where $\mathcal{E}$ is the index set as $s\to\infty$ and $\mathcal{F}$ is the index set as $\tau \to 0$. Moreover, if $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ satisfies $\operatorname{supp}\chi \Subset (-1,1)$, then $\chi(2s\Sigma)I_{-,\mathrm{osc}}[\varphi](\tau,s,\theta) \in \mathcal{A}_{\mathrm{loc}}^{\infty,\infty}([0,\infty)_\tau \times (0,\infty]_s \times \partial X_\theta)$. ■

Proof. Let $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ satisfy $\operatorname{supp}\psi \Subset (-1/2,1/2)$ and $0 \notin \operatorname{supp}(1-\psi)$. Now define

$$I_{-,\mathrm{osc}}[\varphi,\psi](\tau,s,\theta) = \frac{2}{r^2}\int_\Lambda^\infty e^{i\lambda^2\tau - i\lambda}\psi(2\lambda\tau-1)\varphi\Big(\frac{\lambda\tau}{s},\lambda,\theta\Big)\lambda\,\mathrm{d}\lambda, \tag{163}$$

$$I_{-,\mathrm{phg}}[\varphi,\psi](\tau,r,\theta) = \frac{2}{r^2}\int_\Lambda^\infty e^{i\lambda^2\tau - i\lambda}(1-\psi(2\lambda\tau-1))\varphi\Big(\frac{\lambda}{r},\lambda,\theta\Big)\lambda\,\mathrm{d}\lambda. \tag{164}$$

Then eq. (161) holds. We just need to check that each of these integrals lies in the expected function spaces. We begin with $I_{-,\mathrm{phg}}$. Let

$$\varphi(\sigma,\lambda,\theta) \sim \sum_{(j,k)\in\mathcal{E},\Re j\le\alpha} \varphi_{j,k}(\lambda,\theta)\sigma^j\log^k\sigma \tag{165}$$

denote the $\sigma\to 0^+$ expansion of φ, so $\varphi_{j,k}(\lambda,\theta) \in \mathcal{A}_{\mathrm{c}}^{(\mathcal{F},\beta)}((\Lambda,\infty]_\lambda \times \partial X_\theta)$. Consider the function φ_γ defined by

$$\varphi_\gamma(\sigma,\lambda,\theta) = \sigma^{-\gamma}\Big[\varphi(\sigma,\lambda,\theta) - \sum_{(j,k)\in\mathcal{E},\Re j\le\gamma}\varphi_{j,k}(\lambda,\theta)\sigma^j\log^k\sigma\Big] \in \mathcal{A}_{\mathrm{c}}^{0,(\mathcal{F},\beta)}([0,\Sigma)_\sigma \times (\Lambda,\infty]_\lambda \times \partial X_\theta). \tag{166}$$

Then, we have

$$\frac{r^2}{2}I_{-,\mathrm{phg}}[\varphi,\psi] = \sum_{(j,k)\in\mathcal{E},\Re j\le\gamma}\sum_{\kappa=0}^k \binom{k}{\kappa}\Big(\frac{1}{r}\Big)^j\log^{k-\kappa}\Big(\frac{1}{r}\Big)I_{-,\mathrm{phg},1+j,\kappa}[\varphi_{j,k},\psi] + \Big(\frac{1}{r}\Big)^\gamma I_{-,\mathrm{phg},1+\gamma,0}[\varphi_\gamma,\psi], \tag{167}$$

where the quantity $I_{-,\mathrm{phg},\gamma,\kappa}$ is defined by Lemma 4.9. That lemma then gives that each term on the right-hand side of eq. (167), and therefore $I_{-,\mathrm{phg}}[\varphi,\psi]$ itself, has the required form, except with a conormal error of order γ. But since $\gamma\le\alpha$ was arbitrary, eq. (162) follows.

Moving on to the other oscillatory integral, we introduce the coordinate $\delta = \lambda\tau - 1/2$. In terms of this coordinate,

$$s^2 I_{-,\mathrm{osc}}[\varphi,\psi](\tau,s,\theta) = e^{-i/4\tau}\tilde{I}_{-,\mathrm{osc}}[\varphi,(\delta+1)\psi(\delta)](\tau,s,\theta) \tag{168}$$

and

$$\tilde{I}_{-,\mathrm{osc}}[\varphi,\psi](\tau,s,\theta) = \int_{-\infty}^\infty e^{i\delta^2/\tau}\psi(2\delta)\varphi\Big(\frac{1}{s}\Big(\delta+\frac{1}{2}\Big),\frac{1}{\tau}\Big(\delta+\frac{1}{2}\Big),\theta\Big)\,\mathrm{d}\delta. \tag{169}$$

We can split this, for each $\gamma\in\mathbb{R}$, as

$$\tilde{I}_{-,\mathrm{osc}}[\varphi,\psi] = \sum_{(j,k)\in\mathcal{E},\Re j\le\gamma}\sum_{\kappa=0}^k\binom{k}{\kappa}(-1)^{k-\kappa}s^{-j}\log^{k-\kappa}(s)\tilde{I}_{-,\mathrm{osc}}[\varphi_{j,k},\psi_{j,\kappa}] + s^{-\gamma}\tilde{I}_{-,\mathrm{osc}}[\varphi_\gamma,\psi_\gamma], \tag{170}$$

where $\psi_{j,\kappa}(2\delta) = (\delta+1/2)^j\log^\kappa(\delta+1/2)\psi(2\delta)$ and $\psi_\gamma(2\delta) = (\delta+1/2)^\gamma\psi(2\delta)$. Now let

$$\varphi_{j,k}(\lambda,\theta) \sim \sum_{(j',k')\in\mathcal{F},\Re j'\le\beta}\lambda^{-j'}\log^{k'}(\lambda)\varphi_{j,k}^{j',k'}(\theta) \tag{171}$$

denote the expansion of $\varphi_{j,k}(\lambda,\theta)$ as $\lambda\to\infty$, and let $\lambda^{-\gamma}\varphi^{\gamma}_{j,k}(\lambda,\theta)$ denote the error from truncating the expansion to $\Re j'\leq\gamma$. We use similar notation for φ_γ. Then, for each $\gamma,\gamma'\in\mathbb{R}$,

$$\tilde{I}_{-,\mathrm{osc}}[\varphi_{j,k},\psi_{j,k}] = \sum_{(j',k')\in\mathcal{F},\Re j'\leq\gamma}\sum_{\kappa=0}^{k'}\binom{k'}{\kappa}(-1)^{k'-\kappa}\tau^{j'}\log^{k'-\kappa}(\tau)\tilde{I}_{-,\mathrm{osc}}[\varphi^{j',k'}_{j,k},\psi^{j',k'}_{j,k}] + \tau^{\gamma}\tilde{I}_{-,\mathrm{osc}}[\varphi^{\gamma}_{j,k},\psi^{\gamma}_{j,k}], \quad (172)$$

$$\tilde{I}_{-,\mathrm{osc}}[\varphi_{\gamma},\psi_{\gamma}] = \sum_{(j',k')\in\mathcal{F},\Re j'\leq\gamma'}\sum_{\kappa=0}^{k'}\binom{k'}{\kappa}(-1)^{k'-\kappa}\tau^{j'}\log^{k'-\kappa}(\tau)\tilde{I}_{-,\mathrm{osc}}[\varphi^{j',k'}_{\gamma},\psi^{j',k'}_{j,k}] + \tau^{\gamma'}\tilde{I}_{-,\mathrm{osc}}[\varphi^{\gamma'}_{\gamma},\psi^{\gamma'}_{\gamma}], \quad (173)$$

where $\psi^{j',\kappa}_{j,k}(2\delta)=(\delta+1/2)^{j}\log^{\kappa}(\delta+1/2)\psi_{j,k}(2\delta)$, and similarly for the other undefined terms.

The result then follows from Proposition 4.10. Indeed, we have that $\tilde{I}_{-,\mathrm{osc}}$ is a sum of four types of terms:

- First, consider the "main" terms proportional to $s^{-j}\log^{k}(s)\tau^{j'}\log^{k'}(\tau)\tilde{I}_{-,\mathrm{osc}}[\varphi^{j',\varkappa}_{j,\kappa},\psi^{j',\varkappa}_{j,\kappa}]$.

  Noting that $\tilde{I}_{-,\mathrm{osc}}[\varphi^{j',\varkappa}_{j,\kappa},\psi^{j',\varkappa}_{j,\kappa}]$ is completely independent of σ, the last clause of Proposition 4.10 yields
$$\tilde{I}_{-,\mathrm{osc}}[\varphi^{j',\varkappa}_{j,\kappa},\psi^{j',\varkappa}_{j,\kappa}]\in\tau^{1/2}C^{\infty}([0,\infty)_{\tau};C^{\infty}(\partial X_{\theta})). \quad (174)$$
- Now consider the terms $s^{-j}\log^{k}(s)\tau^{\gamma}\tilde{I}_{-,\mathrm{osc}}[\varphi^{\gamma}_{j,k},\psi^{\gamma}_{j,k}]$. Noting that $\tilde{I}_{-,\mathrm{osc}}[\varphi^{\gamma}_{j,k},\psi^{\gamma}_{j,k}]$ does not depend on s, we have
$$\tilde{I}_{-,\mathrm{osc}}[\varphi^{\gamma}_{j,k},\psi^{\gamma}_{j,k}]\in\tau^{1/2}\mathcal{A}^{0}_{\mathrm{loc}}([0,\infty)_{\tau}\times\partial X_{\theta}). \quad (175)$$
- Now consider the terms $s^{-\gamma}\tau^{j}\log^{k}(\tau)\tilde{I}_{-,\mathrm{osc}}[\varphi^{j,k}_{\gamma},\psi^{j,k}_{\gamma}]$.

  By the last clause of Proposition 4.10, $\tilde{I}_{-,\mathrm{osc}}[\varphi^{j,k}_{\gamma},\psi^{j,k}_{\gamma}]\in\tau^{1/2}\mathcal{A}^{0,(0,0)}_{\mathrm{loc}}([0,\infty)_{\tau}\times(0,\infty]_{s}\times\partial X_{\theta})$.
- Finally, in $\tau^{\gamma}s^{-\gamma'}\tilde{I}_{-,\mathrm{osc}}[\varphi^{\gamma'}_{\gamma},\psi^{\gamma'}_{\gamma}]$, $\tilde{I}_{-,\mathrm{osc}}[\varphi^{\gamma'}_{\gamma},\psi^{\gamma'}_{\gamma}]\in\tau^{1/2}\mathcal{A}^{0,0}_{\mathrm{loc}}([0,\infty)_{\tau}\times(0,\infty]_{s}\times\partial X_{\theta})$ by Proposition 4.10.

Putting this all together, $\tilde{I}_{-,\mathrm{osc}}[\varphi](\tau,s,\theta)\in\tau^{1/2}\mathcal{A}^{(\mathcal{E},\gamma),(\mathcal{F},\gamma')}_{\mathrm{loc}}([0,\infty)_{\tau}\times(0,\infty]_{s}\times\partial X_{\theta})$. Taking $\gamma\to\alpha$ and $\gamma'\to\beta$ completes the proof that $\tilde{I}_{-,\mathrm{osc}}$ lies in the desired function spaces.

If $\chi\in C^{\infty}_{\mathrm{c}}(\mathbb{R})$ satisfies $\operatorname{supp}\chi\Subset(-1,1)$, then $\chi(2s\Sigma)I_{-,\mathrm{osc}}[\varphi](\tau,s,\theta)\in\mathcal{A}^{\infty,\infty}_{\mathrm{loc}}([0,\infty)_{\tau}\times(0,\infty]_{s}\times\partial X_{\theta})$, as the corresponding clause of Proposition 4.10 shows that each of the terms $\tilde{I}_{-,\mathrm{osc}}$ appearing above is Schwartz when multiplied by $\chi(2s\Sigma)$. □

Remark 4.8. The proof shows that the $r\to\infty$ expansion of $I_{-,\mathrm{phg}}[\varphi,\psi]$ is given by

$$I_{-,\mathrm{phg}}[\varphi,\psi]\sim\frac{2}{r^{2}}\sum_{(j,k)\in\mathcal{E},\Re j\leq\alpha}\Big(\frac{1}{r}\Big)^{j}\log^{k}\Big(\frac{1}{r}\Big)\sum_{K\geq k,(j,K)\in\mathcal{E}}\binom{K}{k}\int_{\Lambda}^{\infty}e^{i\lambda^{2}\tau-i\lambda}\varphi_{j,K}(\lambda,\theta)\lambda^{1+j}\log^{K-k}(\lambda)\,\mathrm{d}\lambda, \quad (176)$$

analogously to eq. (148). The $s\to\infty$ expansion of $I_{-,\mathrm{osc}}[\varphi,\psi]$ is given by

$$I_{-,\mathrm{osc}}[\varphi,\psi]\sim\frac{e^{-i/4\tau}}{r^{2}}\sum_{(j,k)\in\mathcal{E},\Re j\leq\alpha}s^{-j}\log^{k}(s)(-1)^{k}\sum_{K\geq k,(j,K)\in\mathcal{E}}\binom{K}{k}\tilde{I}_{-,\mathrm{osc}}[\varphi_{j,K},(\delta+1)\psi_{j,K-k}(\delta)], \quad (177)$$

where

$$\tilde{I}_{-,\mathrm{osc}}[\varphi_{j,K},\psi_{j,K-k}] = \int_{-\infty}^{\infty} e^{i\delta^2/\tau}(\delta+1/2)^{1+j}\log^{K-k}(\delta+1/2)\psi(2\delta)\varphi_{j,K}\Big(\frac{1}{\tau}\Big(\delta+\frac{1}{2}\Big),\theta\Big)\,\mathrm{d}\delta. \quad (178)$$

We do not write the $\tau\to 0$ expansion.

Lemma 4.9. *Let $\psi\in C^\infty_{\mathrm{c}}(\mathbb{R})$ satisfy $\operatorname{supp}\psi\Subset(-1/2,1/2)$ and $0\notin\operatorname{supp}(1-\psi)$. Fix $\Sigma,\Lambda>0$. Let $\phi\in\mathcal{A}^{0,0,0,0}_{\mathrm{c}}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times(0,\infty]_r\times[0,\infty)_\tau\times\partial X_\theta)$, and consider, for $\gamma\in\mathbb{C}$ and $\kappa\in\mathbb{N}$,*

$$I_{-,\mathrm{phg},\gamma,\kappa}[\phi,\psi](\tau,r,\theta) = \int_\Lambda^\infty e^{i\lambda^2\tau-i\lambda}(1-\psi(2\lambda\tau-1))\phi\Big(\frac{\lambda}{r},\lambda,r,\tau,\theta\Big)\lambda^\gamma\log^\kappa(\lambda)\,\mathrm{d}\lambda. \quad (179)$$

Then, we have $I_{-,\mathrm{phg},\gamma,\kappa}[\varphi,\psi](\tau,r,\theta)\in\mathcal{A}^{0,0}_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta)$. If $\phi(\sigma,\lambda,r,\tau,\theta)=\phi(\lambda,\theta)$ depends only on λ,θ, then this can be improved to

$$I_{-,\mathrm{phg},\gamma,\kappa}[\phi,\psi](\tau,r,\theta)\in\mathcal{A}^{(0,0),(0,0)}_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta), \quad (180)$$

and $I_{-,\mathrm{phg},\gamma,\kappa}[\phi,\psi](\tau,s,\theta)$ does not depend on r in this case. ■

Proof. We first prove that $I_{-,\mathrm{phg},\gamma,\kappa}[\varphi](\tau,r,\theta)\in L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta)$. If $\Re\gamma<-1$, this follows immediately from an ML-bound. Otherwise, we use an integration-by-parts argument as usual:$\exp(i\lambda^2\tau-i\lambda)=-i(2\lambda\tau-1)^{-1}\partial_\lambda\exp(i\lambda^2\tau-i\lambda)$, so

$$I_{-,\mathrm{phg},\gamma,\kappa}[\phi,\psi] = \int_\Lambda^\infty\Big[\frac{-i}{2\lambda\tau-1}\frac{\partial}{\partial\lambda}e^{i\lambda^2\tau-i\lambda}\Big](1-\psi(2\lambda\tau-1))\phi\Big(\frac{\lambda}{r},\lambda,r,\tau,\theta\Big)\lambda^\gamma\log(\lambda)^\kappa\,\mathrm{d}\lambda. \quad (181)$$

Note that the integrand is well-defined, since the factor $1-\psi(2\lambda\tau-1)$ vanishes when $2\lambda\tau-1$ is sufficiently small. Integrating-by-parts, noting that no boundary terms arise,

$$I_{-,\mathrm{phg},\gamma,\kappa}[\phi,\psi] = I_{-,\mathrm{phg},\gamma-1,\kappa}[\phi_0,\psi_0]-\gamma I_{-,\mathrm{phg},\gamma-1,\kappa}[\phi_1,\psi_0]-\kappa I_{-,\mathrm{phg},\gamma-1,\kappa-1}[\phi_1,\psi_0], \quad (182)$$

where $\psi_0\in C^\infty_{\mathrm{c}}((-1/2,+1/2))$ is identically 1 near the origin and satisfies $\operatorname{supp}\psi_0\Subset\psi^{-1}(\{1\})$, and where

$$\begin{aligned}\phi_0(\sigma,\lambda,r,\tau,\theta) &= (1-\psi(2\lambda\tau-1))\Big[-\frac{2i\lambda\tau}{(2\lambda\tau-1)^2}+\frac{i}{2\lambda\tau-1}(\sigma\partial_\sigma+\lambda\partial_\lambda)\Big]\phi(\sigma,\lambda,r,\tau,\theta)\\ &\quad -\frac{2i\lambda\tau}{2\lambda\tau-1}\psi'(2\lambda\tau-1)\phi(\sigma,\lambda,r,\tau,\theta), \quad (183)\end{aligned}$$

$$\phi_1(\sigma,\lambda,r,\tau,\theta) = (1-\psi(2\lambda\tau-1))\frac{-i\phi(\sigma,\lambda,r,\tau,\theta)}{2\lambda\tau-1}. \quad (184)$$

Since the three functions $(1-\psi(2\lambda\tau-1))/(2\lambda\tau-1),(1-\psi(2\lambda\tau-1))\lambda\tau/(2\lambda\tau-1)^2,\psi'(2\lambda\tau-1)\lambda\tau/(2\lambda\tau-1)$ all lie in $\mathcal{A}^{0,0,0,0}_{\mathrm{c}}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times(0,\infty]_r\times[0,\infty)_\tau\times\partial X_\theta)$, we have

$$\phi_0(\sigma,\lambda,r,\tau,\theta),\phi_1(\sigma,\lambda,r,\tau,\theta)\in\mathcal{A}^{0,0,0,0}_{\mathrm{c}}([0,\Sigma)_\sigma\times(\Lambda,\infty]_\lambda\times(0,\infty]_r\times[0,\infty)_\tau\times\partial X_\theta). \quad (185)$$

Thus, each term on the right-hand side of eq. (182) has the same form as the original oscillatory integral, except with $\gamma-1$ in place of γ and possibly $\kappa-1$ in place of κ, if $\kappa>0$. So, eq. (182) can be used inductively to conclude the desired L^∞-bound from the $\Re\gamma<-1$ case.

In order to prove conormality, we want to prove that

$$(\tau\partial_\tau)^j(r\partial_r)^k L I_{-,\mathrm{phg},\gamma,\kappa}[\varphi](\tau,s,\theta)\in L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_r\times\partial X_\theta) \quad (186)$$

for all $j,k\in\mathbb{N}$ and $L\in\mathrm{Diff}(\partial X)$. The angular derivatives L are handled via differentiation under the integral sign as elsewhere, and for the other directions we use the following identities:

$$\tau\partial_\tau I_{-,\mathrm{phg},\gamma,\kappa}[\phi,\psi] = i\tau I_{-,\mathrm{phg},\gamma+2,\kappa}[\phi,\psi]+I_{-,\mathrm{phg},\gamma,\kappa}[\varphi,\psi]-I_{-,\mathrm{phg},\gamma,\kappa}[\varpi,\psi_0] \quad (187)$$

for $\varphi(\sigma,\lambda,r,\tau,\theta)=\tau\partial_\tau\phi(\sigma,\lambda,r,\tau,\theta)$ and $\varpi(\sigma,\lambda,r,\tau,\theta)=2\lambda\tau\psi'(\lambda\tau-1)\phi(\sigma,\lambda,r,\tau,\theta)$, and

$$r\partial_r I_{-,\mathrm{phg},\gamma,\kappa}[\phi,\psi](\tau,r,\theta) = I_{-,\mathrm{phg},\gamma,\kappa}[\varsigma,\psi] \quad (188)$$

for $\varsigma(\sigma,\lambda,r,\tau,\theta) = (-\sigma\partial_\sigma + r\partial_r)\phi(\sigma,\lambda,r,\tau,\theta)$. Each of these has the same form as the original integral, so, a similar inductive argument to the above shows that the bounds eq. (186) follow from those already proven.

Finally, suppose that $\phi(\sigma,\lambda,r,\tau,\theta) = \phi(\lambda,\theta)$ depends only on λ,θ. Then, eq. (187) can be improved to

$$\partial_\tau I_{-,\mathrm{phg},\gamma,\kappa}[\phi,\psi](\tau,r,\theta) = iI_{-,\mathrm{phg},\gamma+2,\kappa}[\phi,\psi](\tau,r,\theta) - I_{-,\mathrm{phg},\gamma+1,\kappa}[\mathit{\Pi},\psi_0](\tau,r,\theta) \tag{189}$$

for $\mathit{\Pi} = (\lambda\tau)^{-1}\varpi$, and now we simply have $\partial_s I_{-,\mathrm{phg},\gamma,\kappa}[\phi,\psi] = 0$. Noting that each term on the right-hand side of eq. (189) has the same form as the original integral, the usual inductive argument yields polyhomogeneity. □

Proposition 4.10. *Let $\psi \in C_c^\infty(\mathbb{R})$ satisfy $\operatorname{supp}\psi \Subset (-1/2,1/2)$. Fix $\Sigma,\Lambda > 0$. Let $\phi \in \mathcal{A}_c^{0,0,0,0}([0,\Sigma)_\sigma \times (\Lambda,\infty]_\lambda \times (0,\infty]_s \times [0,\infty)_\tau \times \partial X_\theta)$, and consider, for $\gamma \in \mathbb{C}$. Consider*

$$\mathcal{J}_k[\phi,\psi](s,\tau,\theta) = \int_{-\infty}^{+\infty} e^{i\delta^2/\tau}\delta^k\psi(2\delta)\phi\Big(\frac{1}{s}\Big(\delta+\frac{1}{2}\Big),\frac{1}{\tau}\Big(\delta+\frac{1}{2}\Big),s,\tau,\theta\Big)\,\mathrm{d}\delta. \tag{190}$$

Then, $\mathcal{J}_k[\phi,\psi](\tau,s,\theta) \in \tau^{1/2+k}\mathcal{A}^{0,0}_{\mathrm{loc}}([0,\infty)_\tau \times (0,\infty]_s \times \partial X_\theta)$. If $\chi \in C_c^\infty(\mathbb{R})$ satisfies $\operatorname{supp}\chi \Subset (-1,1)$, then $\chi(2s\Sigma)\mathcal{J}_k[\phi,\psi]$ is Schwartz.

If $\phi(\sigma,\lambda,s,\tau,\theta) = \phi(\sigma,s,\theta)$ does not depend on λ,τ, then $\mathcal{J}_k[\phi,\psi](\tau,\theta) \in \tau^{1/2+k}\mathcal{A}^{0,(0,0)}_{\mathrm{loc}}([0,\infty)_\tau \times (0,\infty]_s \times \partial X_\theta)$, where the $(0,0)$ is the index set as $\tau \to 0$. ■

Proof. (I) We begin by proving the weaker claim that

$$\mathcal{J}_{2k}[\phi,\psi] \in \tau^k L^\infty_{\mathrm{loc}}([0,\infty)_\tau \times (0,\infty]_s \times \partial X_\theta). \tag{191}$$

As with the other integrals analyzed elsewhere in this paper, this is proven using integration-by-parts: if $k=0$, then this bound is immediate, and otherwise, if $k \geq 1$, use

$$\mathcal{J}_{2k}[\phi,\psi] = -\frac{i\tau}{2}\int_{-\infty}^{+\infty}\Big[\frac{\partial}{\partial\delta}e^{i\delta^2/\tau}\Big]\delta^{2k-1}\psi(2\delta)\phi\Big(\frac{1}{s}\Big(\delta+\frac{1}{2}\Big),\frac{1}{\tau}\Big(\delta+\frac{1}{2}\Big),s,\tau,\theta\Big)\,\mathrm{d}\delta. \tag{192}$$

So,

$$\begin{aligned}-\frac{2i}{\tau}\mathcal{J}_{2k}[\phi,\psi] &= (2k-1)\mathcal{J}_{2k-2}[\phi,\psi] + 2\mathcal{J}_{2k-2}[\phi,\Delta\psi'(\Delta)] \\ &\quad + \mathcal{J}_{2k-2}[\sigma\partial_\sigma\phi,(\Delta+1/2)^{-1}\Delta\psi(\Delta)] + \mathcal{J}_{2k-2}[\lambda\partial_\lambda\phi,(\Delta+1/2)^{-1}\Delta\psi(\Delta)].\end{aligned} \tag{193}$$

Using this identity inductively, $\mathcal{J}_{2k}[\phi,\psi] \in \tau^k L^\infty_{\mathrm{loc}}([0,\infty)_\tau \times (0,\infty]_s \times \partial X_\theta)$ follows from the $k=0$ case.

(II) The next goal is to improve this to the optimal

$$\mathcal{J}_k[\phi,\psi] \in \tau^{1/2+k}L^\infty_{\mathrm{loc}}([0,\infty)_\tau \times (0,\infty]_s \times \partial X_\theta) \tag{194}$$

In order to improve upon these bounds when $k \geq 1$, we expand ϕ in Taylor series around $\delta = 0$:

$$\begin{aligned}\phi\Big(\frac{1}{s}\Big(\delta+\frac{1}{2}\Big),\frac{1}{\tau}\Big(\delta+\frac{1}{2}\Big),s,\tau,\theta\Big) &= \sum_{j_1+j_2\leq J}\frac{\delta^{j_1+j_2}}{j_1!j_2!s^{j_1}\tau^{j_2}}\phi^{(j_1,j_2)}\Big(\frac{1}{2s},\frac{1}{2\tau},s,\tau,\theta\Big) \\ &\quad + \int_0^\delta(\delta-\Delta)^J\sum_{j_1+j_2=J+1}\frac{1}{j_1!j_2!s^{j_1}\tau^{j_2}}\phi^{(j_1,j_2)}\Big(\frac{1}{s}\Big(\Delta+\frac{1}{2}\Big),\frac{1}{\tau}\Big(\Delta+\frac{1}{2}\Big),s,\tau,\theta\Big)\,\mathrm{d}\Delta,\end{aligned} \tag{195}$$

where $\phi^{(j_1,j_2)}(\sigma,\lambda,s,\tau,\theta) = \partial_\sigma^{j_1}\partial_\lambda^{j_2}\phi(\sigma,\lambda,s,\tau,\theta)$. So, for any $J \in \mathbb{N}$,

$$\mathcal{J}_k[\phi,\psi] = \mathcal{J}_{k,J+1}[\phi,\psi] + \sum_{j_1+j_2\leq J}\mathcal{J}_{k,j_1,j_2}[\phi,\psi] \tag{196}$$

where

$$\mathcal{J}_{j_1,j_2}[\phi,\psi] = \frac{1}{j_1!j_2!s^{j_1}\tau^{j_2}}\phi^{(j_1,j_2)}\Big(\frac{1}{2s},\frac{1}{2\tau},s,\tau,\theta\Big)\int_{-\infty}^{+\infty} e^{i\delta^2/\tau}\delta^{k+j_1+j_2}\psi(2\delta)\,\mathrm{d}\delta \tag{197}$$

and

$$\mathcal{J}_{k,J+1}[\phi,\psi] = \int_{-\infty}^{+\infty} e^{i\delta^2/\tau}\delta^k\psi(2\delta)\Big[\int_0^\delta(\delta-\Delta)^J \times \sum_{j_1+j_2=J+1}\frac{1}{j_1!j_2!s^{j_1}\tau^{j_2}}\phi^{(j_1,j_2)}\Big(\frac{1}{s}\Big(\Delta+\frac{1}{2}\Big),\frac{1}{\tau}\Big(\Delta+\frac{1}{2}\Big),s,\tau,\theta\Big)\,\mathrm{d}\Delta\Big]\,\mathrm{d}\delta. \tag{198}$$

The method of stationary phase (which in this case amounts to Parseval–Plancherel), applied to the integral

$$\hat{\mathcal{J}}_j[\psi] = \int_{-\infty}^{+\infty} e^{i\delta^2/\tau}\delta^j\psi(2\delta)\,\mathrm{d}\delta, \tag{199}$$

yields that $\hat{\mathcal{J}}_j[\psi]\in\tau^{1/2+j}C^\infty[0,\infty)_\tau$. So,

$$\mathcal{J}_{j_1,j_2}[\phi,\psi]\in\tau^{1/2+k+j_1+j_2}\mathcal{A}^{0,0}_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta). \tag{200}$$

In order to estimate $\mathcal{J}_{k,J+1}[\phi,\psi]$, we use a similar integration-by-parts argument as before. We prove, via induction on k, that

$$\mathcal{J}_{k,J+1}[\phi,\psi]\in\tau^{\lfloor(k+J+1)/2\rfloor}L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta), \tag{201}$$

which is trivial in the $k=0$ case. Integrating-by-parts, we can write $-2i\tau^{-1}\mathcal{J}_{k,J+1}[\phi,\psi] = (k-1)\mathcal{J}_{k-2,J+1}[\phi,\psi]+2\mathcal{J}_{k-1,J+1}[\phi,\psi']+J\mathcal{J}_{k-1,J}[\phi,\psi]$ if $J\geq 1$ and, otherwise,

$$-\frac{2i}{\tau}\mathcal{J}_{k,1}[\phi,\psi] = (k-1)\mathcal{J}_{k-2,1}[\phi,\psi]+2\mathcal{J}_{k-1,1}[\phi,\psi']+\mathcal{J}_{k-1}[\varphi,\tilde{\psi}] \tag{202}$$

for $\tilde{\psi}(2\delta)=(\delta+1/2)^{-J-1}\psi(2\delta)$ and

$$\varphi(\sigma,\lambda,s,\tau,\theta)=\sum_{j_1+j_2\leq J+1}\frac{\sigma^{j_1}\lambda^{j_2}}{j_1!j_2!}\frac{\partial^{j_1}}{\partial\sigma^{j_1}}\frac{\partial^{j_2}}{\partial\lambda^{j_2}}\phi(\sigma,\lambda,s,\tau,\theta), \tag{203}$$

where $\mathcal{J}_{k-1}[\varphi,\psi]$ is defined by eq. (190). The method of nonstationary phase shows that

$$\mathcal{J}_{k-1,J+1}[\phi,\psi]\in\tau^\infty L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta). \tag{204}$$

Moreover, by eq. (191), we have $\mathcal{J}_{k+J-1}[\varphi,\tilde{\psi}]\in\tau^{\lfloor(k+J-1)/2\rfloor}L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta)$. So, if we know that

$$\mathcal{J}_{k-2,J+1}[\phi,\psi],\mathcal{J}_{k-1,J}[\phi,\psi]\in\tau^{\lfloor(k+J-1)/2\rfloor}L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta), \tag{205}$$

then we can conclude that eq. (201) holds. As the sum of the subscripts of these integrals are 2 smaller than $J+K$, this sets up an inductive algorithm to deduce the claim from the $k+J=0$ base case which is already known.

Returning now to eq. (194), this follows from eq. (196) combined with eq. (200) and eq. (201), as long as J is sufficiently large.

(III) Having now proven the optimal L^∞-bounds eq. (201), we upgrade this to conormality by estimating derivatives. We just need to do this for $\mathcal{J}_{k,J+1}[\phi,\psi]$. We can do this using the usual argument: $L\mathcal{J}_{k,J+1}[\phi,\psi]=\mathcal{J}_{k,J+1}[L\phi,\psi]$ for $L\in\mathrm{Diff}(\partial X_\theta)$, $s\partial_s\mathcal{J}_{k,J+1}[\phi,\psi](\tau,s,\theta)=\mathcal{J}_{k,J+1}[(\sigma\partial_\sigma+s\partial_s)\varphi(\sigma,\lambda,s,\tau,\theta),\tilde{\psi}]$, where $\varphi,\tilde{\psi}$ are as above, and

$$\tau\partial_\tau\mathcal{J}_{k,J+1}[\phi,\psi](\tau,s,\theta)=-\tau^{-1}\mathcal{J}_{k+2,J+1}[\phi,\psi](\tau,s,\theta)+\mathcal{J}_{k,J+1}[(\lambda\partial_\lambda+\tau\partial_\tau)\varphi(\sigma,\lambda,s,\tau,\theta),\tilde{\psi}]. \tag{206}$$

So, via the usual inductive argument, $(s\partial_s)^m(\tau\partial_\tau)^n L\mathcal{J}_{k,J+1}[\phi,\psi] \in \tau^{\lfloor (k+J+1)/2\rfloor} L^\infty_{\mathrm{loc}}([0,\infty)_\tau\times (0,\infty]_s\times\partial X_\theta)$ for all $m,n\in\mathbb{N}$, which means that

$$\mathcal{J}_{k,J+1}[\phi,\psi] \in \tau^{\lfloor (k+J+1)/2\rfloor}\mathcal{A}^{0,0}_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta). \tag{207}$$

(IV) Consider now $\chi(2s\Sigma)\mathcal{J}_k[\phi,\psi]$, where χ is as in the statement of the proposition. Then, $\chi(2s\Sigma)\mathcal{J}_{k,j_1,j_2}[\phi,\psi]=0$ identically, for all $j_1,j_2\in\mathbb{N}$. So, the estimates above give Schwartz behavior as $\tau\to\infty$.

(V) Suppose now that $\phi(\sigma,\lambda,s,\tau,\theta)=\phi(\sigma,s,\theta)$ does not depend on λ,τ. Then, $\mathcal{J}_k[\phi,\psi]=\sum_{j=0}^J\mathcal{J}_{k,j,0}[\phi,\psi]+\mathcal{J}_{k,J+1}[\phi,\psi]$, where

$$\mathcal{J}_{j,0}[\phi,\psi]=\frac{1}{j!s^j}\phi^{(j)}\Big(\frac{1}{2s},s,\theta\Big)\int_{-\infty}^{+\infty}e^{i\delta^2/\tau}\delta^{k+j}\psi(2\delta)\,\mathrm{d}\delta=\frac{1}{j!s^j}\phi^{(j)}\Big(\frac{1}{2s},s,\theta\Big)\hat{\mathcal{J}}_j[\psi], \tag{208}$$

$$\mathcal{J}_{k,J+1}[\phi,\psi]=\int_{-\infty}^{+\infty}e^{i\delta^2/\tau}\delta^k\psi(2\delta)\Big[\int_0^\delta\frac{(\delta-\Delta)^J}{(J+1)!s^{J+1}}\phi^{(J+1)}\Big(\frac{1}{s}\Big(\Delta+\frac{1}{2}\Big),s,\theta\Big)\,\mathrm{d}\Delta\Big]\,\mathrm{d}\delta. \tag{209}$$

where $\phi^{(j)}(\sigma,s,\theta)=\partial_\sigma^j\phi(\sigma,s,\theta)$. Instead of eq. (200), we now have

$$\mathcal{J}_{j,0}[\phi,\psi]\in\tau^{1/2+k+j}\mathcal{A}^{0,(0,0)}_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta), \tag{210}$$

since $\hat{\mathcal{J}}_j[\psi](\tau)\in\tau^{1/2+k}C^\infty([0,\infty)_\tau)$. On the other hand, we can improve eq. (206) to

$$\partial_\tau\mathcal{J}_{k,J+1}[\phi,\psi]=\tau^{-2}\mathcal{J}_{k+2,J+1}[\phi,\psi](\tau,s,\theta). \tag{211}$$

Note that the the growth of the τ^{-2} factor is cancelled out by the extra τ^2 decay of $\mathcal{J}_{k+2,J+1}$ versus $\mathcal{J}_{k,J+1}$. So, the usual inductive argument yields

$$\mathcal{J}_{k,J+1}[\phi,\psi]\in\tau^{\lfloor (k+J+1)/2\rfloor}\mathcal{A}^{0,(0,0)}_{\mathrm{loc}}([0,\infty)_\tau\times(0,\infty]_s\times\partial X_\theta). \tag{212}$$

Combining the estimates above, we conclude the final clause of the proposition.

□

5. Proof of main lemma

We now turn to the proof of Theorem C, our "main lemma." Let $\mathcal{E},\mathcal{F},\mathcal{G},\alpha,\beta,\gamma,\phi$ be as in the statement of that theorem. Let $\chi_{\mathrm{low}},\chi_{\mathrm{tf}\cap\mathrm{bf}},\chi_{\mathrm{high}}\in C^\infty(X^{\mathrm{sp}}_{\mathrm{res}})$ be a partition of unity as in the introduction, so $\chi_{\mathrm{low}}+\chi_{\mathrm{tf}\cap\mathrm{bf}}+\chi_{\mathrm{high}}=1$ and

$$\operatorname{supp}\chi_{\mathrm{low}}\cap(\mathrm{bf}\cup\infty\mathrm{f})=\varnothing,\ \operatorname{supp}\chi_{\mathrm{tf}\cap\mathrm{bf}}\cap(\mathrm{zf}\cup\infty\mathrm{f})=\varnothing,\ \operatorname{supp}\chi_{\mathrm{high}}\cap(\mathrm{zf}\cup\mathrm{tf})=\varnothing, \tag{213}$$

and we can choose that $\operatorname{supp}\phi_{\mathrm{low}}\cap(\operatorname{supp}\chi_{\mathrm{tf}\cap\mathrm{bf}}\cup\operatorname{supp}\chi_{\mathrm{high}})=\varnothing$. Moreover, $\chi_{\mathrm{tf}\cap\mathrm{bf}}$ can be chosen to be supported over $\dot{X}[R]$ for some R, that is over a collar neighborhood of the boundary of X.

We split $I_\pm[\phi]$ as in eq. (36). We analyze each piece separately. First of all, according to Proposition 2.1,

$$I_\pm[\chi_{\mathrm{low}}\phi]\in\mathcal{A}^{(\mathcal{E}/2+1,\alpha/2+1),(\mathcal{F}+2,\beta+2),(0,0)}(C_1). \tag{214}$$

On the other hand, Proposition 3.1 says that $I_+[\chi_{\mathrm{high}}\phi]\in\mathcal{A}^{\infty,\infty,(0,0)}$, and, together, Proposition 3.1 and Proposition 3.3 say that

$$\begin{aligned}I_-[\chi_{\mathrm{high}}\phi]&\in\mathcal{A}^{\infty,\infty,(0,0)}(C)+e^{-i(1-\chi(t))r^2/4t}\mathcal{A}^{\infty,\infty,(\mathcal{G}+1/2,\gamma+1/2),\infty,\infty}(M)\\&=e^{-i(1-\chi(t))r^2/4t}\mathcal{A}^{\infty,\infty,(\mathcal{G}+1/2,\gamma+1/2),\infty,(0,0)}(M).\end{aligned} \tag{215}$$

Regarding $I_+[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]$, says Proposition 4.1 that $I_+[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]\in\mathcal{A}^{\infty,(\mathcal{F}+2,\beta+2),(0,0)}(C_1)$. Regarding $I_-[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]$, the same proposition says that $I_-[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]=\exp(-ir^2/4t)\tilde{I}_-[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]+I_{-,\mathrm{phg}}[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]$ for some

$$\tilde{I}_-[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]\in\mathcal{A}^{\infty,(\mathcal{F}+2,\beta+2),(\mathcal{G}+1/2,\gamma+1/2),\infty,\infty}(M). \tag{216}$$

and $I_{-,\mathrm{phg}}[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]\in\mathcal{A}^{\infty,(\mathcal{F}+2,\beta+2),(0,0)}(C_1)$. So, summing up $I_\pm[\phi]=I_\pm[\chi_{\mathrm{low}}\phi]+I_\pm[\chi_{\mathrm{tf}\cap\mathrm{bf}}\phi]+I_\pm[\chi_{\mathrm{high}}\phi]$, we conclude Theorem C.

Appendix A. Microlocal supplement

The goal of this appendix is to sketch a microlocal proof of the following proposition:

Proposition A.1. *Suppose that $f \in \mathcal{S}(X)$, and let $\varphi_\pm(\sigma, x) = e^{\mp i\sigma r} R(\sigma^2 \pm i0) f(x)$. Then, letting $I_-[\varphi_-] = \exp(-i(1-\chi(t))r^2/4t) I_{\mathrm{osc}}[\varphi] + I_{-,\mathrm{phg}}[\varphi]$ denote the decomposition of $I_-[\varphi_-]$ provided in Theorem C, it must be the case that the linear combination*

$$I_{\mathrm{phg}} = I_+[\varphi_+] - I_{-,\mathrm{phg}}[\varphi_-] \tag{217}$$

is Schwartz at dilF. ■

Proof. First of all, note that because I_{phg} is already polyhomogeneous on C_1, it suffices to prove that I_{phg} is Schwartz in dilF°. Let M_0 denote the manifold-with-boundary resulting from taking $[\mathbb{R}_t \times X; \{\pm\infty\} \times \partial X]$ subtracting the boundary hypersurfaces corresponding to $\{\pm\infty\}\times X$ and blowing down the boundary hypersurfaces corresponding to $\mathbb{R}_t \times X$. Concretely, this mwc is identifiable with $\mathbb{R}_{t/r} \times [0,\infty)_\rho \times \partial X_\theta$ near the boundary. We utilize Melrose's sc-calculus on M_0. See [Vas18] for an introduction.

To show that I_{phg} is Schwartz at dilF° means to show that $\mathrm{WF}_{\mathrm{sc}}(I_{\mathrm{phg}}) = \varnothing$, where $\mathrm{WF}_{\mathrm{sc}}$ is Melrose's notion of sc-wavefront set. Let ${}^{\mathrm{sc}}o^* M_0$ denote the zero section of the sc-cotangent bundle over dilF°. Because I_{phg} is conormal,

$$\mathrm{WF}_{\mathrm{sc}}(I_{\mathrm{phg}}) \subseteq {}^{\mathrm{sc}}o^* M_0, \tag{218}$$

as follows e.g. via repeated applications of ellipticity. In order to study $\mathrm{WF}_{\mathrm{sc}}(I_{\mathrm{phg}})$ further, we use the relation of $I[\varphi] = I_+[\varphi_+] - I_-[\varphi_-]$ to $u(t,x) = (U(t)f)(x)$ given by eq. (16). Indeed,

$$\mathrm{WF}_{\mathrm{sc}}(e^{iEt}\varphi(x)) = \varnothing \tag{219}$$

for any $E \in \mathbb{R}$ and $\varphi \in \mathcal{S}(X)$. (If this looks strange, recall that ∂M_0 does not contain any points where $r \neq \infty$.) So, $\mathrm{WF}_{\mathrm{sc}}(u) = \mathrm{WF}_{\mathrm{sc}}(I[\varphi])$. Because $I[\varphi] = \exp(-i(1-\chi(t))r^2/4t) I_{\mathrm{osc}}[\varphi_-] + I_{\mathrm{phg}}$, we have

$$\begin{aligned} \mathrm{WF}_{\mathrm{sc}}(I_{\mathrm{phg}}) &\subseteq \mathrm{WF}_{\mathrm{sc}}(\exp(-i(1-\chi(t))/4t\rho^2) I_{\mathrm{osc}}[\varphi_-]) \cup \mathrm{WF}_{\mathrm{sc}}(I[\varphi]) \\ &= \mathrm{WF}_{\mathrm{sc}}(\exp(-i(1-\chi(t))/4t\rho^2) I_{\mathrm{osc}}[\varphi_-]) \cup \mathrm{WF}_{\mathrm{sc}}(u). \end{aligned} \tag{220}$$

By the conormality of I_{osc}, we have $\mathrm{WF}_{\mathrm{sc}}(\exp(-i(1-\chi(t))/4t\rho^2) I_{\mathrm{osc}}) \subseteq \mathrm{graph}_{\partial M_0}(2(r/t)\,\mathrm{d}r + (r/t)^2\,\mathrm{d}t)$ the right-hand side being the graph over ∂M_0 of the 1-form $-\mathrm{d}(r^2/t) = -2(r/t)\,\mathrm{d}r + (r/t)^2\,\mathrm{d}t$, which is a smooth, *nonvanishing* section of ${}^{\mathrm{sc}}T^* M_0$. Because it is nonvanishing,

$$\mathrm{WF}_{\mathrm{sc}}(\exp(-i(1-\chi(t))/4t\rho^2) I_{\mathrm{osc}}) \cap {}^{\mathrm{sc}}o^* M_0 = \varnothing. \tag{221}$$

Combining this with eq. (218) and eq. (220), we have $\mathrm{WF}_{\mathrm{sc}}(I_{\mathrm{phg}}) \subseteq \mathrm{WF}_{\mathrm{sc}}(u) \cap {}^{\mathrm{sc}}o^* M_0$. To summarize, to prove that $\mathrm{WF}_{\mathrm{sc}}(I_{\mathrm{phg}}) = \varnothing$, it suffices to prove that $\mathrm{WF}_{\mathrm{sc}}(u) \cap {}^{\mathrm{sc}}o^* M_0 = \varnothing$.

In order to accomplish this, one can use a standard argument based on the splitting $u = u_+ + u_-$, where $u_\pm(t,x) = 1_{\pm t>0} u(t,x)$. As $\mathrm{WF}_{\mathrm{sc}}(u) \subseteq (\mathrm{WF}_{\mathrm{sc}}(u_-) \cup \mathrm{WF}_{\mathrm{sc}}(u_+))$, it suffices to prove $\mathrm{WF}_{\mathrm{sc}}(u_\pm) \cap {}^{\mathrm{sc}}o^* M_0 = \varnothing$ for each choice of sign. To this end, note that $u_\pm$ satisfy the PDE

$$-i\partial_t u_\pm = P u_\pm \mp i\delta(t) f(x) \tag{222}$$

in the sense of distributions. As $\mathrm{WF}_{\mathrm{sc}}(\delta(t)f(x))$ is contained at fiber infinity (as can be seen using the Fourier transform in a local coordinate patch), it is irrelevant as far as sc-wavefront set in the interior of the fibers is concerned. We have $P = \Delta_g \bmod \mathrm{Diff}_{\mathrm{sc}}^{1,-2}(M_0)$, where the '$-2$' indicates two orders of decay. So, the principal symbol of P is

$$p(\tau,\xi) = \tau + g^{-1}(\xi,\xi) \in C^\infty({}^{\mathrm{sc}}T^* M_0). \tag{223}$$

Associated to this function is the Hamiltonian vector field $H_p = (\partial_\tau p)\partial_t + (\partial_\xi p)\cdot \partial_x = \partial_t + 2g^{-1}(\xi, -)$. Let $\mathsf{H}_p = \rho H_p$, which restricts to a vector field on ${}^{\mathrm{sc}}T^*_{\partial M_0} M_0$. The Duistermaat-Hörmander theorem

— see [Vas18] for a precise statement in the context of the sc-calculus — then says that the portion of $\mathrm{WF}_{\mathrm{sc}}(u_\pm)$ in ${}^{\mathrm{sc}}T^*_{\partial M_0}M_0$ consists of maximally extended integral curves of H_p. Those in the zero section ${}^{\mathrm{sc}}o^*M_0$ are of the form

$$\gamma_{\theta_0} = ({}^{\mathrm{sc}}o^*M_0) \cap \{\theta = \theta_0\}, \tag{224}$$

as follows from the explicit formula for H_p. Because $u_\pm$ vanishes when $\pm t < 0$, it has no sc-wavefront set over the corresponding copy of dilF°. So, $\gamma_{\theta_0} \not\subseteq \mathrm{WF}_{\mathrm{sc}}(u_\pm)$, which, by Duistermaat–Hörmander, implies $\mathrm{WF}_{\mathrm{sc}}(u_\pm) \cap {}^{\mathrm{sc}}o^*M_0 = \varnothing$. □

Appendix B. Necessity of nf and dilF

In this appendix, we summarize the role of $\mathrm{nf}, \mathrm{dilF}$ and why it is not possible to blow down either while maintaining the exponential-polyhomogeneous form of solutions of the Schrödinger equation. We only sketch the argument, and, for simplicity, we work in the $\dim X = 1$ case.

Lemma B.1. *If $\theta_1, \theta_2, a_0, a_1, a_2$ are polyhomogeneous functions on a mwc M and $p \in \partial M$ are such that θ_1, θ_2 are real-valued, $a_0 \in C^\infty(M; \mathbb{C}^\times)$, a_1 extends continuously to a neighborhood of p, and this extension vanishes at p, and $e^{i\theta_1}(a_0 + a_1) = e^{i\theta_2}a_2$, then, near p, the difference $\theta_1 - \theta_2$ is, in a neighborhood of p, conormal at each adjacent boundary hypersurface to every negative order.* ■

Proof. The function $a_0 + a_1$ is nonvanishing near p, so $b = (a_0 + a_1)^{-1}$ is well-defined there, and a straightforward argument shows that b is polyhomogeneous, with a continuous extension to the boundary of M near p. Let $\tilde{b}$ be a globally polyhomogeneous function extending continuously to all of ∂M and satisfying $\tilde{b} = b$ near p. Then,

$$e^{i\theta_1 - i\theta_2} = \tilde{b}a_2, \tag{225}$$

near p. Since a_2 is polyhomogeneous, and since $a_0 + a_1$ is uniformly bounded near p, it must be that a_2 extends continuously to a neighborhood of p. So, the right-hand side of eq. (225) extends continuously to a neighborhood of p.

Moreover, the extension of a_2 to the boundary must be nonvanishing near p, and likewise for $\tilde{b}$, so $\tilde{b}a_2$ is nonvanishing near p. This implies that the difference

$$\theta_1 - \theta_2 = -i \log(\tilde{b}a_2) \tag{226}$$

is, near p, polyhomogeneous with all index sets in $\{z \in \mathbb{C} : \Re z \geq 0\} \times \mathbb{N}$, which is equivalent to the desired result. □

The elementary proof is omitted for brevity's sake.

Consider now the Gaussian wavepacket

$$G(t, x) = \frac{1}{\sqrt{1 + 2it}} \exp\Big(- \frac{x^2}{1 + 2it} \Big). \tag{227}$$

This solves the free Schrödinger equation in 1D with Schwartz initial data.

Proposition B.2. *The Gaussian wavepacket above is not of the form $ae^{i\theta}$ for θ real-valued and a, θ polyhomogeneous on M/nf or M/dilF.* ■

Below, we will use the same names to refer to faces of the M/f as the corresponding faces in M.

Proof sketch. First, suppose, to the contrary, that $G = e^{i\varphi}G_0$ for some φ, G_0 polyhomogeneous on M/nf. Looking at the $t \to 0^+$ behavior of G in compact subsets worth of r, it must be the case that

- ⋆ the index set $\mathcal{E}_\Sigma$ of φ at Σ can be taken to contain no terms $(j, k) \in \mathbb{C} \times \mathbb{N}$ with $\Re j < 0$.

Near the corner $\Sigma \cap \mathrm{dilF} \subset M/\mathrm{nf}$, we can use $\rho = 1/r$ and $s = t/r$ as a coordinate system. In terms of these coordinates,

$$G = \frac{\rho^{1/2}}{\sqrt{\rho + 2is}} \exp\Big(- \frac{1}{\rho^2 + 2is\rho} \Big). \tag{228}$$

For $s > 0$, this has the form $\rho^{1/2}\exp(i/(2s\rho))C^\infty(\mathbb{R}^+_s \times [0,\infty)_\rho; \mathbb{C}^\times)$ near $\rho = 0$. Consequently, applying Lemma B.1,

$$\varphi = 1/(2s\rho) \bmod \mathcal{A}^{0-}(\mathbb{R}^+_s \times [0,\infty)_\rho). \tag{229}$$

But, for no index set $\mathcal{E}_{\mathrm{nf}}$ can $\varphi \in \mathcal{A}^{\mathcal{E}_{\mathrm{nf}},\mathcal{E}_\Sigma}(([0,\infty)_s \times [0,\infty)_\rho)$ be consistent with eq. (229), since this forces $(-1,0) \in \mathcal{E}_\Sigma$, in conflict with our earlier observation ($\star$). So, the supposition that G has the stated form on M/nf is not tenable.

We now turn to M/dilF. Since $(1+2it)^{-1/2}$ *is* polyhomogeneous on C, and therefore on M/dilF, in order to prove the desired result it suffices to prove that we do not have $(1+2it)^{1/2}G = e^{i\varphi}a$ for φ real valued and a, φ polyhomogeneous on M/dilF. Near the corner $\mathrm{nf} \cap \mathrm{parF}$, we can use coordinates $\varrho = 1/t^{1/2}$ and $\tau = t/r^2$. In terms of these coordinates, $(1+2it)^{1/2}G = \exp(-1/\tau(\varrho^2+2i))$. So, for $\varrho > 0$, $(1+2it)^{1/2}G$ is Schwartz as $\tau \to 0^+$. It follows that a is Schwartz at nf, however, restricting to $\varrho = 0$, $(1+2it)^{1/2}G$ has magnitude 1, and therefore so does a. This contradicts the joint expandability of a at the corner, and therefore polyhomogeneity.

□

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