Contents

CHAPTER 1

Relativistic symmetry

[S]ome good news: quantum field theory is based on the same quantum mechanics that was invented by Schrödinger, Heisenberg, Pauli, Born, and others in 1925–26, and has been used ever since in atomic, molecular, nuclear, and condensed matter physics. – Steven Weinberg in [Wei05, §2.1]

Physicists often describe quantum field theory as the inevitable consequence of reconciling the formalism of quantum mechanics with the strictures of special relativity. This applies most clearly to the description of isolated particles – a kinematical problem whose solution will be covered in later lectures. Electrons, neutrinos, and quarks have many different properties, from charge to color, but they are all "spin-1/2." The force carriers in the standard model of particle physics all are "spin-1." The notion of particle spin, which (despite the suggestive terminology) has no true analogue in classical mechanics, arises naturally from the conjunction of quantum mechanics and a fragment of relativistic covariance.

For our purposes, the conjunction can be encoded in a single definition:

A *relativistic quantum mechanical system* consists of a separable Hilbert space $\mathcal{H}$ together with a strongly-continuous projective unitary representation

$$\rho : \mathrm{P}(1,d) \to \mathrm{PU}(\mathcal{H}) = \mathrm{U}(\mathcal{H})/(\mathrm{U}(1)I) \tag{1.1}$$

of the (restricted) Poincaré group

$$\mathrm{P}(1,d) = \mathbb{R}^{1,d} \rtimes \mathrm{SO}(1,d), \tag{1.2}$$

where $d \in \mathbb{N}^+$ is the number of spatial dimensions.

Isolated particles (whether elementary or composite) are relativistic quantum mechanical systems in this sense, as are full-fledged quantum fields.

The goal of this lecture is to unpack the definition above. We do not assume familiarity with the Poincaré group.

1. The Poincaré group

Special relativity is most easily summarized as the requirement that the laws of physics admit as a group of symmetries the (restricted) Poincaré group $\mathrm{P}(1,d)$. From a modern perspective, the central insight contained in Einstein's groundbreaking 1905 paper [Ein05] is that the Poincaré group is among the symmetries of Maxwellian electrodynamics, and, if the same applies to all other fundamental laws of physics, then any observer moving at constant velocity ("inertial frame of reference," see §A.1) will observe no departure from Maxwell's theory. In particular, the speed of light

$$c \approx 2.998 \times 10^8 \text{ m/s} \tag{1.3}$$

will appear the same to all. We will work in units where $c = 1$.

The Poincaré group is a particular subgroup of the group

$$\mathrm{Aff}(\mathbb{R}^{1,d}) = \{\text{bijective affine } T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}\} \tag{1.4}$$

of affine transformations of Minkowski spacetime,

$$\mathbb{R}^{1,d} = \mathbb{R}_t \times \mathbb{R}^d_{\mathbf{x}}. \tag{1.5}$$

A spacetime coordinate $x = (t, \mathbf{x}) \in \mathbb{R}^{1,d}$ consists of two components, a "temporal" component $t = x^0$, saying *when* some event occurs, and a "spatial" component $\mathbf{x} \in \mathbb{R}^d$, saying *where*. On Minkowski spacetime is defined the *Minkowski interval*

$$\begin{aligned} d : (\mathbb{R}^{1,d})^2 &\to \mathbb{R} \\ d(x,y) &= (x-y)^2, \end{aligned} \tag{1.6}$$

where $z^2 \overset{\text{def}}{=} -t^2 + \|\mathbf{z}\|^2$ for $z = (t, \mathbf{z}) \in \mathbb{R}^{1,d}$. The Minkowski interval should be compared with the Euclidean interval $(\mathbf{x}, \mathbf{y}) \mapsto \|\mathbf{x} - \mathbf{y}\|^2$. Note the signs.

Just as the Euclidean group is defined to be the group of isometries of Euclidean space, the Poincaré group

$$\mathrm{P}_{\text{full}}(1,d) = \{\text{bijections } T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d} \text{ s.t. } d(T(x), T(y)) = d(x,y)\} \tag{1.7}$$

is defined to be the group of "isometries" of Minkowski spacetime, bijections which preserve the Minkowski interval. Such T are automatically affine, making the Poincaré group a (Lie) subgroup of the affine group $\mathrm{Aff}(\mathbb{R}^{1,d})$. [Problem 1.2]

This group is not connected. The *restricted* Poincaré group $\mathrm{P} = \mathrm{P}(1,d)$ is then defined to be the connected component of $\mathrm{P}_{\text{full}}(1,d)$ containing the identity.

1.1. Basic Poincaré transformations.

Example 1.1 (Spacetime translation). A spacetime translation $T_a : x \mapsto x + a$ is an example of an element of the restricted Poincaré group. This is clear from the translation invariance of the Minkowski interval. The set of translations $T_a, a \in \mathbb{R}^{1,d}$ is a (Lie) subgroup of the restricted Poincaré group, forming a copy of the abelian group $(\mathbb{R}^{1,d}, +)$. Indeed,

$$T_a T_b = T_{a+b}. \tag{1.8}$$

■

Recall the orthogonal group $\mathrm{O}(d) = \{R \in \mathbb{R}^{d\times d} : R^{-1} = R^\intercal\}$. The subgroup $\mathrm{SO}(d) = \{R \in \mathrm{O}(d) : \det R = 1\}$ consists of the orientation-preserving orthogonal transformations, i.e. rotations.

Example 1.2 (Spatial rotations). Let $R \in \mathrm{SO}(d)$. The transformation

$$T_R : \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} t \\ R\mathbf{x} \end{bmatrix} \tag{1.9}$$

which rotates the spatial coordinate $\mathbf{x}$ (but leaves time invariant), is also in the restricted Poincaré group. This is obvious from the rotation invariance of the Minkowski interval. The $T_R, R \in \mathrm{SO}(d)$ form a (Lie) subgroup of the Poincaré group, a copy of $\mathrm{SO}(d)$. Indeed,

$$T_R T_{R'} = T_{RR'}. \tag{1.10}$$

■

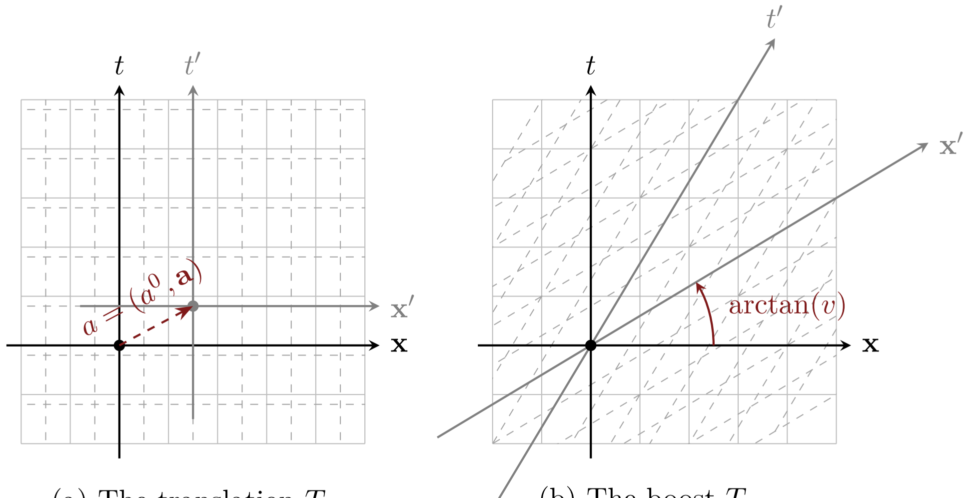
Let $\mathbb{B}^d = \{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| < 1\}$ denote the open unit ball.

Example 1.3 (Boosts). A (Lorentz) *boost* (with velocity $\mathbf{v} \in \mathbb{B}^d$) is a map $T_{\Lambda(\mathbf{v})}$ of the form

$$T_{\Lambda(\mathbf{v})} : \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} \mapsto \Lambda(\mathbf{v}) \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix}, \quad \Lambda(\mathbf{v}) = \frac{1}{\sqrt{1 - \|\mathbf{v}\|^2}} \begin{bmatrix} 1 & -\mathbf{v}^\intercal \\ -\mathbf{v} & \hat{\boldsymbol{\gamma}} \end{bmatrix}, \quad \hat{\boldsymbol{\gamma}} = \sqrt{\frac{1 - \|\mathbf{v}\|^2}{I_d - \mathbf{v}\mathbf{v}^\intercal}}. \tag{1.11}$$

[Exercise 1.2] This lies in the restricted Poincaré group, as a short computation reveals. The factor

$$\gamma = \frac{1}{\sqrt{1 - \|\mathbf{v}\|^2}} \tag{1.12}$$



(a) The translation T_a (b) The boost $T_{\Lambda(\mathbf{v})}$

FIGURE 1.1

is known as the *Lorentz factor*. The matrix $\hat{\boldsymbol{\gamma}}$ is defined via the functional calculus for symmetric matrices. It acts as the identity on $\operatorname{span}_{\mathbb{R}} \mathbf{v}$ and is multiplication by γ^{-1} on $(\operatorname{span}_{\mathbb{R}} \mathbf{v})^{\perp} \subseteq \mathbb{R}^d$, so its matrix elements are

$$\hat{\gamma}^{ij} = \frac{1}{\gamma}\Big(\delta_{ij} + (\gamma - 1)\frac{v_i v_j}{v^2}\Big). \tag{1.13}$$

Thus,

$$\Lambda(\mathbf{v}) = \begin{bmatrix} \gamma & -\gamma \mathbf{v}^{\mathsf{T}} \\ -\gamma \mathbf{v} & I + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}}^{\mathsf{T}} \end{bmatrix}. \tag{1.14}$$

When $\mathbf{v} = (v, 0, \cdots, 0)$,

$$\Lambda(\mathbf{v}) = \Lambda_{\mathrm{std}}(v) \overset{\mathrm{def}}{=} \begin{bmatrix} \gamma & -\gamma v & 0 \\ -\gamma v & \gamma & 0 \\ 0 & 0 & I_{d-1} \end{bmatrix}. \tag{1.15}$$

That is,

$$\Lambda_{\mathrm{std}}(v) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \ni (t, x, \mathbf{y}) \mapsto \Big(\frac{t - vx}{\sqrt{1 - v^2}}, \frac{x - vt}{\sqrt{1 - v^2}}, \mathbf{y}\Big). \tag{1.16}$$

Restricting attention to $\Lambda_{\mathrm{std}}(v)$ is without essential loss of generality, because any boost $\Lambda(\mathbf{v})$ has the form

$$\Lambda(\mathbf{v}) = T_R T_{\Lambda_{\mathrm{std}}(v)} T_{R^{-1}} \tag{1.17}$$

for some $v \in (-1, 1)$ and $R \in \mathrm{SO}(d)$. If $d \geq 2$, then we can take $v \in [0, 1)$. ■ [Exercise 1.2(b)]

The Lorentz boost $\Lambda_{\mathrm{std}}(v)$ should be contrasted with the Galilei boost:

$$(t, x, \mathbf{y}) \mapsto (t, x - vt, \mathbf{y}). \tag{1.18}$$

This is a symmetry of non-relativistic classical and quantum mechanics. Compared to the Galilei boost, the Lorentz boost has two differences:

(i) the presence of the Lorentz factors $\gamma = 1/\sqrt{1 - v^2}$,
(ii) the correction $-\gamma v x$ to the time coordinate.[1]

[1]When factors of the speed of light c are restored, this is $-\gamma v x / c^2$, so very small in everyday life.

The first difference is responsible for length contraction. A moving object will appear to a stationary observer as squashed along its direction of motion, as compared to its shape at rest, by a factor of γ. A conceptual consequence of the second difference is the relativity of simultaneity; events which are simultaneous in one frame need not be simultaneous in other frames. This can be seen by the tilting of the $\mathbf{x}'$-axis in Section 1.1(b).

[Exercise 1.3]

WARNING: Boosts do not form a subgroup of the Lorentz group (unless $d = 1$). The product of two collinear boosts is a boost, but the product of two non-collinear boosts is a boost times a rotation (a *Wigner–Thomas rotation*). At the level of the Lie algebra, this can be seen from the fact that the commutators of generators of boosts involve the generators of rotations.

[Problem 1.3 and Problem 1.4]

EXAMPLE 1.4 (Time-reversal). Consider $T_{\mathscr{T}} : (t, \mathbf{x}) \mapsto (-t, \mathbf{x})$. This is an element of the full Poincaré group but not the *restricted* group, as we will discuss below. ■

EXAMPLE 1.5 (Parity). Let $\mathscr{R} \in \mathrm{O}(d)$ denote a spatial reflection across an odd number of Cartesian coordinates, say

$$\mathscr{R}(\mathbf{x}) = \begin{cases} (-x^1, x^2, \cdots) & (d \text{ even}), \\ -\mathbf{x} & (d \text{ odd}). \end{cases} \tag{1.19}$$

Consider $T_{\mathscr{P}} : (t, \mathbf{x}) \mapsto (t, \mathscr{R}\mathbf{x})$. This is also in the full Poincaré group, but not the restricted Poincaré group. ■

Spacetime translations, spatial rotations, and boosts together generate the restricted Poincaré group — see the problems at the end of this chapter. Any restricted Poincaré transformation T can be written uniquely as

$$T = T_a T_R T_{\Lambda(\mathbf{v})} \tag{1.20}$$

for some $a \in \mathbb{R}^{1,d}$, $R \in \mathrm{SO}(3)$, and $\mathbf{v} \in \mathbb{B}^d$. Moreover, any Poincaré transformation can be written uniquely as

$$T = T_{\mathscr{T}}^{\eta} T_{\mathscr{P}}^{\xi} T_a T_R T_{\Lambda(\mathbf{v})} \tag{1.21}$$

for $\eta, \xi \in \{0, 1\}$, $a \in \mathbb{R}^{1,d}$, $R \in \mathrm{SO}(3)$, and $\mathbf{v} \in \mathbb{B}^d$. So the time-reversal and parity transformations, together with spacetime translations, spatial rotations, and boosts generate the full Poincaré group. The order in which the operators are listed in the expressions above is not important; they do not generally commute, but things can still be rearranged, owing to the following computation:

PROPOSITION 1.6. *Consider the five types of operator above, $T_{\mathscr{T}}$, $T_{\mathscr{P}}$, T_a for $a \in \mathbb{R}^{1,d}$, T_R for $R \in \mathrm{SO}(d)$, and $T_{\Lambda(\mathbf{v})}$ for $\mathbf{v} \in \mathbb{B}^d$. For any A, B of these types, $AB = B'A'$ for A' as the same type as A and B' as the same type of B.* ■

PROOF. Straightforward casework. □

1.2. Lorentz transformations. Any affine transformation $T \in \mathrm{Aff}(\mathbb{R}^{1,d})$ can be written (uniquely) as

$$T = T_a T_\Lambda, \tag{1.22}$$

where $a = T(0)$ is the image of the spacetime origin under T and $T_\Lambda = T_{-a}T$ is some linear transformation. When T is a Poincaré transformation, T_Λ is known as a *Lorentz transformation*. Equation (1.21) shows that any Lorentz transformation has the form

$$T_\Lambda = T_{\mathscr{T}}^{\eta} T_{\mathscr{P}}^{\xi} T_R T_{\Lambda(\mathbf{v})}, \tag{1.23}$$

a product of a pure boost, a spatial rotation, and possibly some reflections. If T_Λ is in the restricted Poincaré group P, then it is a *restricted* Lorentz transformation. This means that $\eta, \xi = 0$ in eq. (1.23):

$$T_\Lambda = T_R T_{\Lambda(\mathbf{v})}. \tag{1.24}$$

Remark: Physicists often speak of "Lorentz covariance" instead of Poincaré covariance. Insofar as these terms are not used interchangeably, the latter means the former together with translation-invariance.

Being linear, any Lorentz transformation T_Λ has the form $T_\Lambda : \mathbb{R}^{1+d} \ni x \mapsto \Lambda x$ for some (unique) matrix $\Lambda \in \mathbb{R}^{(1+d)\times(1+d)}$. Since $x \mapsto x^2$ is a quadratic form $x^2 = x^\intercal \eta x$ represented by the matrix

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix}, \tag{1.25}$$

the requirement that T_Λ preserves the Minkowski interval can be rewritten

$$\Lambda^\intercal \eta \Lambda = \eta. \tag{1.26}$$

A Lorentz matrix is a matrix satisfying this condition. Spatial rotations, $R \in \mathrm{SO}(d)$,[2] and boosts $\Lambda(\mathbf{v})$, for $\mathbf{v} \in \mathbb{B}^d$, are two sorts of Lorentz matrices. Equation (1.24) says that every restricted Lorentz matrix has the form $\Lambda = R\Lambda(\mathbf{v})$ for unique $R \in \mathrm{SO}(d)$ and $\mathbf{v} \in \mathbb{B}^d$. The parity and time-reflection matrices $\mathscr{P}, \mathscr{T}$ are examples of non-restricted Lorentz transformations.

If Λ is a Lorentz matrix, then taking the determinant of both sides yields $(\det \Lambda)^2 = 1$. So,

$$\det \Lambda = \pm 1. \tag{1.27}$$

In particular, all Lorentz matrices are invertible.

Proposition 1.7. *The Lorentz matrices form a subgroup of the group of invertible $(1+d)$-by-$(1+d)$-matrices. That is:*

(a) I_{1+d} is a Lorentz matrix.
(b) The product of two Lorentz matrices is a Lorentz matrix.
(c) If Λ is a Lorentz matrix, then Λ^{-1} is a Lorentz matrix.

■

Proof. (a) Obvious.
(b) If Λ, A are both Lorentz, then $(\Lambda A)^\intercal \eta (\Lambda A) = A^\intercal \Lambda^\intercal \eta \Lambda A = A^\intercal \eta A = \eta$.
(c) Multiplying $\eta = \Lambda^\intercal \eta \Lambda$ by Λ^{-1} on the right and $(\Lambda^\intercal)^{-1}$ on the left yields

$$(\Lambda^\intercal)^{-1} \eta \Lambda^{-1} = \eta. \tag{1.28}$$

Since $(\Lambda^{-1})^\intercal = (\Lambda^\intercal)^{-1}$, this says

$$(\Lambda^{-1})^\intercal \eta \Lambda^{-1} = \eta. \tag{1.29}$$

□

The group of Lorentz matrices is denoted

$$\mathrm{O}(1,d) = \{\Lambda \in \mathbb{R}^{(1+d)\times(1+d)} : \Lambda^\intercal \eta \Lambda = \eta\}. \tag{1.30}$$

This is a matrix Lie group, as is easily checked. Since

$$T_\Lambda T_{\Lambda'} = T_{\Lambda\Lambda'} \tag{1.31}$$

for all $\Lambda, \Lambda' \in \mathrm{O}(1,d)$, it is canonically isomorphic to the subgroup of the Poincaré group consisting of Lorentz transformations. Either group is called *the Lorentz group*, and no confusion will arise from conflating the two. We will identify $T_\bullet$ with $\bullet$.

We use $\mathrm{SO}(1,d) \subseteq \mathrm{O}(1,d)$ to denote the subgroup of restricted Lorentz matrices. This is precisely the connected component of the Lorentz group containing the identity matrix I_{1+d}. (This is usually how the restricted Lorentz group is defined. We defined it via connectivity in the Poincaré group P, but this is the same thing — two Lorentz transformations are connected by a path in $\mathrm{O}(1,d)$ if and only if they are connected by a path in the Poincaré group.)

[2]Strictly speaking, $1 \oplus R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix}\right)$ is the Lorentz matrix describing a spatial rotation, but we will abuse notation and just write this as R.

Warning: This notation is not standard; "$\mathrm{O}^+_\uparrow(1,d)$" is more common, but we will use the less-decorated notation.

Proposition 1.8. *If Λ is a Lorentz matrix, then $\Lambda^\intercal$ is a Lorentz matrix.* ■

Proof. Taking the inverse of both sides of $\Lambda^\intercal\eta\Lambda = \eta$ yields $\Lambda^{-1}\eta(\Lambda^\intercal)^{-1} = \eta$, having used $\eta^{-1} = \eta$. Plugging in $\Lambda^{-1} = (\Lambda^\intercal)^{-1\intercal}$, this reads

$$(\Lambda^\intercal)^{-1\intercal}\eta(\Lambda^\intercal)^{-1} = \eta. \tag{1.32}$$

This means that $(\Lambda^\intercal)^{-1}$ is Lorentz; applying Proposition 1.7(c), we deduce that $\Lambda^\intercal$ is Lorentz. □

Multiplying both sides of eq. (1.26) by the invertible matrix η on the left yields $\eta\Lambda^\intercal\eta\Lambda = I_{1+d}$, so

$$\Lambda \in \mathrm{O}(1,d) \iff \Lambda^{-1} = \eta\Lambda^\intercal\eta. \tag{1.33}$$

This is reminiscent of the relationship $R^{-1} = R^\intercal$ that characterizes orthogonal matrices $R \in \mathrm{O}(d)$.

1.3. Semidirect product structure. In summary, the Poincaré group consists of all maps $T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}$ of the form

$$T : \mathbb{R}^{1,d} \ni x \mapsto a + \Lambda x \tag{1.34}$$

for $a \in \mathbb{R}^{1+d}$ and $\Lambda \in \mathrm{O}(1,d)$ a Lorentz matrix.

Proposition 1.9. *Let $T = T_a T_\Lambda$ and $T' = T_{a'}T_{\Lambda'}$. Then, $TT' = T_{a+\Lambda a'}T_{\Lambda\Lambda'}$.* ■

Proof. $TT'(x) = \Lambda(\Lambda' x + a') + a = \Lambda\Lambda' x + (a + \Lambda a') = T_{a+\Lambda a'}T_{\Lambda\Lambda'}(x)$. □

Proposition 1.10. $T_\Lambda T_a = T_{\Lambda a}T_\Lambda$. ■

Proof. $T_\Lambda T_a(x) = \Lambda(x+a) = \Lambda x + \Lambda a = T_{\Lambda a}T_\Lambda(x)$. □

Proposition 1.11. *The subgroup of* $\mathrm{P}(1,d)$ *consisting of translations is normal.* ■

Proof. Immediate from above: $T_\Lambda T_a (T_\Lambda)^{-1} = T_{\Lambda a}T_\Lambda(T_\Lambda)^{-1} = T_{\Lambda a}$. □

This should be "obvious," because everyone agrees what a translation is — turning your head upside down or taking two steps back does not make a translation look like something else. It may change your description of the direction of the translation, but not whether or not it is a translation. In contrast, two non-collocated observers will not agree about whether an affine transformation of spacetime is linear; being linear means fixing the origin, but there is no objectively correct choice of spacetime origin. Mathematically, this means that the Lorentz subgroup $\mathrm{O}(1,d) \subset \mathrm{P}(1,d)$ is not normal. Indeed, if $a \in \mathbb{R}^{1,d}$ is not fixed by Λ, then

$$T_a T_\Lambda T_{-a}(x) = \Lambda x + (a - \Lambda a) \tag{1.35}$$

is affine, but not linear, so not in the Lorentz group.

Proposition 1.12. *The Poincaré group is an inner semidirect product of the subgroup of translations with the subgroup of Lorentz transformations:*

$$\mathrm{P}_{\mathrm{full}}(1,d) = \mathbb{R}^{1,d} \rtimes \mathrm{O}(1,d). \tag{1.36}$$

The multiplication law is $(a,\Lambda)(a',\Lambda') = (a + \Lambda a', \Lambda\Lambda')$. ■

Proof. Immediate from above. □

Consequently, $\mathrm{P}(1,d) = \mathbb{R}^{1,d} \rtimes \mathrm{SO}(1,d)$.

Remark: It is easy to forget which of '$\rtimes$,' '$\ltimes$' is correct. You can figure it out if you remember two things:

- The tip in the triangle points towards the normal subgroup, as it does in "$N \lhd G$,"
- The subgroup of translations is the normal one, not the Lorentz group (and see above for the intuition why).

2. Quantum building blocks

Compared to special relativity, quantum mechanics is a bit harder to define. One can easily spend an inordinate amount of time debating what its essentials are — wave/particle duality, superposition, entanglement, or something else? — but that runs orthogonal to our concerns here. The most philosophically vexing features of quantum mechanics, those having to do with the wavefunction's apparent collapse upon measurement by an external observer, are no more vexing when relativity is in play than when it is not. So, we will focus on what quantum mechanics says about the state space.

2.1. The state space. Physical theories typically come with a set whose elements are identified with the states of the system being modeled. This is the state space. In Hamiltonian mechanics, the state space is some finite-dimensional symplectic manifold. One of the defining features of quantum mechanics is that states can be *superposed*. The Schrödinger cat thought experiment involves a superposition of a state in which the cat lives and a state in which the cat dies. As far as we are concerned, quantum mechanics consists of the following principle: *in order to accommodate superposition, each quantum mechanical system is associated with a* (separable[3]) *Hilbert space* $\mathcal{H}$*, and the state space is built from it.* More precisely, the distinct (pure) states of the system consist of different *lines*

$$\mathbb{C}\Psi \subseteq \mathcal{H}, \quad \Psi \in \mathcal{H}\backslash\{0\} \tag{1.37}$$

in $\mathcal{H}$ (not unit vectors!). Thus, each pure state $\omega \subseteq \mathcal{H}$ is described – non-uniquely – by a nonzero vector in $\mathcal{H}$, namely a spanning vector $\Psi \in \omega$,

$$\omega = \mathbb{C}\Psi \tag{1.38}$$

The ability to add vectors corresponds to the ability to form quantum superpositions. If $\Phi, \Psi \in \mathcal{H}$ are linearly independent vectors, then their linear combination $\Phi + \Psi$ describes the quantum superposition of the two states described by Φ, Ψ, respectively.

We will also use $[\Psi]$ to denote $\mathbb{C}\Psi$.

The set

$$P\mathcal{H} = \{\mathbb{C}\Psi : \Psi \in \mathcal{H}\backslash\{0\}\} \tag{1.39}$$

of complex lines in $\mathcal{H}$ is known as the *projectivization* of $\mathcal{H}$. Thus, quantum states are identified with points in $P\mathcal{H}$. When $\mathcal{H} = \mathbb{C}^N$ is finite-dimensional, then

$$P\mathcal{H} = \mathbb{C}P^{N-1} \tag{1.40}$$

is a familiar complex manifold, the complex projective space with real dimension $2N-2$. The infinite-dimensional case can be thought of as a "Hilbert manifold," a topological space locally homeomorphic to $\ell^2(\mathbb{N})$, but this is not necessary. We have no need to consider $P\mathcal{H}$ as anything more than a set.

Not all superpositions need be allowed — model builders are allowed to forbid certain superpositions by fiat. (Whether or not nature adheres to that restriction is another matter.) Then, the state space will be some proper subset of $P\mathcal{H}$. This is the case of *superselection rules.* Superselection rules are discussed briefly in §D. For now, the reader may assume the absence of superselection rules. Then, the state space is the entirety of $P\mathcal{H}$.

[3]In these notes, all Hilbert spaces are separable, non-trivial (i.e. not zero dimensional), and, unless stated otherwise, over the complex numbers.

2.2. Wigner morphisms. Many different lines of reasoning converge as to what the natural notion of a symmetry of $P\mathcal{H}$ is: a permutation $[U]$ of the state space induced by a unitary operator $U \in \mathrm{U}(\mathcal{H})$. In the absence of superselection rules, this means

$$\begin{aligned} [U] &= U \bmod \mathrm{U}(1)I \\ [U] &\in \mathrm{PU}(\mathcal{H}). \end{aligned} \tag{1.41}$$

One sometimes begins with a more basic notion of symmetry and then proves that all symmetries are unitarizable in this way (barring special symmetries that invert the arrow of time). This is called *Wigner's theorem* [Wig59]. See [Wei05, §2.A] for an exposition.

The group $\mathrm{PU}(\mathcal{H})$ inherits from $\mathrm{U}(\mathcal{H})$ a topology which makes it into a topological group. Specifically, the topology on the former is the quotient of the strong operator topology on the latter.

[Exercise 1.4] The strong and weak operator topologies agree on $\mathrm{U}(\mathcal{H})$. The uniform (a.k.a. norm) topology is

[Exercise 1.5] too strong to be useful. Whenever we reference topologies on these groups, we are referring to the strong/weak operator topology or its quotient.

Remark: When $\mathcal{H}$ is finite-dimensional, then $\mathrm{PU}(\mathcal{H}) = \mathrm{PSU}(\mathcal{H})$, where $\mathrm{SU}(\mathcal{H}) = \{U \in \mathrm{U}(\mathcal{H}) : \det U = 1\}$ is the special unitary group. However, "$\det U$" does not make sense when $\mathcal{H}$ is infinite-dimensional, except for special classes of unitary operators. So, we refrain from writing "$\mathrm{PSU}(\mathcal{H})$."

Finally: the natural notion of a (topological) group G of quantum symmetries is a (continuous) homomorphism

$$\rho : G \to \mathrm{PU}(\mathcal{H}). \tag{1.42}$$

Applied to $G = \mathrm{P}(1, d)$, the result is the definition of relativistic quantum mechanical system given at the beginning of the lecture.

There exist two, dual, ways of interpreting the Poincaré action on the state space $P\mathcal{H}$. Let Larry be a scientist working in the laboratory frame, and Moe be a scientist working in some other inertial frame of reference. (**M**oe for **m**oving.) Let $T \in \mathrm{P}(1, d)$ denote the Poincaré transformation such that, if Larry perceives a spacetime event at coordinates x, Moe will perceive the same event at coordinates $T(x)$. Then:

(i) If Larry perceives some quantum system in state $\mathbb{C}\Psi \in P\mathcal{H}$, Moe will perceive the same system in state $\mathbb{C}\rho(T)\Psi$.

The dual way of interpreting the Poincaré-action is this:

(ii) Any state of the system can be translated, rotated, and/or boosted.
- Given some state which Larry labels $\mathbb{C}\Psi$, there exists another state $\mathbb{C}\rho(T_{-a})\Psi$ which Larry sees as a translated version of the original state.
- Similarly, if Larry describes the initial state as possessing energy-momentum $p = (E, \mathbf{p}) \in \mathbb{R}^{1,d}$, there exists another state $\mathbb{C}\rho(T_\Lambda^{-1})\Psi$ which Larry perceives as possessing energy-momentum $\Lambda^{-1}p$.

This distinction is sometimes referred to as that between the *passive* and *active* interpretations of group elements. More colorful phraseology is "alias versus alibi" — if a crime was committed at what Larry describes as $x \in \mathbb{R}^{1,d}$, but Larry saw me at $T(x)$, then I am exonerated (assuming $T(x) \neq x$), for I have an alibi. But if Moe takes the witness stand and says that he saw me at $T(x)$, and T relates Moe's coordinates to Larry's, then I am in trouble.

2.3. Anti-unitary maps and Wigner representations ($\star$). An anti-unitary operator on $\mathcal{H}$ is a complex anti-linear bijection $V : \mathcal{H} \to \mathcal{H}$ such that

$$\langle \phi, \psi \rangle = \langle V\psi, V\phi \rangle \tag{1.43}$$

for all $\phi, \psi \in \mathcal{H}$. This is equivalent to being of the form $V = C \circ U$ (or equivalently $U \circ C$, for some other U), where $U \in \mathrm{U}(\mathcal{H})$ is unitary and $C : \mathcal{H} \to \mathcal{H}$ is a map that conjugates the coefficients of vectors when expanded in some fixed orthonormal basis.

Together, the unitary and anti-unitary operators form a group $\mathrm{UaU}(\mathcal{H}) \cong C_2 \ltimes \mathrm{U}(\mathcal{H})$.

If V is anti-unitary, then

$$\begin{aligned} [V] : P\mathcal{H} &\to P\mathcal{H} \\ [\Psi] &\mapsto [V\Psi] \end{aligned} \tag{1.44}$$

is a well-defined permutation of $P\mathcal{H}$. Together with the unitarizable automorphisms $[U] \in \mathrm{PU}(\mathcal{H})$, these form a group

$$\mathrm{Aut}(P\mathcal{H}) = [\mathrm{UaU}(\mathcal{H})] \cong C_2 \ltimes \mathrm{PU}(\mathcal{H}). \tag{1.45}$$

Wigner's theorem also allows anti-unitary maps as symmetries. However, these are always associated with "time-reversal symmetry." When considering the symmetries of a quantum system with time-reversal symmetry, one has a group

$$G \cong C_2 \ltimes G_0 \tag{1.46}$$

arising as a semidirect product of $C_2 = \{1, \mathsf{T}\}$ and a group G_0. Then, T is interpreted as time-reversal. Rather than a projective representation of G, the incarnation of G as a group of quantum symmetries takes the form of a (continuous) homomorphism

$$\rho : G \to \mathrm{Aut}(P\mathcal{H}) \tag{1.47}$$

in which the elements of G_0 are mapped to unitarizable symmetries and T is mapped to an anti-unitarizable symmetry, and hence the other elements of $G \backslash G_0$ are as well. We will call these *Wigner representations* of (G, G_0), or just of G for short, leaving the designated subgroup G_0 of unitarizable symmetries implicit.

3. Parity and time-reversal ($\star$)

Above, we were careful to stipulate only that relativistic systems have the restricted Poincaré group $\mathrm{P}(1, d)$ as a symmetry group, not the full Poincaré group

$$\mathrm{P}_{\text{full}}(1, d) \ni T_{\mathscr{T}}, T_{\mathscr{P}} \tag{1.48}$$

for the simple reason that the current reigning theory of particle physics, the standard model, has neither time-reversal nor reflection symmetry, nor a combination thereof.

However, many simplified theories, including QED and QCD, *do* have these fundamental symmetries, in which case the symmetry group is augmented from $\mathrm{P}(1, d)$ to some larger subgroup of $\mathrm{P}_{\text{full}}(1, d)$. This possibility is the topic of this section. The simplest case is where the symmetry group is that generated by $\mathscr{P}$ and P. Such theories have *chiral* symmetry — they look the same as their mirror image, modulo a relabeling of states. A theory with $\mathscr{T}$ symmetry looks the same when run in reverse, modulo a relabeling of states. It has *time-reversal* symmetry. It is also possible for a system to have $\mathscr{P}\mathscr{T}$ symmetry – a "particle-hole" symmetry – without having $\mathscr{P}$ or $\mathscr{T}$ symmetry individually. Understanding these possibilities, and the interplay between parity and time-reversal symmetry, is surprisingly involved.

3.1. The $\mathscr{P}, \mathscr{T}, \mathscr{C}$ matrices. The parity and time-reversal matrices $\mathscr{P}, \mathscr{T} \in \mathrm{O}(1, d)$ are

$$\mathscr{P} = \begin{bmatrix} 1 & 0 \\ 0 & \mathscr{R} \end{bmatrix} \qquad \mathscr{T} = \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix}, \tag{1.49}$$

respectively; $\mathscr{R} \in \mathrm{O}(d)$ is as in eq. (1.19). The product of $\mathscr{P}, \mathscr{T}$ is a third Lorentz matrix, $\mathscr{C} = \mathscr{P}\mathscr{T}$,

$$\mathscr{C} = \begin{bmatrix} -1 & 0 \\ 0 & \mathscr{R} \end{bmatrix} = \begin{cases} \begin{bmatrix} -I_2 & 0 \\ 0 & I_{d-2} \end{bmatrix} & (d \text{ even}), \\ -I_{1+d} & (d \text{ odd}). \end{cases} \tag{1.50}$$

We also use $\mathscr{I} = I_{1+d}$ to denote the identity matrix. The matrices $\mathscr{I}$, $\mathscr{P}$, $\mathscr{T}$, $\mathscr{C}$ constitute a subgroup of $\mathrm{O}(1,d)$ isomorphic to the Klein four-group, $V_4 = C_2 \times C_2$. Together with the identity component, which – as a reminder – we are calling $\mathrm{SO}(1,d)$, they generate the full Lorentz group.

We provide the proof below that $\mathscr{P}, \mathscr{T}, \mathscr{C}$ are not in the identity component of the Lorentz group. From this it follows that the full Lorentz group $\mathrm{O}(1,d)$, as well as the full Poincaré group, have exactly *four* connected components – one component for each

$$\mathscr{A} \in \{\mathscr{I}, \mathscr{P}, \mathscr{T}, \mathscr{C}\}, \tag{1.51}$$

namely the connected component containing that $\mathscr{A}$. No two of these components can coincide, since that would imply that one of $\mathscr{P}, \mathscr{T}, \mathscr{C}$ lies in the identity component. No other components can exist, by the last sentence of the previous paragraph.

It follows from the above that the full Lorentz group $\mathrm{O}(1,d)$ arises as the (inner) semidirect product

$$\mathrm{O}(1,d) = V_4 \ltimes \mathrm{SO}(1,d) \tag{1.52}$$

of its identity component $\mathrm{SO}(1,d)$ and the subgroup $\{\mathscr{I}, \mathscr{P}, \mathscr{T}, \mathscr{C}\} \cong V_4$. Here $\mathrm{SO}(1,d)$ is the normal subgroup. The identity component of a topological group is always normal.

The four components of the Lorentz group can be denoted

$$\mathrm{SO} = \mathrm{O}^+_\uparrow \ni \mathscr{I}, \quad \mathrm{O}^+_\downarrow \ni \mathscr{T}, \quad \mathrm{O}^-_\uparrow \ni \mathscr{P}, \text{ and } \mathrm{O}^-_\downarrow \ni \mathscr{C}. \tag{1.53}$$

From these components, one can form the following three index-two subgroups of $\mathrm{O}(1,d)$:

$$\begin{aligned} \mathrm{O}_\uparrow(1,d) &= \mathrm{SO}(1,d) \sqcup \mathrm{O}^-_\uparrow(1,d), \\ \mathrm{O}^+(1,d) &= \mathrm{SO}(1,d) \sqcup \mathrm{O}^+_\downarrow(1,d), \\ \mathrm{sO}(1,d) &= \mathrm{SO}(1,d) \sqcup \mathrm{O}^-_\downarrow(1,d). \end{aligned} \tag{1.54}$$

This information is summarized in Figure 1.2.

WARNING: "$\mathrm{SO}(1,d)$" is often used to denote $\mathrm{sO}(1,d)$.

Similar notation can be used for the components of $\mathrm{P}(1,d)$, and the corresponding index-two subgroups.

3.2. Relativistic quantum mechanical systems with parity and/or time-reversal. Let G denote one of P, $\mathrm{P}_\uparrow$, P^+, sP, $\mathrm{P}_{\mathrm{full}}$. Each of these has a distinguished subgroup $G_\uparrow = G \cap \mathrm{P}_\uparrow$. A relativistic quantum mechanical system with the reflection symmetries

$$Z = G \cap \{\mathscr{I}, \mathscr{P}, \mathscr{T}, \mathscr{C}\} \tag{1.55}$$

is a Wigner representation of G, i.e. a continuous homomorphism $\rho : G \to \mathrm{Aut}(P\mathcal{H})$ in which the elements of $G_\uparrow$ are mapped to unitarizable symmetries and the elements of $G \backslash G_\uparrow$ are mapped to anti-unitarizable symmetries. The possible pairs $(Z, Z_\uparrow = Z \cap \mathrm{P}_\uparrow)$ are

- Z, Z_0 both trivial,
- $Z = \{\mathscr{I}, \mathscr{P}\} \cong \mathbb{Z}_2$, and $Z_0 = Z$,
- $Z = \{\mathscr{I}, \mathscr{A}\} \cong \mathbb{Z}_2$ for $\mathscr{A} \in \{\mathscr{T}, \mathscr{C}\}$, and Z_0 trivial,
- $Z = \{\mathscr{I}, \mathscr{P}, \mathscr{T}, \mathscr{C}\} \cong V_4$, and $Z_0 = \{\mathscr{I}, \mathscr{P}\}$.

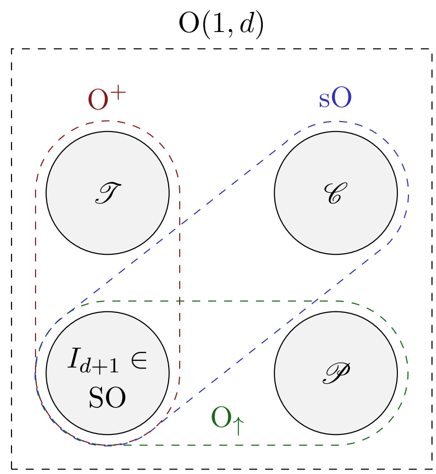


FIGURE 1.2. The various components of the full Lorentz group $\mathrm{O}(1,d)$, and the important subgroups thereof.

In the first case, we have neither chiral symmetry nor time-reversal symmetry. In the second and third cases, we have one of $\mathscr{P},\mathscr{T},\mathscr{C}$. In the fourth case, we have all three.

For each $\mathscr{A} \in Z$, let

$$\mathsf{A} = \rho(\mathscr{A}). \tag{1.56}$$

The map $\mathscr{A} \mapsto \mathsf{A}$ is a Wigner representation of Z. Since each Poincaré transformation $T \in G$ can be written $T = \mathscr{A}T_0$ for exactly one such $\mathscr{A}$ and restricted Poincaré transformation $T_0 \in \mathrm{P}$, and since

$$\rho(\mathscr{A}T_0) = \mathsf{A}\rho(T_0), \tag{1.57}$$

the full Wigner representation is determined by $\rho|_{\mathrm{P}}$ and the various A.

Conversely, suppose we are given

- a projective unitary representation $\rho : \mathrm{P} \to \mathrm{PU}(\mathcal{H})$,
- together with a Wigner representation $\mathscr{A} \mapsto \mathsf{A} \in \mathrm{Aut}(P\mathcal{H})$ of Z.

We can attempt to extend ρ to all of G by taking eq. (1.57) as a definition for $\mathscr{A} \neq \mathscr{I}$. This is a well-defined continuous function $G \to \mathrm{Aut}(P\mathcal{H})$. It is a homomorphism, and therefore Wigner representation, if and only if

$$\rho(\mathscr{A}_1 T \mathscr{A} T_2) = \mathsf{A}_1\rho(T)\mathsf{A}\rho(T_2) \tag{1.58}$$

for all $\mathscr{A}_1, \mathscr{A} \in Z$ and $T, T_2 \in \mathrm{P}$. Note that $T\mathscr{A} = \mathscr{A}\tilde{T}_1$, where $\tilde{T} = \mathscr{A}T\mathscr{A} \in \mathrm{P}$. So,

$$\begin{aligned} \rho(\mathscr{A}_1 T \mathscr{A} T_2) = \rho((\mathscr{A}_1\mathscr{A})\tilde{T}T_2) &= \mathsf{A}_1\mathsf{A}\rho(\tilde{T}T_2) \\ &= \mathsf{A}_1\mathsf{A}\rho(\tilde{T})\rho(T_2), \end{aligned} \tag{1.59}$$

where the second equality used that $\mathscr{A} \mapsto \mathsf{A}$ is a homomorphism, and the third used that $\rho|_{\mathrm{P}}$ is a homomorphism. So, the desired equality holds if and only if $\rho(T)\mathsf{A} = \mathsf{A}\rho(\tilde{T})$, i.e.

$$\mathsf{A}\rho(T)\mathsf{A} = \rho(\mathscr{A}T\mathscr{A}) \tag{1.60}$$

for all $T, \mathscr{A}$.

To summarize, *a Wigner representation of G is the same thing as a projective unitary representation of the restricted Poincaré group* P *together with a Wigner representation of Z, such that eq.* (1.60) *holds.*

The rest of this section is devoted to the study of the Wigner representations of the group Z. A Wigner representation of the group Z is the same thing as an assignment to each $\mathscr{A} \in Z$ a Wigner automorphism $\mathsf{A} \in \mathrm{Aut}(P\mathcal{H})$ such that

- P is unitarizable and involution (if defined),

- T, C (whichever are defined) are anti-unitarizable and involutive,
- $\mathsf{C} = \mathsf{PT} = \mathsf{TP}$ if all three are defined.

3.3. Wigner representations of $\mathscr{P}, \mathscr{T}, \mathscr{C}$ individually.

PROPOSITION 1.13. *If $\mathsf{P} \in \mathrm{PU}(\mathcal{H})$ is an involution, then there exists a unitary involution $\mathcal{P} \in \mathrm{U}(\mathcal{H})$ such that $\mathsf{P} = [\mathcal{P}]$.* ■

PROOF. By the assumption of unitarity, we have $\mathsf{P} = [\mathcal{P}_0]$ for a unitary operator $\mathcal{P}_0$. Since $\mathsf{P}^2 = \mathrm{id}$, this must satisfy $\mathcal{P}_0^2 = cI$ for some $c \in \mathrm{U}(1)$. Let $\mathcal{P} = c^{-1/2}\mathcal{P}_0$. (The branch of the square root does not matter.) Then,

$$\begin{aligned} \mathcal{P}^2 &= c^{-1/2}\mathcal{P}_0(c^{-1/2}\mathcal{P}_0) \\ &= c^{-1}\mathcal{P}_0^2 = I, \end{aligned} \tag{1.61}$$

which is what it means to be an involution. □

The proof above would not work if $\mathcal{P}$ were *anti*-unitary, because the equality going between the two lines in eq. (1.61) would break. Indeed, if $\mathcal{A}$ is anti-unitary, then

$$(c^{-1/2}\mathcal{A})^2 = c^{-1/2}\mathcal{A}(c^{-1/2}\mathcal{A}) = |c|^{-1}\mathcal{A}^2 = \mathcal{A}^2 \tag{1.62}$$

whenever $c \in \mathrm{U}(1)$ is a phase. Instead:

PROPOSITION 1.14. *Suppose that $\mathcal{A}$ is an anti-unitary operator such that $\mathsf{A} = [\mathcal{A}]$ satisfies $\mathsf{A}^2 = 1$. Then, $\mathcal{A}^2 = \pm I$.* ■

It is automatic that $\mathcal{A}^2 = \omega I$ for some $\omega \in \mathrm{U}(1)$. Equation (1.62) shows that we cannot get rid of the phase ω by replacing $\mathcal{A}$ with $c^{-1/2}\mathcal{A}$ for some $c \in \mathrm{U}(1)$. So, ω is actually uniquely determined by A.

PROOF. As mentioned above, we have $\mathcal{A}^2 = \omega I$ for some $\omega \in \mathrm{U}(1)$. We get a restriction on ω from computing $\mathcal{A}^3$ in two different ways:

$$\begin{aligned} \mathcal{A}^3 &= \mathcal{A}(\mathcal{A}^2) = \mathcal{A}(\omega I) = \omega^{-1}\mathcal{A}, \\ \mathcal{A}^3 &= (\mathcal{A}^2)\mathcal{A} = (\omega I)\mathcal{A} = \omega\mathcal{A}. \end{aligned} \tag{1.63}$$

So, $\omega^{-1} = \omega$, which means that $\omega^2 = 1$, so $\omega \in \{-1, +1\}$. □

If $\mathcal{A}^2 = I$, then $\mathcal{A}$ is a *real structure* on $\mathcal{H}$. If $\mathcal{A}^2 = -I$, it is a *quaternionic structure*. Because the sign of $\mathcal{A}^2$ is an invariant of A, every anti-unitary Wigner transformation comes from a real structure or a quaternionic structure, but not both. Thus, we have two sorts of anti-unitary involutions.

3.4. The tenfold way. Now suppose that our system has *both* mirror and time-reversal symmetry. So, we have Wigner automorphisms P, T, implementing parity and time-reversal, respectively, satisfying the requirements above, including $\mathsf{PT} = \mathsf{TP}$. These constitute a Wigner representation of $(V_4, C_2 \times \{1\})$. By the discussion above, we have $\mathcal{P} \in \mathrm{U}(\mathcal{H})$ and $\mathcal{T} \in \mathrm{aU}(\mathcal{H})$ related to P, T by $\mathsf{P} = [\mathcal{P}]$, $\mathsf{T} = [\mathcal{T}]$ and satisfying

$$\mathcal{P}^2 = I, \text{ and } \mathcal{T}^2 = \varepsilon_{\mathsf{T}} I \tag{1.64}$$

for some $\varepsilon_{\mathsf{T}} \in \{-1, +1\}$.

The condition $\mathsf{PT} = \mathsf{TP}$ is equivalent to the existence of $\theta \in [0, 2\pi)$ such that

$$\mathcal{P}\mathcal{T} = e^{i\theta}\mathcal{T}\mathcal{P}. \tag{1.65}$$

But computing $\mathcal{P}\mathcal{T}\mathcal{P}\mathcal{T}$ in two ways, we get a constraint on θ:

$$\begin{aligned} \mathcal{P}\mathcal{T}\mathcal{P}\mathcal{T} &= e^{i\theta}\mathcal{T}\mathcal{P}^2\mathcal{T} = e^{i\theta}\varepsilon_{\mathsf{T}} I \\ \mathcal{P}\mathcal{T}\mathcal{P}\mathcal{T} &= \mathcal{P}(e^{-i\theta}\mathcal{P}\mathcal{T})\mathcal{T} = e^{-i\theta}\mathcal{P}^2\mathcal{T}^2 = e^{-i\theta}\varepsilon_{\mathsf{T}} I, \end{aligned} \tag{1.66}$$

so that $e^{i\theta} = \pm 1$. Let us call this ϵ_{PT}. To summarize: for some $\varepsilon_{\mathsf{T}}, \epsilon_{\mathsf{PT}} \in \{-1, +1\}$,

$$\boxed{\mathcal{P}^2 = I, \quad \mathcal{T}^2 = \varepsilon_{\mathsf{T}} I, \quad \mathcal{P}\mathcal{T} = \epsilon_{\mathsf{PT}}\mathcal{T}\mathcal{P}.} \tag{1.67}$$

The operator $\mathcal{C} = \mathcal{P}\mathcal{T}$ also has to square to $\pm I$:

$$\mathcal{C}^2 = \mathcal{P}\mathcal{T}\mathcal{P}\mathcal{T} = \epsilon_{\mathsf{PT}}\mathcal{T}\mathcal{P}^2\mathcal{T} = \epsilon_{\mathsf{PT}}\varepsilon_{\mathsf{T}} I. \tag{1.68}$$

So, let

$$\boxed{\varepsilon_{\mathsf{C}} = \epsilon_{\mathsf{PT}}\varepsilon_{\mathsf{T}}} \tag{1.69}$$

denote the sign of $\mathcal{C}^2$. (Note the typographical difference between 'ε' and 'ϵ.') We can equally well use $(\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}})$, instead of $(\varepsilon_{\mathsf{T}}, \epsilon_{\mathsf{PT}})$, to describe the situation. It should be emphasized that $\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}}$ are *invariants* — they depend only on $\mathsf{P}, \mathsf{T} \in \operatorname{Aut}(P\mathcal{H})$ (as the notation indicates) and not on the operators $\mathcal{P}, \mathcal{T}$ used to represent them. Thus, they are invariants of the given Wigner representation, and can be used to classify them.

Each of the four possible cases of $(\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}})$ can be realized, already with $\mathcal{H} = \mathbb{C}^2$.

What eq. (1.67) tells us is that $\mathcal{P}, \mathcal{T}$ constitute a (faithful) representation of some group related to the Klein 4-group. We will call the group generated by $\mathcal{P}, \mathcal{T}$ the "$\mathcal{P}\mathcal{T}$-group." Doing the casework, and using conventional labels [nLab25] for the various cases:

- (Class BDI.) If $\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}} = 1$, then $\mathcal{P}, \mathcal{T}$ generate the group $\{I, \mathcal{P}, \mathcal{T}, \mathcal{P}\mathcal{T}\}$, which is just a copy of the Klein 4-group.
- (Class DIII.) If $\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}} = -1$, then $\mathcal{P}, \mathcal{T}$ commute but generate a slightly larger group: the $\mathcal{P}\mathcal{T}$-group is
$$\{I, \mathcal{T}, -I, -\mathcal{T}, \mathcal{P}, \mathcal{P}\mathcal{T}, -\mathcal{P}, -\mathcal{P}\mathcal{T}\}. \tag{1.70}$$
This is a copy of $C_4 \times C_2$, with $\mathcal{T}$ generating the $\mathbb{Z}_4$ factor $\{I, \mathcal{T}, -I, -\mathcal{T}\}$ and $\mathcal{P}$ generating the C_2 factor $\{I, \mathcal{P}\}$.
- (Class CII.) If $\varepsilon_{\mathsf{T}} = -1$ but $\varepsilon_{\mathsf{C}} = 1$, then $\mathcal{P}, \mathcal{T}$ no longer commute, but rather anti-commute:
$$\mathcal{P}\mathcal{T} = -\mathcal{T}\mathcal{P}. \tag{1.71}$$
As a *set*, the $\mathcal{P}\mathcal{T}$-group generated is still given by eq. (1.70), but now the group structure is non-abelian. It turns out to be isomorphic to the dihedral group D_8:
$$\begin{aligned} D_8 &= \langle x, a : a^4 = x^2 = 1, xax = a^{-1} \rangle \\ &= \{1, x, a, xa, a^2, xa^2, a^3, xa^3\}. \end{aligned} \tag{1.72}$$
The group D_8 can be interpreted as the group of symmetries of a square; x is a reflection across a median, and a is a $90°$ rotation. An isomorphism with the $\mathcal{P}\mathcal{T}$-group is given by $\mathcal{T} \mapsto a$ and $\mathcal{P} \mapsto x$.
- (Class CI.) If $\varepsilon_{\mathsf{T}} = 1$ and $\varepsilon_{\mathsf{C}} = -1$, then everything in the previous item applies. Just interchange $\mathscr{T}, \mathscr{C}$. The isomorphism between the $\mathcal{P}\mathcal{T}$-group and D_8 is now different, because of the interchange (in particular, $\mathcal{T}^2 = I$). An explicit isomorphism is given by $\mathcal{T} \mapsto xa$ and $\mathcal{P} \mapsto x$.

Altogether, we have ten possibilities, including those without some of $\mathscr{P}, \mathscr{T}, \mathscr{C}$- symmetry. These possibilities are summarized in Table 1.1. The cases where Z is trivial or the parity group $\{1, \mathsf{P}\}$ each contribute a single possibility. The cases where $Z = \{1, \mathsf{A}\}$ for $\mathsf{A} \in \{\mathsf{T}, \mathsf{C}\}$ each contribute *two* possibilities, one for each possible sign ε_{A}. The case where Z is maximal, $Z = V_4$, contributes four possibilities, one for each possible pair $(\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}})$. This sort of tenfold classification is known as a *tenfold way* [Bae20]. Recently, condensed matter theorists working on topological superconductivity have popularized closely related tenfold ways. The one here goes back to Wigner [Wig59].

All of the listed classes appear in physically significant examples. For examples, fermions tend to have $\mathcal{T}^2 = -1$, whereas bosons have $\mathcal{T}^2 = 1$. Breaking symmetries is easy — *chiral* terms in the standard model Lagrangian break parity. Some of these, like the axial vector term in the weak sector,

Class	ε_{T}	ε_{C}	ε_{P}	Cover	Example
A	0	0	0	Trivial	Standard model
AI	+1	0	0	C_2	Fermi theory, bosonic sector
AII	-1	0	0	C_4	Fermi theory, fermionic sector
D	0	+1	0	C_2	Yang–Mills θ-term,
C	0	-1	0	C_4	$\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi F_{\mu\nu}$-theory
AIII	0	0	1	C_2	$2\Re(e^{i\theta}\phi)$-theory, $\theta \in (0, \pi/2)$
BDI	+1	+1	1	$V_4 = C_2 \times C_2$	Free spin-0 particle
CI	+1	-1	1	D_8	Symplectic boson
CII	-1	+1	1	D_8	Majorana fermion
DIII	-1	-1	1	$C_4 \times C_2$	Electron

Table 1.1. The tenfold classification of Wigner representations of the PT-group $(V_4, C_2 \times \{1\})$ and subgroups thereof. For each generator $\mathscr{A}$, $\varepsilon_{\mathsf{A}} = \pm 1$ if $\mathscr{A}$ is present and $\varepsilon_{\mathsf{A}} = 0$ otherwise. For $\mathscr{A} = \mathscr{T}, \mathscr{C}$, the sign of ε_{A}, if nonzero, denotes whether A is implementable by an involution. For $\mathsf{A} = \mathsf{P}$, it always is. The covering group is the group that the Wigner representation lifts to an ordinary unitary/anti-unitary representation of.

$W_\mu \bar{\psi}\gamma^\mu\gamma^5\psi$, break parity without breaking time-reversal. Others, like the θ-term $\theta\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ break both.

Warning: Different conventions exist for defining $\mathcal{P}$, given P. In classes CI, CII, some authors prefer to use $\mathcal{P}_{\text{alt}} = i\mathcal{P}$ instead of $\mathcal{P}$. This only satisfies $\mathcal{P}_{\text{alt}}^4 = I$, not $\mathcal{P}_{\text{alt}}^2 = I$, but it has the advantage that it commutes with time-reversal:

$$\mathcal{T}\mathcal{P}_{\text{alt}} = \mathcal{T}i\mathcal{P} = -i\mathcal{T}\mathcal{P} = i\mathcal{P}\mathcal{T} = \mathcal{P}_{\text{alt}}\mathcal{T}. \tag{1.73}$$

For example, when physicists say that the parity of a Majorana spinor is $\pm i$, this is the convention they are following.

3.5. Classification of Wigner representations within each way. The tenfold way here is *not* a complete classification of the Wigner representations of V_4 or subgroups thereof. A single "way" in the tenfold way can apply to more than one possible representation of Z via Wigner automorphisms. An obvious exception is class A, describing a system with none of the extra symmetries $\mathscr{P}, \mathscr{T}, \mathscr{C}$. We will see below that all of the classes except AIII, BDI, and DIII have a unique representation, modulo equivalence, on each $\mathcal{H}$, if a representation exists at all.

Consider class AIII, a system with chiral symmetry but neither $\mathscr{T}$ nor $\mathscr{C}$ symmetry. We saw that any Wigner representation in this class lifts to an ordinary unitary representation of $C_2 = \{I, \mathcal{P}\}$. There are two different irreps of C_2, the one-dimensional representations where $\mathcal{P} = \pm 1$. Call these $\mathbf{1}_\pm$. The two irreps are said to differ in terms of parity; $\mathbf{1}_+$ is even parity, $\mathbf{1}_-$ is odd. The most general finite-dimensional representation, modulo equivalence, is

$$\mathbf{1}_-^{N_-} \oplus \mathbf{1}_+^{N_+}, \tag{1.74}$$

where $N_\pm \in \mathbb{N}$. (The infinite-dimensional cases are analogous.)

Consider now classes AI, AII, D, and C. Regardless of $\dim \mathcal{H}$ (recall this is the complex dimension), then the only representation of classes AI and D modulo equivalence is: $\mathcal{H} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^N$, $\mathcal{A} = * \otimes I_N$, where $\mathscr{A} = [\mathcal{A}]$ is whichever of $\mathscr{T}, \mathscr{C}$ is in Z. If $\dim \mathcal{H} = 2N$ is even, then we have the following unique representation in classes AII and C: $\mathcal{H} = \mathbb{H}^N$, $\mathcal{A} = j$ where $\mathbb{H}$ is the quaternions, considered as a complex vector space. Here, $j \in \mathbb{H}$ is a unit quaternion anti-commuting with i and satisfying $j^2 = -1$. This is a complex-antilinear map, acting on $\mathbb{H}^N$, since $jaq = ajq$ and

$j(iaq) = -iajq$ for all $a \in \mathbb{R}$ and $q \in \mathbb{H}$. If $\dim \mathcal{H} < \infty$ is odd, then we have no representation of classes AII and C.

Classes BDI and DIII are similarly easy, since $\mathcal{P}$ and $\mathcal{T}$ commute. A representation in these classes is just a combination (really, a tensoring) of the possibilities above. Regardless of $\dim \mathcal{H}$, we have a representation of class BDI: $\mathcal{H} = \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}^{N_-} \oplus \mathbb{R}^{N_+})$, $\mathcal{T} = * \otimes I$, and

$$\mathcal{P} = 1 \otimes \begin{bmatrix} -I_{N_-} & 0 \\ 0 & 1_{N_+} \end{bmatrix}. \tag{1.75}$$

If $\dim \mathcal{H} = 2N$ is even, then we have the following representation of class DIII: $\mathcal{H} = \mathbb{H}^{N_-} \oplus \mathbb{H}^{N_+}$, $\mathcal{T} = j$, and

$$\mathcal{P} = \begin{bmatrix} -\operatorname{id}_{\mathbb{H}^{N_-}} & 0 \\ 0 & \operatorname{id}_{\mathbb{H}^{N_+}} \end{bmatrix}. \tag{1.76}$$

These are the only possibilities, modulo equivalence. The comments above regarding absolute vs. relative parity apply here as well; $\mathcal{P}, -\mathcal{P}$ define the same Wigner representation, so should not be considered inequivalent.

Classes CI, CII are more interesting, since $\mathcal{P}, \mathcal{T}$ now anti-commute. In terms of the $\mathcal{P}$-action, $\mathcal{H} = \mathbf{1}_-^{N_-} \oplus \mathbf{1}_+^{N_+}$. We can identify this with $\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}^{N_-} \oplus \mathbb{R}^{N_+})$. Then,

$$\mathcal{P} = 1 \otimes \begin{bmatrix} -I_{N_-} & 0 \\ 0 & I_{N_+} \end{bmatrix}. \tag{1.77}$$

The anti-commutation $\mathcal{T}\mathcal{P} = -\mathcal{P}\mathcal{T}$ tells us that $\mathcal{T} = * \otimes \left(\begin{smallmatrix} 0 & \mathcal{T}_{-+} \\ \mathcal{T}_{+-} & 0 \end{smallmatrix}\right)$, where $\mathcal{T}_{\pm\mp} : \mathbf{1}_\pm^{N_\pm} \to \mathbf{1}_\mp^{N_\mp}$ are two *unitary* maps. This forces $N_- = N_+$. Call their shared value N. Without loss of generality, we can assume that $\mathcal{T}_{-+} = I_N$. Thus,

$$\mathcal{T}^2 = 1 \otimes \begin{bmatrix} \mathcal{T}_{+-} & 0 \\ 0 & \mathcal{T}_{-+} \end{bmatrix}. \tag{1.78}$$

We can therefore read off $\mathcal{T}_{+-}$:

- In class CI, $\mathcal{T}^2 = I$, so $T_{+-} = I_N$, and thus $\mathcal{T} = * \otimes \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}$.
- In class CII, $\mathcal{T}^2 = -I$, so $\mathcal{T}_{+-} = -I_N$, and thus $\mathcal{T} = * \otimes \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$.

These indeed define representations in the desired classes, the unique ones of the given dimension, modulo equivalence.

Class	Space hosting irrep	irrep
A	$\mathcal{H} = \mathbb{C}$	Trivial
AI	$\mathcal{H} = \mathbb{C}$	$\mathcal{T} = *$
AII	$\mathcal{H} = \mathbb{H}$	$\mathcal{T} = j$
D	$\mathcal{H} = \mathbb{C}$	$\mathcal{C} = *$
C	$\mathcal{H} = \mathbb{H}$	$\mathcal{C} = j$
AIII	$\mathcal{H} = \mathbb{C}$	$\mathcal{P} = \pm 1$
BDI	$\mathcal{H} = \mathbb{C}$	$\mathcal{T} = *$, $\mathcal{P} = \pm 1$
CI	$\mathcal{H} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2$	$\mathcal{T} = * \otimes \sigma_1$, $\mathcal{P} = 1 \otimes \sigma_3$
CII	$\mathcal{H} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2$	$\mathcal{T} = * \otimes i\sigma_2$, $\mathcal{P} = 1 \otimes \sigma_3$
DIII	$\mathcal{H} = \mathbb{H}$	$\mathcal{T} = j$, $\mathcal{P} = \pm 1$

TABLE 1.2. The irreps found above. Each Wigner representation of the stated classes is induced (possibly non-uniquely) by a direct sum of irreps of the stated forms. In this table, $\sigma_1, \sigma_2, \sigma_3$ are the three Pauli matrices (eq. (A.1)).

Warning: Not all of the possibilities above describe distinct Wigner representations. For example, every case with $\dim \mathcal{H} = 1$ is identical, since then $P\mathcal{H}$ is a singleton.

Also, in the classes AIII, BDI, and DIII, replacing $\mathcal{P}$ with $-\mathcal{P}$ leads to the same Wigner representation, since $[\mathcal{P}] = [-\mathcal{P}]$. As a consequence, only *relative* parity is defined. In quantum field theory, everyone agrees that the vacuum state has even parity, so parities of states involving an even number of spinors are defined relative to that. The univalence superselection rule (see §D) forbids superposing states with an odd number of fermions with the vacuum, so the parities of spinor states are only defined relative to each other.

A. More about special relativity

A.1. Inertial frames of reference. Consider an observer moving at some velocity $\mathbf{v} \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ relative to some fiducial reference frame (the "laboratory frame"). The term *inertial frame of reference* refers to the standpoint of such an observer.

Remark: More accurately, you should imagine a frame of reference as consisting of an army of comoving experimentalists, spread throughout space, each equipped with their own clock (all of which are synchronized from their collective point-of-view) and instructed to record the times of events occurring at their location. Then, for an event to occur at $(t, \mathbf{x}) \in \mathbb{R}^{1,d}$ means that the observer whom they perceive to be at location $\mathbf{x} \in \mathbb{R}^d$ records a hit when their clock reads time $t \in \mathbb{R}$.

Let's call the moving observer Moe, and a scientist working in the lab frame Larry. One difference between the two observers is that they might not agree to synchronize their clocks. Let $a^0 \in \mathbb{R}$ be the time, according to Larry's clock, when Moe's clock reads $t = 0$. Moe's worldline, the path they trace out in spacetime from the perspective of Larry, is

$$\Gamma = \{(t, t\mathbf{v} + \mathbf{a}_0) : t \in \mathbb{R}\} \subset \mathbb{R}^{1,d}, \tag{1.79}$$

where $\mathbf{a}_0$ is their position at time $t = 0$. We are measuring velocity in units relative to the speed of light ("natural units"), so

$$\|\mathbf{v}\| < 1 \tag{1.80}$$

imposes the physical requirement that Moe be moving slower than the speed of light. Let $\mathbf{a} = a^0\mathbf{v} + \mathbf{a}_0$. We combine this with a^0 to form

$$a = (a^0, \mathbf{a}) \in \Gamma. \tag{1.81}$$

This is Larry's description of the point that Moe labels as the spacetime origin.

Conceivably, Larry and Moe could be using different Cartesian directions to coordinate space — what one labels as the "x^1-direction" could be labeled by the other as the "x^2-direction" — but this just amounts to a spatial rotation, and we understand these. So, let us assume the following "no rotation" condition: any event that Larry perceives at $(0, \mathbf{y})$ for $\mathbf{y} \perp \mathbf{v}$ will be perceived by Moe at $(t', \mathbf{y} - \mathbf{a} + w\mathbf{v})$, for some $t', w \in \mathbb{R}$.

In summary: an inertial frame of reference (barring rotations) is specified by two pieces of data, $(\mathbf{v}, a) \in \mathbb{B}^d \times \mathbb{R}^{1,d}$, Moe's velocity and his spacetime origin. **Q.** If Larry perceives an event as occurring at spacetime coordinates $x \in \mathbb{R}^{1,d}$, at what spacetime coordinates will Moe perceive that event?

Let $T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}$ denote the map such that if Larry perceives an event at x, Moe perceives the same event at $T(x)$. We know the following:

- (i) The spacetime point labeled a by Larry will be labeled 0 by Moe, so $T(a) = 0$.
- (ii) The worldline Γ will be described as the time axis $\mathcal{T} = \{(t, \mathbf{0}) : t \in \mathbb{R}\}$ by Moe, so $T(\Gamma) = \mathcal{T}$,
- (iii) If $\mathbf{y} \perp \mathbf{v}$, then $T(0, \mathbf{y}) = (t', \mathbf{y} - \mathbf{a} + w\mathbf{v})$ for some $t', w \in \mathbb{R}$.
- (iv) T preserves the arrow of time and spatial orientations.

These are the only reasonable requirements, besides that Moe should be using the same laws of physics as Larry. Consider the restricted Poincaré transformation

$$T_0 : (t, \mathbf{x}) \mapsto T_{\Lambda(-\mathbf{v})} T_{-a} \tag{1.82}$$

This obeys (i), (ii), (iii), and (iv). In fact, it is the unique Poincaré transformation with these properties. "Covariance" is a technical term meaning retaining form under a coordinate transformation. A consequence of the Poincaré-covariance of the laws of physics is that Moe can adequately describe the world using the coordinates furnished by T_0. A corollary is that the moving observer Moe cannot tell that it is he that is moving, not Larry. There may exist no objectively correct answer to the question as to which observer is moving.

Special relativity is often expressed as the slightly stronger requirement that Moe will *perceive* at $T_0(x)$ an event that Larry perceives at x. This is a consequence of another assumption: that, besides being Poincaré-covariant, the laws of physics are not covariant under a larger group of spacetime symmetries. The map T_0 is not the only affine transformation satisfying the three properties above — consider, for $Z > 0$, the scaling transformation $T_Z : (t, \mathbf{x}) \mapsto (Zt, Z\mathbf{x})$. Unless $Z = 1$, this is *not* a Poincaré transformation. But, like T_0,

$$T_1 \stackrel{\text{def}}{=} T_Z T_{\Lambda(-\mathbf{v})} T_{-a} \tag{1.83}$$

also satisfies properties (i), (ii), and (iii), and therefore serves as a candidate coordinate transformation. Why shouldn't Moe use the coordinate system derived from T_1, instead of T_0, to describe the world?

The answer is: the very fact that we have the ability to measure distances and durations means that nontrivial dilations cannot be symmetries of the laws of physics. This is implicit in the way that we talk about frames of reference. When we ask how an observer perceives the world, we are assuming that there exists an unambiguous answer. Consequently, there exists exactly one adequate T, namely $T = T_0$.

Scaling symmetries are typically symmetries of theories involving only massless particles. A creature made entirely of massless particles would not be able to measure distances or durations. This applies to light. Since light is massless, the symmetry group of classical electrodynamics *without matter* (!) is the full conformal group, including the scaling symmetries T_Z. This means that the behavior of light is not by itself sufficient to derive special relativity. Fortunately, scaling is not a symmetry of theories involving massive particles. The existence of massive charged particles like the electron allows us to design clocks and rulers that, together with the behavior of light, single out Poincaré-covariance.

B. Poincaré Lie algebra [*]

Physicists almost always work at the level of the complexified Poincaré Lie algebra $\mathfrak{p}_{\mathbb{C}}$ rather than the Lie group.

$$\begin{aligned}
[\mathsf{P}^\mu, \mathsf{P}^\nu] &= 0, \\
i[\mathsf{M}^{\mu\nu}, \mathsf{P}^\rho] &= \eta^{\mu\rho}\mathsf{P}^\nu - \eta^{\nu\rho}\mathsf{P}^\mu, \\
i[\mathsf{M}^{\mu\nu}, \mathsf{M}^{\rho\sigma}] &= \eta^{\mu\rho}\mathsf{M}^{\nu\sigma} - \eta_{\mu\sigma}\mathsf{M}^{\nu\rho} - \eta_{\nu\rho}\mathsf{M}^{\mu\sigma} + \eta_{\nu\sigma}\mathsf{M}^{\mu\rho}.
\end{aligned} \tag{1.84}$$

In terms of the rotation generators $\mathsf{J}^{k\ell} = \mathsf{M}^{k\ell}$, boost generators $\mathsf{K}^j = \mathsf{M}^{0j}$, and "Hamiltonian" $\mathsf{H} = \mathsf{P}^0$, this reads

$$\begin{aligned} [\mathsf{H}, \mathsf{P}^j] &= [\mathsf{H}, \mathsf{J}^{km}] = [\mathsf{P}^j, \mathsf{P}^k] = 0, \\ i[\mathsf{J}^{jk}, \mathsf{J}^{lm}] &= \delta^{km}\mathsf{J}^{jl} - \delta^{jm}\mathsf{J}^{kl} - \delta^{kl}\mathsf{J}^{jm} + \delta^{jl}\mathsf{J}^{km}, \\ i[\mathsf{J}^{jk}, \mathsf{P}^l] &= \delta^{kl}\mathsf{P}^j - \delta^{jl}\mathsf{P}^k, \\ i[\mathsf{J}^{jk}, \mathsf{K}^l] &= \delta^{kl}\mathsf{K}^j - \delta^{jl}\mathsf{K}^k, \\ i[\mathsf{K}^j, \mathsf{H}] &= -\mathsf{P}^j, \quad i[\mathsf{K}^j, \mathsf{P}^k] = -\delta^{jk}\mathsf{H}, \quad i[\mathsf{K}^j, \mathsf{K}^k] = \mathsf{J}^{jk}. \end{aligned} \tag{1.85}$$

When $d = 3$, this can be written using $\mathsf{J}^k = \varepsilon k \mathsf{J}^{\ell m}$, where $\varepsilon_{jk\ell}$ is the Levi–Civita symbol. I.e. $(\mathsf{J}^1, \mathsf{J}^2, \mathsf{J}^3) = (\mathsf{J}^{23}, \mathsf{J}^{31}, \mathsf{J}^{12})$. Then, the Lie algebra reads

$$\begin{aligned} [\mathsf{H}, \mathsf{P}^j] &= [\mathsf{H}, \mathsf{J}^{km}] = [\mathsf{P}^j, \mathsf{P}^k] = 0, \\ i[\mathsf{J}^j, \mathsf{J}^k] &= -\epsilon_{jkl}\mathsf{J}^l, \\ i[\mathsf{J}^j, \mathsf{P}^k] &= -\epsilon_{jkl}\mathsf{P}^l, \\ i[\mathsf{J}^j, \mathsf{K}^k] &= -\epsilon_{jkl}\mathsf{K}^l, \\ i[\mathsf{K}^j, \mathsf{H}] &= -\mathsf{P}^j, \\ i[\mathsf{K}^j, \mathsf{P}^k] &= -\delta^{jk}\mathsf{H}, \\ i[\mathsf{K}^j, \mathsf{K}^k] &= \epsilon_{jkl}\mathsf{J}^l. \end{aligned} \tag{1.86}$$

C. More on the Poincaré group

C.1. Dimensions. The dimension of the Lorentz group $\mathrm{O}(1, d)$ is

$$\dim \mathrm{O}(1, d) = \dim \mathrm{P}(1, d) - d - 1 = \frac{d^2}{2} + \frac{d}{2}. \tag{1.87}$$

In the physical case, $d = 3$, this is $\dim \mathrm{O}(1, d) = 6$. Three of these dimensions are from the subgroup of rotations and three from the boosts. Rotations outnumber boosts for $d \geq 4$, and boosts outnumber rotations if $d = 1, 2$.

Since $\mathrm{SO}(d)$ has dimension $d(d-1)/2$, the Poincaré group has dimension

$$\dim \mathrm{P}(1, d) = \frac{d^2}{2} + \frac{3d}{2} + 1. \tag{1.88}$$

In the physical case, $d = 3$, this is $\dim \mathrm{P}(1, d) = 10$. Four of these dimensions are from the subgroup of translations and the remaining six are from the subgroup of Lorentz transformations.

For the Lie algebras, see [Wei05, §2.4].

C.2. Topology. A Lorentz matrix Λ is called

(i) *orthochronous* if ${\Lambda^0}_0 > 0$, where ${\Lambda^0}_0$ is the upper-leftmost entry of Λ,
(ii) *orthochorous* if $\det \boldsymbol{\Lambda} > 0$, where $\boldsymbol{\Lambda} \in \mathbb{R}^{d\times d}$ consists of the bottom-right d-by-d submatrix of Λ,
(iii) *special* if $\det \Lambda > 0$, which, by eq. (1.27), means $\det \Lambda = 1$.

An orthochronous Lorentz matrix is one which preserves the arrow of time, and an orthochorous Lorentz matrix is one which preserves spatial orientation. A special Lorentz matrix is one which preserves spacetime orientations.

Among $\mathscr{P}, \mathscr{T}, \mathscr{C}$, the only orthochronous matrix is $\mathscr{P}$, the only orthochorous matrix is $\mathscr{T}$, and the only special matrix is $\mathscr{C}$.

Proposition 1.15. *For any Lorentz matrix* Λ, $|{\Lambda^0}_0| \geq 1$. ■

Proof. Let $N = (1, 0, 0, \cdots)$. This satisfies $N^2 = -1$. Since Λ is a Lorentz matrix, $(\Lambda N)^2 = N^2 = -1$ as well;

$$\Lambda N = (\Lambda^0{}_0, \vec{\Lambda}) \tag{1.89}$$

is the first column of Λ; letting $\vec{\Lambda} \in \mathbb{R}^d$ denote the spatial part, $(\Lambda N)^2 = -|\Lambda^0{}_0|^2 + \|\vec{\Lambda}\|^2$. Setting the left-hand side to -1, we get

$$|\Lambda^0{}_0|^2 = 1 + \|\vec{\Lambda}\|^2 \geq 1. \tag{1.90}$$

□

Second proof. We know that $\Lambda = \mathscr{T}^\eta R \Lambda(\mathbf{v})$ for some $\eta \in \{0, 1\}$, $R \in \mathrm{O}(d)$, and $\mathbf{v} \in \mathbb{B}^d$. Thus,

$$\Lambda^0{}_0 = N^\intercal \Lambda N = \pm N^\intercal \Lambda(\mathbf{v}) N = \pm\gamma, \tag{1.91}$$

where $\gamma = 1/\sqrt{1 - \|\mathbf{v}\|^2} \geq 1$ is the Lorentz factor. □

Proposition 1.16. *For any Lorentz matrix* Λ, $|\det \mathbf{\Lambda}| \geq 1$. ■

Proof. Let $\mathbf{x} \in \mathbb{R}^d$ be a unit vector, and $x = (0, \mathbf{x}) \in \mathbb{R}^{1,d}$. Then $x^2 = \|\mathbf{x}\|^2 = 1$. Since Λ is Lorentz, $(\Lambda x)^2 = x^2 = 1$ as well. The spatial component of Λx is $\mathbf{\Lambda x}$, so

$$1 = (\Lambda x)^2 \leq \|\mathbf{\Lambda x}\|^2. \tag{1.92}$$

Since $\mathbf{x}$ was an arbitrary unit vector, this implies that every eigenvalue λ of $\mathbf{\Lambda}$ has $|\lambda| \geq 1$. As $\det \mathbf{\Lambda}$ is the product of the eigenvalues of $\mathbf{\Lambda}$ (with multiplicity), $|\det \mathbf{\Lambda}| \geq 1$ follows. □

Proposition 1.17. *Let $\bullet$ stand for "orthochronous," "orthochorous," or "special." Let C be a connected component of* $\mathrm{O}(1, d)$. *If one matrix in C is $\bullet$, then all are.* ■

Proof. The three maps $\Lambda \mapsto \Lambda^0{}_0, \det \mathbf{\Lambda}, \det \Lambda$ are all continuous functions $\mathrm{O}(1, d) \to \mathbb{R}$. By the results above, they are non-vanishing, so they cannot swap signs on any connected component of $\mathrm{O}(1, d)$. □

Since the identity matrix $\mathscr{I}$ is orthochronous, orthochorous, and special, we conclude that matrices in the same connected component are as well. This shows that $\mathscr{P}, \mathscr{T}, \mathscr{C}$ are not in the identity component $\mathrm{SO}(d)$.

Proposition 1.18.

(a) *A Lorentz matrix lies in the identity component of the Lorentz group if it is orthochronous, orthochorous, and special.*
(b) *Any two of these imply the third.*

■

Proof. This follows from the representation theorem $\Lambda = \mathscr{T}^\xi \mathscr{P}^\eta R \Lambda(\mathbf{v})$ holding for general Lorentz matrices. Since $\mathrm{SO}(d)$ is connected, $R\Lambda(\mathbf{v})$ lies in the same connected component as the pure boost $\Lambda(\mathbf{v})$, which lies in the same connected component as $\mathscr{I}$ (just take $\mathbf{v} \rightsquigarrow 0$). Thus, $R\Lambda(\mathbf{v})$ is orthochronous, orthochorous, and special. Multiplying this by one of $\mathscr{T}, \mathscr{P}, \mathscr{C}$ on the left ruins two of these. So, the only way Λ can be two or three is if $\Lambda = R\Lambda(\mathbf{v})$. □

Let

$$\begin{aligned} \mathrm{O}_\uparrow(1, d) &= \{\Lambda \in \mathrm{O}(1, d) : \Lambda^0{}_0 > 0\}, \\ \mathrm{O}^+(1, d) &= \{\Lambda \in \mathrm{O}(1, d) : \det \mathbf{\Lambda} > 0\}, \\ \mathrm{sO}(1, d) &= \{\Lambda \in \mathrm{O}(1, d) : \det \Lambda = 1\}, \end{aligned} \tag{1.93}$$

denote the subsets of $\mathrm{O}(1, d)$ consisting of orthochronous, orthochorous, and special Lorentz matrices, respectively. By what we have discussed so far, each of these subsets consists of a disjoint union of components of the Lorentz group. It follows that they are all subgroups, and subgroups of index two. (This can also be checked algebraically.) Each is generated by the restricted Lorentz matrices together with whichever of $\mathscr{A} \in \{\mathscr{P}, \mathscr{T}, \mathscr{C}\}$ it contains.

The decompositions $T = T_a T_\Lambda$ and $T_\Lambda = T_R T_{\Lambda(\mathbf{v})}$ yield:

Proposition 1.19.
- *$\mathrm{P}(1,d)$ is homeomorphic to $\mathbb{R}^{1+d} \times \mathrm{SO}(1,d)$,*
- *$\mathrm{SO}(1,d)$ is homeomorphic to $\mathbb{R}^d \times \mathrm{SO}(d)$.*

■□

Consequently,

$$\pi_1(\mathrm{P}(1,d)) \cong \pi_1(\mathrm{SO}(d)) \cong \begin{cases} \text{trivial} & (d=1), \\ \mathbb{Z} & (d=2), \\ \mathbb{Z}_2 & (d \geq 3). \end{cases} \tag{1.94}$$

C.3. The Haar measure [*].

Proposition 1.20. *The left Haar measure on each of the groups $G = \mathrm{P}$, P^*, $\mathrm{SO}(1,d)$, $\mathrm{Spin}(1,d)$ is also right-invariant.* ■

D. Superselection rules

If we are given two unitary representations $\rho : G \to \mathrm{U}(\mathcal{V})$, $\varrho : G \to \mathrm{U}(\mathcal{W})$ of some group, then we can form their direct sum

$$\rho \oplus \varrho : G \to \mathrm{U}(\mathcal{V} \oplus \mathcal{W}). \tag{1.95}$$

The Hilbert space $\mathcal{V} \oplus \mathcal{W}$ is used to model a system whose state can lie in either summand, $\mathcal{V}$ or $\mathcal{W}$, *or be a quantum superposition thereof*. The matrices in the image of the combined representation $\rho \oplus \varrho$ are block diagonal:

$$(\rho \oplus \varrho)(g) = \begin{bmatrix} \rho(g) & 0 \\ 0 & \varrho(g). \end{bmatrix} \tag{1.96}$$

That is, they lie in the image of the natural embedding $\mathrm{U}(\mathcal{V}) \times \mathrm{U}(\mathcal{W}) \hookrightarrow \mathrm{U}(\mathcal{V} \oplus \mathcal{W})$.

Projective unitary representations, in contrast, cannot generally be summed (unless we are willing to enlarge our group G). The reason is that we lack a natural embedding

$$\mathrm{PU}(\mathcal{V}) \times \mathrm{PU}(\mathcal{W}) \hookrightarrow \mathrm{PU}(\mathcal{V} \oplus \mathcal{W}). \tag{1.97}$$

The natural attempt, $([V],[W]) \mapsto \left[\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \right]$, is not well-defined because

$$\mathrm{id}_{\mathcal{V}} \oplus e^{i\theta} \, \mathrm{id}_{\mathcal{W}} = \begin{pmatrix} \mathrm{id}_{\mathcal{V}} & 0 \\ 0 & e^{i\theta} \, \mathrm{id}_{\mathcal{W}} \end{pmatrix} \tag{1.98}$$

is not a scalar multiple of $\mathrm{id}_{\mathcal{V} \oplus \mathcal{W}}$ if $e^{i\theta} \neq 1$.

Example 1.21. Consider the double cover $\pi : \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$. Because $\ker \pi = \{I_2, -I_2\}$ is in the center of SU(2) (this is a general property of the kernels of covering maps), Schur's lemma tells us that, in any irreducible unitary representation of SU(2), both $I_2, -I_2$ are mapped to scalars. Consequently, we get a projective unitary representation of SO(3).

We have two classes of SU(2)-irreps:

- those, like the trivial representation, where $-I_2$ is mapped to the identity I,
- those, like the fundamental representation, where $-I_2$ is mapped to $-I$.

If we take a direct sum of irreps in *one* of these classes — it can be either class, but all of the irreps have to be from the same class — then $-I_2$ is still mapped to a scalar, and so the SU(2)-rep factors through to a projective unitary representation of SO(3). *But*, if we take a direct sum of irreps from *both* classes, then $-I_2$ is not mapped to a scalar, and it follows that the resulting SU(2)-rep does not factor through. ■

This is the motivation for superselection rules. It is an empirical fact that spinors exist. One would like to construct a theory in which one-particle spinor states coexist with the vacuum. But the corresponding projective space does not admit a projective representation of the Poincaré group. The problem consists of states involving a quantum superposition of states with an even number of spinors with one with an odd number of spinors. For example, consider the vacuum $|\varnothing\rangle$ and a one-electron state $|\mathrm{e}\rangle$. Under a $360°$ rotation,

$$\frac{1}{\sqrt{2}}(|\varnothing\rangle + |\mathrm{e}\rangle) \rightsquigarrow \frac{1}{\sqrt{2}}(|\varnothing\rangle - |\mathrm{e}\rangle). \tag{1.99}$$

These vectors are *not* linearly dependent, so they span different states $\in P\mathcal{H}$.

Q. So why does the world appear to be Lorentz invariant? **A.** Because the offending superpositions are not seen in nature.

The interactions present in the standard model of particle physics guarantee that spinors are only created in pairs. This is one way in which Lorentz invariance is preserved. So, we have three choices.

(1) Declare that the actual state space consists *either* of only states in which the parity of the spinor number is even, or states in which the parity is odd.
(2) Accept that Lorentz invariance might hold only at the level of experimental predictions, not the state space.
(3) Cut down the state space by excluding the offending superpositions. This is known as the *univalence* superselection rule [WWW52; SW00].

Until we observe some failure of Lorentz invariance, the choice between these possibilities is an aesthetic one. The first is unappealing because it implies the existence of two different theories, one in which the total number of spinors in the universe is even, and one in which the total number is odd. Good luck figuring out which holds! Regarding the second possibility, manifest Lorentz symmetry is a great simplification arguably not worth abandoning due to the conceivability of quantum superpositions never found in nature. Our preference is (3), but no consensus exists [Ear08]. Physicists seem to like (2), replacing manifest Lorentz covariance with manifest $\mathrm{Spin}(1,3)$ covariance. Nonobservability of problematic superpositions then amounts to a conservation law and concomitant restriction on practical observables.

More generally, a model with superselection rules in force is one in which the state space is defined to be a disjoint union of projectivizations of Hilbert spaces:

$$\text{state space} = P\mathcal{H}[1] \sqcup P\mathcal{H}[2] \sqcup \cdots \tag{1.100}$$

The choice between the different components

$$\mathcal{H}[n] \subsetneq \mathcal{H} \tag{1.101}$$

(*superselection sectors*) is a classical OR, not a quantum OR. The system must lie in some definite superselection sector. No quantum superpositions between the different possibilities are allowed. Thus, the univalence superselection rule accommodates possibility (1) above at the level of the state space. The theory of symmetries of quantum systems with superselection rules is developed in parallel to the version without superselection rules — see [SW00, Thm. 1.1]. The only new complication is that symmetries can permute superselection sectors, but we do not need to consider this possibility.

In the C*-algebra framework for quantum mechanics (which we have not broached), the existence of multiple superselection sectors can become a theorem holding for various models. This applies to QED, where the superselection sectors are labeled by the total charge Q. A rigorous result to this effect is due to Strocchi–Wightman [SW74], but the basic idea goes back to Haag [Haa63]: letting ρ

denote the charge density and $\mathbf{E}$ denote the electric field, we should have

$$\begin{aligned} Q = \lim_{R\to\infty} \int_{|\mathbf{x}|<R} \rho(t,\mathbf{x})\,\mathrm{d}^3\mathbf{x} &= \lim_{R\to\infty} \int_{|\mathbf{x}|<R} \nabla\cdot\mathbf{E}(t,\mathbf{x})\,\mathrm{d}^3\mathbf{x} \\ &= \lim_{R\to\infty} \int_{|\mathbf{x}|=R} \mathbf{E}(t,\mathbf{x})\cdot \mathrm{d}A(\mathbf{x}), \end{aligned} \tag{1.102}$$

using $\nabla\cdot\mathbf{E}=\rho$. This is Gauss's law. Consequently, for any observable O, one expects

$$[Q,O] = \lim_{R\to\infty} \int_{|\mathbf{x}|=R} [\mathbf{E}(t,\mathbf{x}),O]\cdot \mathrm{d}A(\mathbf{x}). \tag{1.103}$$

But, if observables are local things, then, if we take R large enough, $[\mathbf{E}(t,\mathbf{x}),O]=0$. So, $[Q,O]=0$. The main task taken up by Strocchi–Wightman is showing that the foregoing reasoning can be rigorized.

In non-abelian gauge theories, topological properties of field configurations are also expected to constitute superselection sectors. An example is the θ-angle in QCD [Col85, §7.3.3].

Exercises and Problems

EXERCISE 1.1: For $v\in(-1,1)$, define the *rapidity* $\beta\in\mathbb{R}$ by $v=\tanh\beta$. Show that the standard boost $\Lambda_{\mathrm{std}}(v)$ defined in eq. (1.15) can be written

$$\Lambda_{\mathrm{std}}(v) = \begin{bmatrix} \cosh\beta & -\sinh\beta & 0 \\ -\sinh\beta & \cosh\beta & 0 \\ 0 & 0 & I_{d-1} \end{bmatrix}. \tag{1.104}$$

This is a "hyperbolic rotation."

EXERCISE 1.2:
(a) Prove that $\Lambda_{\mathrm{std}}(v)$ preserves the Minkowski interval, hence lies in the Lorentz group.
(b) Let $\mathbf{v}\in\mathbb{R}^d\backslash\{\mathbf{0}\}$. Show that

$$\Lambda(\mathbf{v}) = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \Lambda_{\mathrm{std}}(\|\mathbf{v}\|) \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix} \tag{1.105}$$

for $R\in\mathrm{O}(3)$ any orthogonal transformation that takes $\hat{\mathbf{x}}=(1,0,\cdots)$ to $\hat{\mathbf{v}}=\mathbf{v}/\|\mathbf{v}\|$.

So, $\Lambda(\mathbf{v})$ lies in the Lorentz group as well.

EXERCISE 1.3: Let $v,u\in(-1,1)$. Show the following:

- $-1<(v+u)/(1+vu)<1$.
-
$$\Lambda_{\mathrm{std}}(v)\Lambda_{\mathrm{std}}(u) = \Lambda\Big(\frac{v+u}{1+vu}\Big). \tag{1.106}$$

  This is the *velocity addition formula* (for collinear boosts).

In particular, $\Lambda_{\mathrm{std}}(v)^{-1}=\Lambda_{\mathrm{std}}(-v)$.

Hint: the use of rapidities is very convenient here.

EXERCISE 1.4: Let $\mathcal{H}$ denote an infinite-dimensional (separable, as always) Hilbert space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$.

(a) Show that the strong and weak operator topologies agree on $\mathrm{U}(\mathcal{H})$.
(b) Show that $\mathrm{U}(\mathcal{H})$ is a topological group under the strong/weak operator topology.
(c) Explain why, when $\mathrm{PU}(\mathcal{H})$ is endowed with the quotient topology, it is also a topological group.

EXERCISE 1.5: For each $a \in \mathbb{R}$, let $T_a \in \mathrm{U}(L^2(\mathbb{R}))$ be the map

$$T_a\psi(x) = \psi(x-a) \tag{1.107}$$

that translates functions to the right by a units. Is the map $a \mapsto T_a$ continuous with respect to the strong operator topology? Uniform operator topology?

EXERCISE 1.6: Let $\mathcal{T}$ denote an anti-unitary operator on $\mathcal{H}$ such that $\mathcal{T}^2 = \varepsilon I$ for $\varepsilon = \pm 1$. If $\varepsilon = +1$, use $\mathcal{T}$ to write $\mathcal{H}$ as the complexification of some real subspace. If $\varepsilon = -1$, use $\mathcal{T}$ to endow $\mathcal{H}$ with the structure of a left $\mathbb{H}$-module.

EXERCISE 1.7: When working with spinors in 1+3 spacetime dimensions, the time-reversal operator $\mathcal{T}$ satisfies $\mathcal{T}^2 = -I$.

(a) Suppose one has a parity operator $\mathcal{P}$ satisfying $\mathcal{P}^2 = I$ and commuting with $\mathcal{T}$. According to Table 1.1, in which tenfold way class do we land?

(b) When working with Majorana spinors, physicists like to define the parity operator $\mathcal{P}$ so that it commutes with $\mathcal{T}$ but satisfies $\mathcal{P}^2 = -I$. Let $\tilde{\mathcal{P}} = i\mathcal{P}$. Show that $\mathcal{T}, \tilde{\mathcal{P}}$ *anti*-commute. Using Table 1.1, determine which tenfold way class this spinor system falls into, when analyzed using the pair $(\mathcal{T}, \tilde{\mathcal{P}})$.

PROBLEM 1.1: Prove rigorously that

- $\mathrm{O}(1,d)$ is a Lie subgroup of $\mathrm{GL}(1+d,\mathbb{R})$
- $\mathrm{P}(1,d)$ is a Lie subgroup of $\mathrm{Aff}(\mathbb{R}^{1,d})$.

While you are at it, prove eq. (1.87), eq. (1.88).

PROBLEM 1.2:

(a) (Optional.) Show that any bijection $T : \mathbb{R}^d \to \mathbb{R}^d$ preserving the Euclidean distance must have the form $\mathbf{x} \mapsto R\mathbf{x} + \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^d$ and $R \in \mathrm{O}(d)$.

Let $T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}$ denote a bijection preserving the Minkowski interval $d(x,y) = (x-y)^2$, i.e. a Minkowski isometry. Our goal is to show that T is affine.

(b) Reduce the general case to the case where T fixes the spacetime origin.

Now assume that T fixes the spacetime origin. Our goal is to show that T is *linear*.

(c) Let $\mathbf{e} = (1, \mathbf{0})$. Show that there exists a boost $\Lambda(\mathbf{v})$ such that $\mathscr{T}^j\Lambda(\mathbf{v})\mathbf{e} = T(\mathbf{e})$, for $j = 0, 1$. So, $\Lambda(\mathbf{v})^{-1}T$ is a Minkowski isometry fixing $\mathbf{e}$.

(d) Let $\Sigma = \{(0,\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$. Show that if T is any Minkowski isometry fixing $\mathbf{e}$ and the spacetime origin, then there exists some $R \in \mathrm{O}(d)$ such that $T|_\Sigma = R$.

(e) Then,

$$T_0 \stackrel{\text{def}}{=} R^{-1}\Lambda(\mathbf{v})^{-1}\mathscr{T}^j T$$

is a Minkowski isometry which fixes the spacetime origin, $\mathbf{e}$, and Σ. Show that T_0 is the identity.

So, $T = \mathscr{T}^j\Lambda(\mathbf{v})R$ is linear. This also shows that the time-reflection and parity operators, spacetime translations, boosts, and rotations together generate the full Poincaré group.

PROBLEM 1.3: Let $\mathbb{B}^d = \{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| < 1\}$.

(a) (Velocity addition formula.) Show that, for any $\mathbf{v}, \mathbf{u} \in \mathbb{B}^d$ (not necessarily collinear!),

$$\Lambda(\mathbf{v})\Lambda(\mathbf{u}) = W\Lambda(\mathbf{v} \oplus \mathbf{u}) \tag{1.108}$$

for some $W \in \mathrm{SO}(3)$, where

$$\mathbf{v} \oplus \mathbf{u} = \frac{1}{1+\mathbf{u}\cdot\mathbf{v}}\Big[\Big(1 + \frac{\gamma}{\gamma+1}\mathbf{u}\cdot\mathbf{v}\Big)\mathbf{v} + \frac{\mathbf{u}}{\gamma}\Big] \tag{1.109}$$

and $\gamma = \gamma(\mathbf{v})$ is the Lorentz factor associated to $\mathbf{v}$. Prove that $\mathbf{v} \oplus \mathbf{u} \in \mathbb{B}^d$.

(b) Show that

$$\gamma(\mathbf{u} \oplus \mathbf{v}) = \gamma(\mathbf{u})\gamma(\mathbf{v})(1 + \mathbf{u} \cdot \mathbf{v}). \tag{1.110}$$

(c) Show that $\mathbf{u} \oplus \mathbf{v} \neq \mathbf{v} \oplus \mathbf{u}$ in general, but they have the same magnitude.

PROBLEM 1.4: This problem continues Problem 1.3. The rotation W is called a *Thomas–Wigner rotation*. Prove that, if $\mathbf{v}, \mathbf{u}$ are non-collinear, then W is a rotation of the plane in which $\mathbf{u}, \mathbf{v}$ lie by an angle $\epsilon \in (-\pi, \pi)$ satisfying

$$\cos \epsilon = \frac{(1 + \gamma(\mathbf{u}) + \gamma(\mathbf{v}) + \gamma(\mathbf{v} \oplus \mathbf{u}))^2}{(1 + \gamma(\mathbf{v} \oplus \mathbf{u}))(1 + \gamma(\mathbf{u}))(1 + \gamma(\mathbf{v}))} - 1. \tag{1.111}$$

Moreover, the correct sign is the one such that W rotates $\mathbf{u}$ towards $\mathbf{v}$.

PROBLEM 1.5: Consider the matrix exponential

$$e^{\Lambda} = \sum_{j=1} \frac{\Lambda^j}{j!}.$$

Prove that this is a surjective map $\mathfrak{o}(1,3) \to \mathrm{SO}(1,3)$. *Hint*: consensus is that no short proof of this exists. You may wish to use the double cover $\mathrm{SL}(2,\mathbb{C}) \twoheadrightarrow \mathrm{SO}(1,3)$ that we discuss in the next lecture.

CHAPTER 2

Deprojectivization

In the previous lecture, we defined the notion of a relativistic quantum system in $d \in \mathbb{N}^+$ spatial dimensions: a (continuous) projective unitary representation $\rho : \mathrm{P}(1, d) \to \mathrm{PU}(\mathcal{H})$ of the restricted Poincaré group $\mathrm{P}(1, d)$. In this lecture, we present:

> THEOREM (Wigner–Bargmann [Wig39; Bar54]). *If $d \geq 2$, then ρ can be lifted to an ordinary unitary representation of the universal cover $\pi : \mathrm{P}^*(1, d) \twoheadrightarrow \mathrm{P}(1, d)$.* ■

Classifying the projective unitary representations of the restricted Poincaré group is thereby reduced to the more amenable problem of classifying *ordinary* unitary representations of a slightly bigger group,

$$\mathrm{P}^*(1, d) = \mathbb{R}^{1,d} \rtimes \widetilde{\mathrm{SO}(1, d)}. \tag{2.1}$$

To this latter problem, the ample mathematical tools of linear representation theory apply. This will be the topic of the next chapter.

We will call $\mathrm{P}^*(1, d)$ the *universal* Poincaré group, for lack of standard terminology. The Lorentz part of this is

$$\widetilde{\mathrm{SO}(1, d)} \cong \begin{cases} \widetilde{\mathrm{SL}(2, \mathbb{R})} & (d = 2), \\ \mathrm{Spin}(1, d) & (d \geq 3). \end{cases} \tag{2.2}$$

"Universal" is synonymous with *spinorial* if $d \geq 3$, in which case the relevant covers are all double covers. For each $d \geq 2$, there exists a unique connected double cover

$$\mathrm{Spin}(1, d) \twoheadrightarrow \mathrm{SO}(1, d). \tag{2.3}$$

As the name indicates, the existence of this cover has to do with spinors.

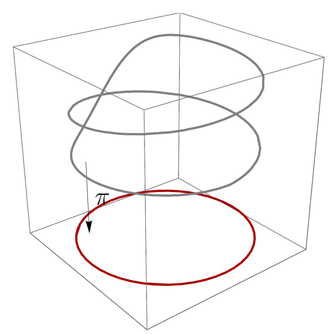


FIGURE. The double cover $\mathrm{Spin}(2) \twoheadrightarrow \mathrm{SO}(2)$.

In the physical $d = 3$ case, $\mathrm{Spin}(1, 3)$ is isomorphic to the group

$$\mathrm{SL}(2, \mathbb{C}) = \{S \in \mathbb{C}^{2\times 2} : \det S = 1\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ ad - bc = 1 \right\} \tag{2.4}$$

of unimodular two-by-two matrices with complex entries. This isomorphism, which plays the same role that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ plays in non-relativistic quantum mechanics, is the basis for many computations involving spinors.

REMARK: Conversely, unitary representations of $\mathrm{P}^*(1, d)$ induce projective unitary representations of $\mathrm{P}(1, d)$ as long as the elements of $\ker \pi$ are mapped to c-numbers.

WARNING: The notation "$\mathrm{Spin}(1, d)$" is also used (by some authors, not us) to refer to a particular double cover of the *full* Lorentz group $\mathrm{O}(1, d)$.

WARNING: It is sometimes said that spinors do not return to themselves upon performing a 360° rotation, only after a 720° rotation. This is misleading. The state of a spinor, as element of $P\mathcal{H}$, does return to itself upon a full rotation. The confusion arises from mistaking vectors in $\mathcal{H}$ as states.

1. The Wigner–Bargmann theorem

Let $\mathcal{H}$ denote an arbitrary Hilbert space. Suppose we are given a (continuous, as always) projective unitary representation $\rho : G \to \mathrm{PU}(\mathcal{H})$ of some topological group G.

Definition. Let E denote a set and

$$\pi : E \twoheadrightarrow G \tag{2.5}$$

a surjection. A *lift* of ρ (to E, via π) is a map of sets $\varrho : E \to \mathrm{U}(\mathcal{H})$ such that

$$[\varrho(g)] = \rho(\pi(g)) \tag{2.6}$$

for all $g \in E$. In other words:

$$\begin{tikzcd} E \arrow[r, "\varrho"] \arrow[d, "\pi"'] & \mathrm{U}(\mathcal{H}) \arrow[d, "U\mapsto[U]"] \\ G \arrow[r, "\rho"'] & \mathrm{PU}(\mathcal{H}) \end{tikzcd} \tag{2.7}$$

commutes.

Note that, in this definition, we are not requiring anything of ϱ other than that it be a map of sets. By the axiom of choice, at least one lift exists. The phase of the operator $\varrho(g)$ is however completely arbitrary:

$$\varrho_{\mathrm{alt}}(g) = e^{i\theta(g)}\varrho(g) \tag{2.8}$$

is also a lift of ρ, for any function $\theta : E \to \mathbb{R}$. Consequently, there is no reason why ϱ, given what we have said so far, would be a homomorphism, or even continuous.

The question of deprojectivization is whether there exists a choice of lift that is a continuous homomorphism. In other words, can we, by choosing θ carefully, arrange that ϱ_{alt} is an ordinary unitary representation of E? A far-reaching theorem of Bargmann [Bar54] states:

> **Lemma.** (*Bargmann's theorem.*) Any continuous projective unitary representation of a connected Lie group G lifts to a continuous unitary representation of G itself if the following two conditions are satisfied:
> (I) the Lie group is simply connected: the fundamental group $\pi_1(G)$ is trivial,
> (II) a certain cohomology group $\mathrm{H}^2(\mathfrak{g}) = \mathrm{H}^2(\mathfrak{g};\mathbb{R})$ (defined below) associated to the Lie algebra $\mathfrak{g}$ of G is trivial.

We will provide an outline of a proof of Bargmann's theorem, omitting the difficult technical bits.

What matters for us is that the hypotheses are satisfied when $G = \mathrm{P}^*(1,d)$ for $d \geq 2$. The topological condition, (I), is automatic: the definition of the universal cover includes that π_1 vanish. For (II): the Lie algebra of $\mathrm{P}^*(1,d)$ is the same as the Lie algebra $\mathfrak{p}$ of $\mathrm{P}(1,d)$, so it suffices to check that

$$\mathrm{H}^2(\mathfrak{p}) = \{0\}. \tag{2.9}$$

This calculation is rather involved, but it is just a calculation. We have included it in §A. If the reader is willing to take for granted that Bargmann's theorem applies to $\mathrm{P}^*(1,d)$, then it is not important to understand what $\mathrm{H}^2(\mathfrak{p})$ is.

Proof of the Bargmann–Wigner theorem. Assume that $\mathrm{H}^2(\mathfrak{p}) = \{0\}$. For any continuous projective representation ρ, let

$$\begin{aligned} \bar{\rho} &: \mathrm{P}^*(1,d) \to \mathrm{PU}(\mathcal{H}) \\ \bar{\rho} &= \rho \circ \pi. \end{aligned} \tag{2.10}$$

This is a continuous projective unitary representation *of* $\mathrm{P}^*(1,d)$. By Bargmann's theorem, there exists a (continuous) unitary representation $\varrho : \mathrm{P}^*(1,d) \to \mathrm{U}(\mathcal{H})$ lifting $\bar{\rho}$. That is, $[\varrho(g)] = \bar{\rho}(g)$. Since $\bar{\rho}(g) = \rho(\pi(g))$, we conclude eq. (2.6). ∎

The two hypotheses of Bargmann's theorem each reflect an obstruction to lifting a projective representation to an ordinary representation:

(I) The first hypothesis indicates a global topological obstruction to choosing a continuous lift.
(II) The second indicates an "infinitesimal" algebraic obstruction to making the lift a homomorphism.

We discuss these in turn.

Since the calculation of $\mathrm{H}^2(\mathfrak{p})$ is rather involved, let us say a few words about what would happen were it *non*trivial. Projective unitary representations ρ of simply connected Lie groups G with nontrivial $\mathrm{H}^2(\mathfrak{g})$ lift to unitary representations not of G itself but of some U(1)-fiber bundle $E \twoheadrightarrow G$ (which can depend on ρ). This Lie group has dimension $\dim E = \dim G + 1$ rather than $\dim G$. In physicists' language, projective representations of G involve a "central charge." Examples of G where this happens include the Galilean group, where the additional central charge is interpreted as the *mass* of the representation, the Weyl algebra (the symmetry algebra of conformal field theory on the worldsheet $\mathbb{R} \times \mathbb{S}^1$ of a string), and the Poincaré group in $d = 1$ spatial dimensions, $\mathrm{P}^*(1, 1)$, where the central charge can be interpreted as the slope of an energy gradient across space[1]. So, *what the proof of eq. (2.9) is doing is ruling out the presence of spurious central charges in projective representations of the Poincaré group.*

1.1. The topological obstruction, $\pi_1(G)$. Topological obstructions show up as a matter of course in lifting problems, so the presence of a topological obstruction here is unsurprising. Some lifting problems are topologically trivial, but this is not one of those: for each $N \in \mathbb{N}^+$, the covering map $\mathrm{U}(N) \twoheadrightarrow \mathrm{PU}(N)$ is a nontrivial U(1)-bundle. It has no continuous sections. So, if a projective unitary representation ρ is to be lifted, this imposes a topological constraint on ρ.

The ur-example, essential to understanding spinors, is:

EXAMPLE 2.1 ($\mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$). Famously, the two groups

$$\begin{aligned} \mathrm{SU}(2) &= \{U \in \mathrm{U}(2) : \det U = 1\} \\ \mathrm{SO}(3) &= \{R \in \mathrm{O}(3) : \det R = 1\} \end{aligned} \tag{2.11}$$

are related by a (smooth, homomorphic) double cover $\pi : \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$. Correspondingly, while SU(2) is simply connected, SO(3) is doubly connected:

$$\pi_1(\mathrm{SU}(2)) = \text{trivial}, \qquad \pi_1(\mathrm{SO}(3)) \cong C_2. \tag{2.12}$$

The kernel of π is $\ker \pi = \{I_2, -I_2\}$. Consequently,

$$\mathrm{SO}(3) \cong \mathrm{PU}(2), \tag{2.13}$$

with an explicit isomorphism $\rho : \mathrm{SO}(3) \to \mathrm{PU}(2)$ being $\rho(\pi(U)) = [U]$. The map ρ is a projective unitary representation of SO(3).

Q. Does this lift to a continuous map $\mu : \mathrm{SO}(3) \to \mathrm{SU}(2)$? **A.** No.

Suppose, to the contrary, that there did exist such a μ. This would mean that $[\mu(R)] = \rho(R)$ for all $R \in \mathrm{SO}(3)$. I.e.

$$[\mu(\pi(U))] = [U] \tag{2.14}$$

for all $U \in \mathrm{SU}(2)$. That is, $\mu(\pi(U)), U$ differ by a phase. But $U, \mu(\pi(U))$ are *special* two-by-two unitary matrices, not just general unitary matrices. So differing by a phase implies $\mu(\pi(U)) = \pm U$. That is,

$$U^{-1}\mu(\pi(U)) = \pm I_2. \tag{2.15}$$

A priori, the sign $\pm$ could depend on U — but it is easy to see that it cannot, because the left-hand side depends continuously on U. Since SU(2) is connected, this means that the sign is constant.

[1]When $d \geq 2$, this sort of thing is ruled out by rotation covariance.

Without loss of generality, we can assume the + case, which means $\mu(I_3) = I_2$. Then,

$$\mu(\pi(U)) = U. \tag{2.16}$$

But this is impossible, because $\pi(-I_2) = \pi(I_2)$. ■

REMARK 2.2. The reader may have noticed an artificial restriction in the previous example: we required $\mu : \mathrm{SO}(3) \to \mathrm{SU}(2)$. But why not weaken this to land in U(2)? The distinction does not matter: *there does not exist any continuous map* $\mathrm{SO}(3) \to \mathrm{U}(2)$ *lifting* ρ, but proving this requires
[Problem 2.1] a bit more topology. ■

REMARK 2.3. If we demand that $\mu : \mathrm{SO}(3) \to \mathrm{U}(2)$ be a homomorphism, then the lack of a lift follows from representation theoretic facts: the only two-dimensional representation of SO(3) is the trivial one.

One way of proving this is the argument above. Note that $\det \circ \mu : \mathrm{SO}(3) \to \mathrm{U}(1)$ is a one-dimensional representation of SO(3), and the only one-dimensional representation of SO(3) is the trivial one (see Exercise 2.1). Consequently, any two-dimensional unitary representation of SO(3) must land in SU(2), and then the argument above applies. ■

The fundamental group $\pi_1(G)$ is, as a set, the set of homotopy classes of continuous maps $\mathbb{S}^1 \to G$. (Recall that G is connected, so we do not need to worry about the choice of base point.) In the proof of Bargmann's theorem, the vanishing of the fundamental group is used to reduce the problem to one about representations of the Lie *algebra* $\mathfrak{g}$. This is what we discuss next.

1.2. The algebraic obstruction, $\mathrm{H}^2(\mathfrak{g})$**.** Given a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the vector space $\mathrm{H}^2(\mathfrak{g}) = \mathrm{H}^2(\mathfrak{g}; \mathbb{K})$ is defined as follows. Identify elements of $(\wedge^2\mathfrak{g})^*$ with anti-symmetric bilinear maps $\mathfrak{g}^2 \to \mathbb{K}$. Then,

$$\mathrm{B}^2(\mathfrak{g}) = \{\omega \in (\wedge^2\mathfrak{g})^* : \exists \lambda \in \mathfrak{g}^* \text{ s.t. } \omega(X, Y) = \lambda([X, Y])\}, \tag{2.17}$$

and

$$\mathrm{Z}^2(\mathfrak{g}) = \{\omega \in (\wedge^2\mathfrak{g})^* : \omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0\} \tag{2.18}$$

are the spaces of "coboundaries" and "cocycles" respectively. These are both vector subspaces of $(\wedge^2\mathfrak{g})^*$. Note that $\mathrm{B}^2(\mathfrak{g}) \subseteq \mathrm{Z}^2(\mathfrak{g})$. Indeed, if $\omega \in \mathrm{B}^2(\mathfrak{g})$,

$$\begin{aligned}\omega([X, Y], Z) &+ \omega([Y, Z], X) + \omega([Z, X], Y) \\ &= \lambda([[X, Y], Z]) + \lambda([[Y, Z], X]) + \lambda([[Z, X], Y]) = \lambda(J),\end{aligned} \tag{2.19}$$

where $J = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$. But $J = 0$ is the Jacobi identity. So, $\omega \in \mathrm{Z}^2(\mathfrak{g})$. Then,

$$\mathrm{H}^2(\mathfrak{g}; \mathbb{K}) = \mathrm{Z}^2(\mathfrak{g}) / \mathrm{B}^2(\mathfrak{g}) \tag{2.20}$$

is the quotient of the vector space of cocycles by the subspace of coboundaries.

The general theory of Lie algebra cohomology is due to Chevalley–Eilenberg [CE48]. The special case of $\mathrm{H}^\bullet(\mathfrak{g}; \mathbb{K})$ goes further back, to Cartan, in his work on the de Rham cohomology of compact Lie groups.

The relevance of $\mathrm{H}^2(\mathfrak{g})$ to lifting projective representations has to do with its role in classifying central extensions of $\mathfrak{g}$ via $\mathbb{K}$. Recall that a short exact sequence of Lie algebras consists of the data

$$\mathbb{K} \hookrightarrow \mathfrak{e} \overset{\pi}{\twoheadrightarrow} \mathfrak{g}, \tag{2.21}$$

where $\ker \pi$ is exactly the image of $\mathbb{R}$ under the first embedding. If $\ker \pi$ lies in the *center* of $\mathfrak{e}$, then this is called a central extension.

EXAMPLE 2.4. The *trivial* extension is $\mathfrak{e} = \mathfrak{g} \oplus \mathbb{K}$. More precisely, it is the short exact sequence in which

- the embedding $\mathbb{K} \hookrightarrow \mathfrak{e}$ is $x \mapsto (0, x)$ and
- the projection $\mathfrak{e} \twoheadrightarrow \mathfrak{g}$ is $(X, x) \mapsto X$.

■

Two central extensions $\mathbb{K} \hookrightarrow \mathfrak{e}_\bullet \twoheadrightarrow \mathfrak{g}$ are called *equivalent* if they fit together into a commutative diagram

$$\begin{tikzcd} & \mathfrak{e}_1 \arrow[dd, "\phi"] \arrow[dr, twoheadrightarrow] & \\ \mathbb{K} \arrow[ur, hookrightarrow] \arrow[dr, hookrightarrow] & & \mathfrak{g} \\ & \mathfrak{e}_2 \arrow[ur, twoheadrightarrow] & \end{tikzcd} \tag{2.22}$$

where ϕ is an isomorphism of Lie algebras. A central extension $\mathfrak{e}$ is also called trivial if it is equivalent to the trivial extension $\mathfrak{g} \oplus \mathbb{K}$. Equivalently, a central extension is trivial if and only if it splits, which means that there exists a map $\mathfrak{g} \ni X \mapsto \bar{X} \in \mathfrak{e}$ of Lie algebras such that $\pi(\bar{X}) = X$.

Given any central extension $\mathbb{K} \hookrightarrow \mathfrak{e} \twoheadrightarrow \mathfrak{g}$, let $Z \in \mathfrak{e}$ denote a nonzero element in the image of the first embedding. Choose an embedding of vector spaces $\mathfrak{g} \hookrightarrow \mathfrak{e}$, and denote this $X \mapsto \bar{X}$. Note that this will typically not be a map of Lie algebras. Measure its failure to be a map of Lie algebras by defining $\omega : \mathfrak{g}^2 \to \mathbb{K}$ by

$$\omega(X,Y)Z = [\bar{X},\bar{Y}] - \overline{[X,Y]}, \tag{2.23}$$

where the first Lie bracket is $\mathfrak{e}$'s and the second is $\mathfrak{g}$'s. The right-hand side lies in the kernel of the projection $\mathfrak{e} \twoheadrightarrow \mathfrak{g}$ (this being a map of Lie algebras), so, by exactness, lies in the image of $\mathbb{K} \hookrightarrow \mathfrak{e}$, which is why it has the form $\mathbb{K}Z$.

Proposition 2.5. *(a) The map ω defined above is a cocycle.*
(b) It is a coboundary if and only if the extension is trivial.

■

Proof sketch. The first part of the proposition is the Jacobi identity for ω, as defined by eq. (2.23). This will follow from combining the Jacobi identities for $\mathfrak{e}, \mathfrak{g}$. The second part of the proposition comes from the following observation: an alternative to the lift $X \mapsto \bar{X}$ is $X \mapsto \bar{X} + \lambda(X)Z$, for any $\lambda \in \mathfrak{g}^*$. This alternative has a different ω, one differing from the original by a coboundary. So, we can choose a lift eliminating ω (in which case the lift is a splitting map) if and only if ω is a coboundary. □

Proof. (a) We want to show that ω satisfies the Jacobi identity $\omega([W,X],Y)+\omega([X,Y],W)+\omega([Y,W],X) = 0$. Note that

$$\begin{aligned} \omega([W,X],Y) &= [\overline{[W,X]},\bar{Y}] - \overline{[[W,X],Y]} \\ &= [[\bar{W},\bar{X}] - \omega(W,X)Z,\bar{Y}] - \overline{[[W,X],Y]} \\ &= [[\bar{W},\bar{X}],\bar{Y}] - \overline{[[W,X],Y]}. \end{aligned} \tag{2.24}$$

Likewise,

$$\begin{aligned} \omega([X,Y],W) &= [[\bar{X},\bar{Y}],\bar{W}] - \overline{[[X,Y],W]} \\ \omega([W,X],Y) &= [[\bar{W},\bar{X}],\bar{Y}] - \overline{[[W,X],Y]}. \end{aligned} \tag{2.25}$$

The Jacobi identity for ω therefore follows from three things: the Jacobi identity for $\mathfrak{e}$'s Lie bracket, the Jacobi identity for $\mathfrak{g}$'s Lie bracket, and the linearity of the map $X \mapsto \bar{X}$.

(b)
- ('If.') It suffices to consider the case where $\mathfrak{e} = \mathfrak{g} \oplus \mathbb{K}$ is literally the trivial extension, in which case we can choose $Z = (0,1)$. Because $X \mapsto \bar{X}$ is linear, there must exist

some $\lambda \in \mathfrak{g}^*$ such that $\bar{X} = (X, \lambda(X))$. Then,

$$\begin{aligned} \omega(X,Y)Z = [\bar{X}, \bar{Y}] - \overline{[X,Y]} &= ([X,Y],0) - ([X,Y], \lambda([X,Y])) \\ &= (0, \lambda([X,Y])). \end{aligned} \tag{2.26}$$

We conclude that $\omega(X,Y) = \lambda([X,Y])$, which is what it means to be a coboundary.

- ('Only if.') Suppose that $\omega(X,Y) = \lambda([X,Y])$ for some $\lambda \in \mathfrak{g}^*$. Now define $\phi : \mathfrak{g} \oplus \mathbb{K} \to \mathfrak{e}$ by

$$\phi(X,s) = \bar{X} + (s + \lambda(X))Z. \tag{2.27}$$

This is an isomorphism of vector spaces, fitting into a commutative diagram of vector spaces, as in eq. (2.22). It is also a map of Lie algebras:

$$[\phi(X,s), \phi(Y,t)] = [\bar{X}, \bar{Y}] \tag{2.28}$$

$$\begin{aligned} \phi([(X,s),(Y,t)]) = \phi([X,Y],0) &= \overline{[X,Y]} + \lambda([X,Y])Z \\ &= \overline{[X,Y]} + \omega(X,Y)Z = [\bar{X}, \bar{Y}]. \end{aligned} \tag{2.29}$$

□

It turns out that any continuous projective unitary representation of a Lie group induces an ordinary representation not of $\mathfrak{g}$ but of a one-dimensional central extension thereof. Indeed, consider the topological group

$$E = \{(U,g) \in \mathrm{U}(\mathcal{H}) \times G : [U] = \rho(g)\}, \tag{2.30}$$

whose (natural, continuous) unitary representation $\varrho : (U,g) \mapsto U$ lifts ρ. This sits in a central extension

$$\mathrm{U}(1) \hookrightarrow E \twoheadrightarrow G \tag{2.31}$$

in the category of topological groups. It turns out that E is canonically a Lie group of dimension $\dim G + 1$, and the maps in eq. (2.31) are smooth.[2] Consequently, they can be "differentiated" to yield a central extension $\mathbb{R} \hookrightarrow \mathfrak{e} \twoheadrightarrow \mathfrak{g}$ of the corresponding Lie algebras. The representation ϱ induces a representation of $\mathfrak{e}$ on $\mathcal{H}$.

If the central extension $\mathbb{R} \hookrightarrow \mathfrak{e} \twoheadrightarrow \mathfrak{g}$ is trivial, so that $\mathfrak{e} \cong \mathbb{R} \oplus \mathfrak{g}$, then we can ignore the extra $\mathbb{R}$ factor and get a representation on $\mathcal{H}$ of $\mathfrak{g}$ itself. Exponentiating this (using Lie's second theorem) yields an ordinary unitary representation of the universal cover of G. If G is already simply connected, this is G itself. So:

$$\rho \text{ is guaranteed to lift} \iff \text{all 1D central extensions of } \mathfrak{g} \text{ are trivial.}$$

Conversely, nontrivial central extensions of $\mathfrak{g}$ typically yield projective representations of G which only lift to ordinary unitary representations of some central extension of G by $\mathrm{U}(1)$, namely E.

EXAMPLE 2.6 (The Heisenberg group). In this example, we exhibit a projective representation of the abelian group $(\mathbb{R}^2, +)$ that does not lift to a unitary representation of $(\mathbb{R}^2, +)$ but rather of a three-dimensional Lie group H known as the "Heisenberg group." The existence of such an intrinsically projective representation has to do with the fact that the abelian Lie algebra $\mathfrak{t} = \mathbb{R}^2$ has nontrivial cohomology,

$$\mathrm{H}^2(\mathfrak{t}) \cong \mathbb{R} \tag{2.32}$$

(see Exercise 2.3) and therefore admits a nontrivial central extension, namely the Lie algebra of the Heisenberg group.

---

[2]Establishing this is the main technical step in the proof of Bargmann's theorem. First, one shows straightforwardly that E is a topological manifold [Sim71], being a $\mathrm{U}(1)$-bundle over G. Then, the solution of Hilbert's fifth problem (the Montgomery–Zippin theorem) guarantees that E is canonically a Lie group. The special case relevant here – that of a central extension of a Lie group by another Lie group – is considerably easier than the full problem, and a complete exposition can be found in [Tao14, §2.6].

The *Heisenberg group* H is the group

$$H = \left\{ \underbrace{\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}}_{M(a,b,c)} : a, b, c \in \mathbb{R} \right\} \tag{2.33}$$

of 3-by-3 matrices $M \in \mathbb{R}^{3\times 3}$ such that $M - I_3$ is upper triangular. The group law is

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+a' & c+c'+ab' \\ 0 & 1 & b+b' \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.34}$$

So, if we define $\pi : H \twoheadrightarrow \mathbb{R}^2$ by $M(a,b,c) \mapsto (a,b)$, this is a homomorphism onto the abelian group $(\mathbb{R}^2, +)$. This fits into a central extension

$$(\mathbb{R}, +) \hookrightarrow H \overset{\pi}{\twoheadrightarrow} (\mathbb{R}^2, +) \tag{2.35}$$

of groups, where the first map is $c \mapsto M(0,0,c)$. Differentiating yields a central extension

$$\mathbb{R} \hookrightarrow \mathfrak{h} \twoheadrightarrow \mathfrak{t} = \mathbb{R}^2 \tag{2.36}$$

of Lie algebras. Concretely, $\mathfrak{h} \subset \mathbb{R}^{3\times 3}$ is the Lie algebra of upper-triangular 3-by-3 matrices

$$m(a,b,c) = \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.37}$$

in which the Lie bracket is the usual commutator. The first map in eq. (2.36) is $c \mapsto m(0,0,c)$ and the second map is $m(a,b,c) \mapsto (a,b)$. The central extension $\mathfrak{h}$ is nontrivial, because $\mathfrak{h}$ is not abelian.

The *Schrödinger representation* of the Heisenberg group H is defined on $\mathcal{H} = L^2(\mathbb{R})$. Actually, we have many different Schrödinger representations, one for each $\hbar > 0$. Specifically, let $\varrho_\hbar : H \to \mathrm{U}(\mathcal{H})$ be defined by

$$\varrho_\hbar\left(\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \right)\psi(x) = e^{i\hbar c} e^{ibx} \psi(x + \hbar a) \text{ for all } \psi \in L^2(\mathbb{R}) \tag{2.38}$$

for all $\psi \in L^2(\mathbb{R})$. This is indeed a continuous unitary representation. [Problem 2.3(b)]

Let $\rho_\hbar : \mathbb{R}^2 \to \mathrm{PU}(\mathcal{H})$ be defined by $\rho_\hbar(a,b) = \varrho_\hbar(M(a,b,0)) \bmod \mathrm{U}(1)$. This is a (continuous) projective unitary representation of $(\mathbb{R}^2, +)$. While it lifts to an ordinary representation of H, by construction, it does not lift to an ordinary representation of $\mathbb{R}^2$. The physical interpretation of $\rho_\hbar$ [Problem 2.3(c)] is that it describes the Galilean symmetry of Schrödinger's wave mechanics on the real line. One factor of $\mathbb{R}^2$ is carrying out a translation, and the other is carrying out a Galilean boost. ■

2. Parity and time-reversal ($\star$)

We now discuss how the considerations above are modified in the presence of parity and/or time-reversal symmetry. In §3, we discussed how these additional symmetries lift to unitary/anti-unitary transformations of the ambient Hilbert space. The possibilities were classified according to a "tenfold way." Excluding class A, a system with no additional symmetries beyond the restricted Poincaré group, there were nine cases: AI, AII, AIII, BDI, C, CI, CII, D, DIII. The goal of this section is to discuss how, in each class, a Wigner representation of the relevant subgroup of the full Poincaré group lifts to an ordinary unitary/anti-unitary representation of some cover thereof.

In each of the ten ways, we have a subgroup $Z \subseteq \{\mathscr{I}, \mathscr{P}, \mathscr{T}, \mathscr{C} = \mathscr{P}\mathscr{T}\}$ telling us which symmetries are present. In Table 1.1 is listed a covering group $c : E \twoheadrightarrow Z$. This is either an isomorphism or a double cover, depending on which way we are in. For each $\mathscr{A} \in Z$, we have an automorphism

$$T \mapsto T_{\mathscr{A}} T T_{\mathscr{A}} \tag{2.39}$$

of $\mathrm{P}(1,d)$. This induces an automorphism of the Lie algebra $\mathfrak{p}(1,d)$, which exponentiates (by Lie's second theorem) to an automorphism $\Phi_{\mathscr{A}} \in \mathrm{Aut}(\mathrm{P}^*(1,d))$ of $\mathrm{P}^*(1,d)$. The map $\mathscr{A} \mapsto \Phi_{\mathscr{A}}$ associating $\mathscr{A}$ with this automorphism is a homomorphism

$$Z \to \mathrm{Aut}(\mathrm{P}^*(1,d)). \tag{2.40}$$

Precomposing with c yields a homomorphism $E \to \mathrm{Aut}(\mathrm{P}^*(1,d))$. Let $E \ltimes \mathrm{P}^*(1,d)$ denote the corresponding semidirect product. This is a double or quadruple cover of the subgroup G of the full Poincaré group containing the identity component and the members of Z.

Recall that we have been assuming $d \geq 2$.

Proposition 2.7. *Every Wigner representation of G lifts to an ordinary unitary/anti-unitary representation of $E \ltimes \mathrm{P}^*(1,d)$.* ■

Proof. By the discussion in the previous lecture, and by the Wigner–Bargmann theorem, we have:

- an ordinary unitary representation $\varrho : \mathrm{P}^*(1,d) \to \mathrm{U}(\mathcal{H})$, lifting the restriction of the given Wigner representation to $\mathrm{P}(1,d)$,
- for each $\mathscr{A} \in Z$, an operator $\mathcal{A}$ (as described in the tenfold way) such that

$$\mathcal{A}\varrho(T)\mathcal{A} = \varepsilon_{\mathsf{A}}\varrho(\Phi_{\mathscr{A}}(T))e^{i\theta(T;\mathscr{A})} \tag{2.41}$$

  for each $T \in \mathrm{P}^*(1,d)$, where $\theta(T;\mathscr{A})$ is some phase, and where $\varepsilon_{\mathsf{A}} \in \{-1,+1\}$ is the sign of $\mathcal{A}^2 \propto I$.

The group generated by the $\mathcal{A}$'s is the covering group E.

Let us restrict $\theta(T;\mathscr{A})$. Note that $e^{i\theta(I;\mathscr{A})} = 1$. Compute

$$\begin{aligned}\varepsilon_{\mathsf{A}}e^{i\theta(T_1T_2;\mathscr{A})} = \mathcal{A}\varrho(T_1T_2)\mathcal{A}\varrho(\Phi_{\mathscr{A}}(T_2^{-1}T_1^{-1})) &= \varepsilon_{\mathsf{A}}\mathcal{A}\varrho(T_1)\mathcal{A}\mathcal{A}\varrho(T_2)\mathcal{A}\varrho(\Phi_{\mathscr{A}}(T_2^{-1}))\varrho(\Phi_{\mathscr{A}}(T_1^{-1})) \\ &= \mathcal{A}\varrho(T_1)\mathcal{A}e^{i\theta(T_2;\mathscr{A})}\varrho(\Phi_{\mathscr{A}}(T_1^{-1})) \\ &= e^{i\theta(T_2;\mathscr{A})}\mathcal{A}\varrho(T_1)\mathcal{A}\varrho(\Phi_{\mathscr{A}}(T_1^{-1})) \\ &= \varepsilon_{\mathsf{A}}e^{i\theta(T_1;\mathscr{A})+i\theta(T_2;\mathscr{A})}\end{aligned} \tag{2.42}$$

for all $T_1, T_2 \in \mathrm{P}^*(1,d)$. That is, $T \mapsto e^{i\theta(T;\mathscr{A})}$ is a one-dimensional unitary representation of $\mathrm{P}^*(1,d)$. Every such representation must be trivial (skip ahead to Proposition 3.8 for the proof).

Having now shown that $e^{i\theta(T;\mathscr{A})} = 1$, it suffices to observe that what we have in the $\mathcal{A}$'s, ϱ is a unitary/anti-unitary representation of $E \ltimes \mathrm{P}^*(1,d)$. Explicitly,

$$E \ltimes \mathrm{P}^*(1,d) \ni (\mathcal{A}, \varrho(T)) \mapsto \mathcal{A}\varrho(T) \in \mathrm{UaU}(\mathcal{H}) \tag{2.43}$$

is a unitary/anti-unitary representation lifting the given Wigner representation. □

3. $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$

The trick to connect the Lorentz group with $\mathrm{SL}(2,\mathbb{C})$ is to identify $\mathbb{R}^{1,3}$ with the real vector space $H_2 = \{M \in \mathbb{C}^{2\times 2} : M = M^\dagger\}$ of Hermitian 2-by-2 matrices with complex entries. Let $\Sigma : \mathbb{R}^{1,3} \to H_2$ denote the *Bloch map*,

$$\Sigma(t,x^1,x^2,x^3) = tI_2 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = \begin{pmatrix} t+x^3 & x^1 - ix^2 \\ x^1 + ix^2 & t - x^3 \end{pmatrix}, \tag{2.44}$$

where $\sigma_\bullet$ are the three Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.45}$$

which satisfy $\sigma_j\sigma_k = \delta_{jk}I_2 + i\varepsilon_{jk\ell}\sigma_\ell$, where $\varepsilon_{jk\ell} \in \{-1, 0, +1\}$ is the Levi-Civita symbol. The Bloch map is a linear isomorphism between $\mathbb{R}^{1,3}$ and H_2. Its inverse is

$$\boldsymbol{\Sigma}^{-1} : \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \mapsto \Big(\frac{\alpha+\gamma}{2}, \Re\beta, -\Im\beta, \frac{\alpha-\gamma}{2}\Big). \tag{2.46}$$

Let $S \in \mathbb{C}^{2\times 2}$. Whenever $M \in H_2$, then $SMS^\dagger \in H_2$. Consequently, if $x \in \mathbb{R}^{1,3}$, then $\boldsymbol{\Sigma}^{-1}(S\boldsymbol{\Sigma}(x)S^\dagger) \in \mathbb{R}^{1,3}$ is well-defined. The map

$$x \mapsto \boldsymbol{\Sigma}^{-1}(S\boldsymbol{\Sigma}(x)S^\dagger) \tag{2.47}$$

is linear and so can be represented by a matrix, which we denote $\pi(S) \in \mathbb{R}^{4\times 4}$. Evidently, $\pi : \mathbb{C}^{2\times 2} \to \mathbb{R}^{4\times 4}$ is smooth. (An explicit formula will be below.) We can now state:

PROPOSITION 2.8. $\pi : \mathrm{SL}(2,\mathbb{C}) \twoheadrightarrow \mathrm{SO}(1,3)$ *is a surjective 2-to-1 homomorphism of Lie groups.* ■

We will prove this below.

PROPOSITION 2.9. $\pi(S^\dagger) = \pi(S)^\intercal$. ■

PROOF. Note that $\boldsymbol{\Sigma}(x)^{-1}$ exists if $x^2 \neq 0$, and $\boldsymbol{\Sigma}(x)^{-1} \propto \boldsymbol{\Sigma}(\mathscr{P}x)$. By definition, $\pi(S^{-1}) : x \mapsto \boldsymbol{\Sigma}^{-1}(S^{-1}\boldsymbol{\Sigma}(x)(S^\dagger)^{-1})$, and the right-hand side is, if $x^2 \neq 0$,

$$\boldsymbol{\Sigma}^{-1}((S^\dagger\boldsymbol{\Sigma}(x)^{-1}S)^{-1}) = \mathscr{P}\boldsymbol{\Sigma}^{-1}(S^\dagger\boldsymbol{\Sigma}(\mathscr{P}x)S) = \mathscr{P}\pi(S^\dagger)\mathscr{P}x. \tag{2.48}$$

So, $\pi(S^{-1}) = \mathscr{P}\pi(S^\dagger)\mathscr{P}$. Equivalently, $\pi(S^\dagger) = \mathscr{P}\pi(S^{-1})\mathscr{P}$. The right-hand side is $\mathscr{P}\pi(S)^{-1}\mathscr{P}$ (since π is a homomorphism, which implies $\pi(S^{-1}) = \pi(S)^{-1}$). Now we can use the identity $\Lambda^{-1} = \eta\Lambda^\intercal\eta$ that holds for all Lorentz matrices to get $\pi(S^\dagger) = (-I_4)\pi(S)^\intercal(-I_4) = \pi(S^\intercal)$. □

WARNING: Many facts about the representation theory of $\mathrm{SL}(2,\mathbb{C})$ can be deduced from facts about the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. For example, it is possible to prove Proposition 2.8 by exhibiting an explicit isomorphism

$$\mathfrak{o}(1,3) \cong \mathfrak{sl}(2,\mathbb{C}). \tag{2.49}$$

However, care is required, because the exponential map $\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ is not surjective. For example, $\left(\begin{smallmatrix} -1 & s \\ 0 & -1 \end{smallmatrix}\right)$ is not in the image of the exponential map whenever $s \in \mathbb{C}^\times$. (This does not contradict Problem 1.5.)

3.1. Proof of Proposition 2.8. Observe: if $x \in \mathbb{R}^{1,3}$, then the Minkowski norm $x^2 = -x_0^2 + \|\mathbf{x}\|^2$ is given by $-\det\boldsymbol{\Sigma}(x)$.

LEMMA 2.10. Whenever $S \in \mathrm{SL}(2,\mathbb{C})$, we have $\pi(S) \in \mathrm{SO}(1,3)$. ■

PROOF. For any $x \in \mathbb{R}^{1,3}$, the Minkowski norm of $\pi(S)x = \boldsymbol{\Sigma}^{-1}(S\boldsymbol{\Sigma}(x)S^\dagger)$ is

$$\begin{aligned} \det\boldsymbol{\Sigma}(\boldsymbol{\Sigma}^{-1}(S\boldsymbol{\Sigma}(x)S^\dagger)) &= \det(S\boldsymbol{\Sigma}(x)S^\dagger) \\ &= |\det S|^2\det(\boldsymbol{\Sigma}(x)) = \det\boldsymbol{\Sigma}(x) = -x^2, \end{aligned} \tag{2.50}$$

the Minkowski norm of x. So, $\pi(S)$ preserves the Minkowski norm. This means that $\pi(S) \in \mathrm{O}(1,3)$ is a Lorentz matrix.

Since $\mathrm{SL}(2,\mathbb{C})$ is connected, the image of π must be a connected subset of $\mathrm{O}(1,3)$, and must therefore lie entirely in one of the four connected components of $\mathrm{O}(1,3)$. Because $\pi(I_2) = I_4$, the relevant component is the one containing the identity I_4, i.e. the restricted Lorentz group $\mathrm{SO}(1,d)$. □

LEMMA 2.11. The map $\pi : \mathrm{SL}(2,\mathbb{C}) \ni S \mapsto \pi(S) \in \mathrm{SO}(1,3)$ is a homomorphism. ■

PROOF. Firstly, $\pi(I_2) = I_4$.
If $S, Q \in \mathrm{SL}(2,\mathbb{C})$, then the Lorentz matrices $\pi(SQ)$, $\pi(S)\pi(Q)$ implement the maps

$$\begin{aligned} x &\mapsto \Sigma^{-1}(SQ\Sigma(x)(QS)^\dagger) = \Sigma^{-1}(SQ\Sigma(x)Q^\dagger S^\dagger), \\ x &\mapsto \Sigma^{-1}(S\Sigma(\Sigma^{-1}(Q\Sigma(x)Q^\dagger))S^\dagger) = \Sigma^{-1}(SQ\Sigma(x)Q^\dagger S^\dagger), \end{aligned} \tag{2.51}$$

which agree. So, $\pi(SQ) = \pi(S)\pi(Q)$. □

LEMMA 2.12. Fix $N \in \mathbb{N}^+$. Let $S \in \mathbb{C}^{N\times N}$, and suppose that $SMS^\dagger = M$ for all $M \in H_N$. Then, $S = e^{i\theta} I_N$ for some $\theta \in \mathbb{R}$. ■

PROOF. Try $M = vv^\dagger$ for $v \in \mathbb{C}^N$. Then, $SMS^\dagger = (Sv)(Sv)^\dagger$. This map, whose range is the span of Sv, is equal to M, whose range is the span of v, if and only if

$$Sv = e^{i\theta} v \tag{2.52}$$

for some $\theta \in \mathbb{R}$. So, every nonzero element of $\mathbb{C}^N$ is an eigenvector of S. This implies that S is a scalar multiple of the identity: $S = cI_N$ for some $c \in \mathbb{C}$, which, by eq. (2.52), must be a phase. □

COROLLARY. $\ker \pi = \{-I_2, I_2\}$. ■□

PROOF. Evidently, $\pm I_2 \in \ker \pi$, so the main order of business is the converse.
If $S \in \ker \pi$, then it satisfies the hypotheses of lemma 2.12 with $N = 2$. So $S = e^{i\theta} I$ for some $\theta \in \mathbb{R}$. Then $\det S = e^{2i\theta}$, so S is unimodular only if $S = \pm I_2$. □

LEMMA 2.13. π is onto $\mathrm{SO}(1,3)$. ■

PROOF. This can be shown in several ways. It follows from the fact that $\mathrm{SO}(1,3)$ is connected and $D\pi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{o}(1,3)$ is surjective (which must be true because both Lie algebras have real dimension 6 and $\ker \pi$ is discrete; alternatively, $D\pi$ is computed below). □

This completes the proof of Proposition 2.8.

3.2. $D\pi$**.** The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ can be taken to consist of $X \in \mathbb{C}^{2\times 2}$ such that $e^{sX} \in \mathrm{SL}(2,\mathbb{C})$ for all $s \in \mathbb{R}$. Since

$$e^{(s+\delta s)X} = e^{sX} e^{X\delta s} = e^{sX}(I_2 + X\delta s + O(\delta s^2)), \tag{2.53}$$

$$\begin{aligned} \det e^{(s+\delta s)X} &= (\det e^{sX})(\det(I_2 + X\delta s) + O(\delta s^2)) \\ &= (\det e^{sX})(1 + (\delta s)\operatorname{tr} X + O(\delta s^2)) \end{aligned} \tag{2.54}$$

the derivative

$$\frac{\mathrm{d}}{\mathrm{d}s} \det e^{sX} = (\det e^{sX}) \operatorname{tr} X \tag{2.55}$$

can be computed. So, the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ consists of traceless matrices:

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in \mathbb{C}^{2\times 2} : \operatorname{tr} X = 0\} = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{C} \right\}. \tag{2.56}$$

Over the complex numbers, this is spanned by the three Pauli matrices.
Since $e^{sX} = I_2 + sX + O(s^2)$, we can compute $\pi(e^{sX})$ modulo $O(s^2)$ by throwing out all terms in eq. (2.61) that are quadratic in s. This results in

$$\pi(e^{sX}) = I_4 + s \underbrace{\begin{bmatrix} 0 & \Re(b^* + c) & \Im(b^* + c) & 2\Re a \\ \Re(c^* + b) & 0 & 2\Im a & \Re(c^* - b) \\ -\Im(c^* + b) & -2\Im a & 0 & \Im(b - c^*) \\ 2\Re a & \Re(b^* - c) & \Im(b^* - c) & 0 \end{bmatrix}}_{(D\pi)X} + O(s^2). \tag{2.57}$$

The matrix in the previous line is supposed to lie in the Lie algebra

$$\mathfrak{o}(1,3) = \left\{ \begin{bmatrix} 0 & \mathbf{a}^\intercal \\ \mathbf{a} & \mathbf{X} \end{bmatrix} : \mathbf{a} \in \mathbb{R}^3, \mathbf{X} \in \mathbb{R}^{3\times 3}, \mathbf{X} = -\mathbf{X}^\intercal \right\} \tag{2.58}$$

of $\mathrm{O}(1,3)$, and indeed, it does, with

$$\mathbf{a} = \begin{bmatrix} \Re(b^* + c) \\ \Im(b^* + c) \\ 2\Re a \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} 0 & 2\Im a & \Re(c^* - b) \\ -2\Im a & 0 & \Im(b - c^*) \\ -\Re(c^* - b) & -\Im(b - c^*) & 0 \end{bmatrix}. \tag{2.59}$$

In summary:

Proposition 2.14. *The differential* $D\pi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{o}(1,3)$ *is given by*

$$X = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mapsto \begin{bmatrix} 0 & \mathbf{a}^\intercal \\ \mathbf{a} & \mathbf{X} \end{bmatrix}, \tag{2.60}$$

where $\mathbf{a}$, $\mathbf{X}$ *are as above. So:*

- *The generators of Lorentz boosts, in* $\mathfrak{sl}(2,\mathbb{C})$*, are those* X *with* $a \in \mathbb{R}$ *and* $b = c^*$*, i.e. satisfying* $X \in \mathrm{span}_{\mathbb{R}}\{\sigma_1, \sigma_2, \sigma_3\}$*.*
- *The generators of rotations, in* $\mathfrak{sl}(2,\mathbb{C})$*, are those* X *with* $a \in i\mathbb{R}$ *and* $b = -c^*$*, i.e. satisfying* $X \in \mathrm{span}_{\mathbb{R}}\{i\sigma_1, i\sigma_2, i\sigma_3\}$*.*

■

Corollary. The pre-image $\pi^{-1}(\mathrm{SO}(3))$ of the subgroup $\mathrm{SO}(3) \subseteq \mathrm{SO}(1,3)$ of rotations is the subgroup $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$. ■□

Proof. We know that $\pi^{-1}(\mathrm{SO}(3))$ is a Lie subgroup of $\mathrm{SL}(2,\mathbb{C})$. By the previous proposition, its Lie algebra is $\mathrm{span}_{\mathbb{R}}\{i\sigma_1, i\sigma_2, i\sigma_3\}$, which exponentiates to $\mathrm{SU}(2)$. □

3.3. An explicit formula (⋆).

Proposition 2.15. *For* $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$*, the matrix* $\pi(S)$ *is*

$$\begin{bmatrix} 2^{-1}(|a|^2 + |b|^2 + |c|^2 + |d|^2) & \Re(ab^* + cd^*) & \Im(ab^* + cd^*) & 2^{-1}(|a|^2 - |b|^2 + |c|^2 - |d|^2) \\ \Re(ac^* + bd^*) & \Re(ad^* + bc^*) & \Im(ad^* - bc^*) & \Re(ac^* - bd^*) \\ -\Im(ac^* + bd^*) & -\Im(ad^* + bc^*) & \Re(ad^* - bc^*) & \Im(bd^* - ac^*) \\ 2^{-1}(|a|^2 + |b|^2 - |c|^2 - |d|^2) & \Re(ab^* - cd^*) & \Im(ab^* - cd^*) & 2^{-1}(|a|^2 - |b|^2 - |c|^2 + |d|^2) \end{bmatrix}. \tag{2.61}$$

■

Proof. Concretely,

$$S\Sigma(x)S^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & t - x^3 \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \tag{2.62}$$

for

$$\begin{aligned} \alpha &= |a|^2(t + x^3) + |b|^2(t - x^3) + 2\Re[(x^1 - ix^2)ab^*], \\ \beta &= (t + x^3)ac^* + (t - x^3)bd^* + (x^1 - ix^2)ad^* + (x^1 + ix^2)bc^*, \\ \gamma &= |c|^2(t + x^3) + |d|^2(t - x^3) + 2\Re[(x^1 - ix^2)cd^*]. \end{aligned} \tag{2.63}$$

Consequently, $\Sigma^{-1}(S\Sigma(x)S^\dagger) = (t', (x^1)', (x^2)', (x^3)')$ for

$$\begin{aligned} t' &= (|a|^2+|b|^2+|c|^2+|d|^2)\frac{t}{2} + (|a|^2-|b|^2+|c|^2-|d|^2)\frac{x^3}{2} + \Re[(x^1-ix^2)(ab^*+cd^*)], \\ (x^1)'-i(x^2)' &= (t+x^3)ac^* + (t-x^3)bd^* + (x^1-ix^2)ad^* + (x^1+ix^2)bc^*, \\ (x^3)' &= (|a|^2+|b|^2-|c|^2-|d|^2)\frac{t}{2} + (|a|^2-|b|^2-|c|^2+|d|^2)\frac{x^3}{2} + \Re[(x^1-ix^2)(ab^*-cd^*)]. \end{aligned} \tag{2.64}$$

The matrix implementing the transformation $x \mapsto x'$ is eq. (2.61). □

A. Calculation of $\mathrm{H}^2(\mathfrak{p})$

One of the hypotheses of Bargmann's theorem is that the second cohomology group $\mathrm{H}^2(\mathfrak{p}) = \mathrm{H}^2(\mathfrak{p};\mathbb{R})$ of the Lie algebra $\mathfrak{p} = \mathfrak{p}(1,d)$ of $\mathrm{P}(1,d)$ is trivial. It will be, if $d \geq 2$. Carrying out this computation is the purpose of this appendix.

Because $\mathfrak{p}$ is finite-dimensional, $(\wedge^2\mathfrak{p})^*$ is finite-dimensional, so the subspaces $\mathrm{B}^2(\mathfrak{p}), \mathrm{Z}^2(\mathfrak{p})$ of "coboundaries" and "cocycles" are both finite-dimensional. The latter is defined by finitely many linear constraints, and the former is defined as the image of $\mathfrak{p}^*$ under a particular map. So, computing

$$\mathrm{H}^2(\mathfrak{p}) = \mathrm{Z}^2(\mathfrak{p})/\,\mathrm{B}^2(\mathfrak{p}) \tag{2.65}$$

is a matter of finite-dimensional Lie algebra, for each individual d, and could be done with a computer.

Rather than phrasing the computation this way, we revert to the perspective of $\mathrm{H}^2(\mathfrak{p})$ as classifying central extensions (by $\mathbb{R}$). Proving that it is trivial amounts to proving that every central extension is equivalent to the trivial one. Proving that it is *non*-trivial amounts to constructing a central extension that is provably inequivalent to the trivial one. Either way, the computations involved are equivalent to computations done using the cohomological language, the difference being a matter of presentation.

Remark 2.16 (Complexification). Physicists usually prefer to work with the complexification $\mathfrak{p}_\mathbb{C} = \mathbb{C}\otimes\mathfrak{p}$ of the Lie algebra rather than $\mathfrak{p}$ itself. This does not matter because

$$\mathrm{H}^2(\mathfrak{p}_\mathbb{C};\mathbb{C}) \cong \mathbb{C}\otimes_\mathbb{R} \mathrm{H}^2(\mathfrak{p};\mathbb{R}) \tag{2.66}$$

[Exercise 2.4] naturally. So, the triviality of H^2 does not depend on the choice of base field. Below, we work with $\mathfrak{p}_\mathbb{C}$ in place of $\mathfrak{p}$, so as to match physicists' notation (specifically that in [Wei05, §2.7]). Thus, *all Lie algebras in this appendix will be complexified*, and we will not write the '$\mathbb{C}$' subscript. ■

Example 2.17 ($d=1$). If $d=1$, then, the Poincaré algebra $\mathfrak{p}=\mathfrak{p}(1,1)$ is three-dimensional. It is spanned over $\mathbb{C}$ by three generators, $\mathsf{P}^0, \mathsf{P}^1, \mathsf{K}$, the two generators of translations and the generator of boosts, respectively, satisfying

$$i[\mathsf{P}^0,\mathsf{K}] = \mathsf{P}^1, \quad i[\mathsf{P}^1,\mathsf{K}] = \mathsf{P}^0 \tag{2.67}$$

and $[\mathsf{P}^0,\mathsf{P}^1]=0$. Let us try to centrally extend $\mathfrak{p}$. A central extension of $\mathfrak{p}$ consists of a Lie algebra $\mathfrak{e}$ with one more generator than $\mathfrak{p}$ — call it Z (physicists call this a "central charge") — in the center, whose Lie bracket can be written

$$\begin{aligned} &i[\mathsf{P}^0,\mathsf{K}] = \mathsf{P}^1 + D\mathsf{Z}, \quad i[\mathsf{P}^1,\mathsf{K}] = \mathsf{P}^0 + E\mathsf{Z}, \\ &i[\mathsf{P}^0,\mathsf{P}^1] = C\mathsf{Z} \end{aligned} \tag{2.68}$$

for some $C,D,E\in\mathbb{C}$. Usually, the new coefficients would be constrained by the Jacobi identity, but the Jacobi identity is automatically satisfied in this case. So, eq. (2.68) defines a valid Lie algebra.

The extension $\mathfrak{e}$ is trivial if and only if it is possible to replace each of $\mathsf{P}^0, \mathsf{P}^1, \mathsf{K}$ with a linear combination with Z so as to eliminate the C, D, E. This is easy to do for D, E. Indeed, letting

$$\tilde{\mathsf{P}}^1 = \mathsf{P}^1 + D\mathsf{Z}, \qquad \tilde{\mathsf{P}}^0 = \mathsf{P}^0 + E\mathsf{Z}, \tag{2.69}$$

we have

$$\begin{aligned} i[\tilde{\mathsf{P}}^0, \mathsf{K}] &= \tilde{\mathsf{P}}^1, \quad i[\tilde{\mathsf{P}}^1, \mathsf{K}] = \tilde{\mathsf{P}}^0, \\ i[\tilde{\mathsf{P}}^0, \tilde{\mathsf{P}}^1] &= C\mathsf{Z}. \end{aligned} \tag{2.70}$$

Thus, we have eliminated D, E.

But, C cannot be so eliminated — replacing the generators P^j with some linear combinations with Z, we do not change $[\mathsf{P}^0, \mathsf{P}^1]$. So,

$$\mathrm{H}^2(\mathfrak{p}(1,1)) \cong \mathbb{C} \tag{2.71}$$

is one-dimensional. ■

The fact that $\mathrm{H}^2(\mathfrak{p}(1,1))$ is non-trivial should make us appreciate:

Lemma 2.18. For each $d \geq 2$, $\mathrm{H}^2(\mathfrak{p}(1,d)) \cong \{0\}$. ■

Proof. Let $\mathbb{R} \hookrightarrow \mathfrak{e} \twoheadrightarrow \mathfrak{p}$ denote a central extension of $\mathfrak{p}$. Let Z denote a central charge, and let $\mathsf{J}^{\mu\nu}, \mathsf{P}^\mu$ denote arbitrarily chosen elements of the preimages of the identically named generators of $\mathfrak{p}$. The Lie bracket of $\mathfrak{e}$ then takes the form

$$\begin{aligned} i[\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}] &= \eta^{\mu\lambda}\mathsf{J}^{\nu\sigma} - \eta^{\mu\sigma}\mathsf{J}^{\nu\lambda} + \eta^{\nu\sigma}\mathsf{J}^{\mu\lambda} - \eta^{\nu\lambda}\mathsf{J}^{\mu\sigma} + C^{\mu\nu,\sigma\lambda}\mathsf{Z} \\ i[\mathsf{P}^\mu, \mathsf{J}^{\nu\sigma}] &= \eta^{\mu\nu}\mathsf{P}^\sigma - \eta^{\mu\sigma}\mathsf{P}^\nu + C^{\mu,\nu\sigma}\mathsf{Z} \\ i[\mathsf{P}^\mu, \mathsf{P}^\nu] &= C^{\mu,\nu}\mathsf{Z} \end{aligned} \tag{2.72}$$

for some $C^{\mu\nu,\sigma\lambda}, C^{\mu,\nu\sigma}, C^{\mu,\nu} \in \mathbb{C}$. The game is to show that there exist other choices of generators,

$$\tilde{\mathsf{J}}^{\mu\nu} = \mathsf{J}^{\mu\nu} + D^{\mu\nu}\mathsf{Z}, \tilde{\mathsf{P}}^\mu = \mathsf{P}^\mu + D^\mu\mathsf{Z} \in \mathfrak{e}, \quad D^{\mu\nu} = -D^{\nu\mu}, D^\mu \in \mathbb{C}, \tag{2.73}$$

each differing from $\mathsf{J}^{\mu\nu}, \mathsf{P}^\mu$ by a multiple of the central charge Z, such that, when rewritten in terms of these new generators, eq. (2.72) becomes the usual Poincaré algebra,

$$i[\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{J}}^{\sigma\lambda}] = \eta^{\mu\lambda}\tilde{\mathsf{J}}^{\nu\sigma} - \eta^{\mu\sigma}\tilde{\mathsf{J}}^{\nu\lambda} + \eta^{\nu\sigma}\tilde{\mathsf{J}}^{\mu\lambda} - \eta^{\nu\lambda}\tilde{\mathsf{J}}^{\mu\sigma} \tag{2.74}$$

$$i[\tilde{\mathsf{P}}^\mu, \tilde{\mathsf{J}}^{\nu\sigma}] = \eta^{\mu\nu}\tilde{\mathsf{P}}^\sigma - \eta^{\mu\sigma}\tilde{\mathsf{P}}^\nu \tag{2.75}$$

$$i[\tilde{\mathsf{P}}^\mu, \tilde{\mathsf{P}}^\nu] = 0. \tag{2.76}$$

But note that the commutators above do not change when we replace $\mathsf{J}^{\mu\nu}, \mathsf{P}^\mu$ with $\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{P}}^\mu$:

$$[\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}] = [\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{J}}^{\sigma\lambda}], \quad [\mathsf{P}^\mu, \mathsf{J}^{\nu\sigma}] = [\tilde{\mathsf{P}}^\mu, \tilde{\mathsf{J}}^{\nu\sigma}], \quad [\mathsf{P}^\mu, \mathsf{P}^\nu] = [\tilde{\mathsf{P}}^\mu, \tilde{\mathsf{P}}^\nu]. \tag{2.77}$$

So, what we want to do is choose the coefficients $D^{\mu\nu}, D^\mu$ so that, upon rewriting the right-hand side of eq. (2.72) in terms of $\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{P}}^\mu$, the effect is to absorb the terms involving the central charge Z.

- ($C^{\mu,\nu} = 0$.) The most dangerous terms in eq. (2.72) are the $C^{\mu,\nu}$, because there are no terms on the right-hand side of $i[\mathsf{P}^\mu, \mathsf{P}^\nu] = C^{\mu,\nu}\mathsf{Z}$ besides the central charge. If any of these were nonzero, we would not be able to eliminate them by redefining $\mathsf{J}^{\mu\nu}, \mathsf{P}^\mu$.

  Fortunately, $C^{\mu,\nu}$ must be zero, so eq. (2.76) holds automatically. In order to prove this, consider the Jacobi identity

$$[\mathsf{J}^{\mu\nu}, [\mathsf{P}^\sigma, \mathsf{P}^\lambda]] - [\mathsf{P}^\lambda, [\mathsf{P}^\sigma, \mathsf{J}^{\mu\nu}]] + [\mathsf{P}^\sigma, [\mathsf{P}^\lambda, \mathsf{J}^{\mu\nu}]] = 0. \tag{2.78}$$

  The terms here, as computed using eq. (2.72), are $[\mathsf{J}^{\mu\nu}, [\mathsf{P}^\sigma, \mathsf{P}^\lambda]] = 0$ and

$$\begin{aligned} [\mathsf{P}^\lambda, [\mathsf{P}^\sigma, \mathsf{J}^{\mu\nu}]] &= i[\mathsf{P}^\lambda, \eta^{\sigma\nu}\mathsf{P}^\mu - \eta^{\sigma\mu}\mathsf{P}^\nu] = (C^{\lambda,\mu}\eta^{\sigma\nu} - C^{\lambda,\nu}\eta^{\sigma\mu})\mathsf{Z}, \\ [\mathsf{P}^\sigma, [\mathsf{P}^\lambda, \mathsf{J}^{\mu\nu}]] &= i[\mathsf{P}^\sigma, \eta^{\lambda\nu}\mathsf{P}^\mu - \eta^{\lambda\mu}\mathsf{P}^\nu] = (C^{\sigma,\mu}\eta^{\lambda\nu} - C^{\sigma,\nu}\eta^{\lambda\mu})\mathsf{Z}. \end{aligned} \tag{2.79}$$

So, the Jacobi identity eq. (2.78) says

$$C^{\lambda,\mu}\eta^{\sigma\nu} - C^{\lambda,\nu}\eta^{\sigma\mu} = C^{\sigma,\mu}\eta^{\lambda\nu} - C^{\sigma,\nu}\eta^{\lambda\mu}. \tag{2.80}$$

Contracting with $\eta_{\mu\sigma}$ yields $(d-1)C^{\lambda,\nu} = 0$, so $C^{\lambda,\nu} = 0$. (This is the only place in the argument where we use that $d \geq 2$.)

- (**Eliminating** $C^{\mu,\nu\sigma}$.) Next, we check that, by defining $\tilde{\mathsf{P}}^\mu = \mathsf{P}^\mu + D^\mu\mathsf{Z}$ for appropriate D^μ, we can arrange eq. (2.75). However, we only have $1+d$ different D^μ's, and many more $C^{\mu,\nu\sigma}$'s, so it had better be the case that the $C^{\mu,\nu\sigma}$'s are severely restricted by the Jacobi identity. We win if (and only if, as reversing the reasoning shows) the Jacobi identity implies that $C^{\mu,\nu\sigma}$ has the form

$$C^{\mu,\nu\sigma} = \eta^{\mu\nu}\xi^\sigma - \eta^{\mu\sigma}\xi^\nu \tag{2.81}$$

for some $\xi^\bullet \in \mathbb{C}$. Indeed, we can then define $D^\mu = \xi^\mu$, and this has the desired effect:

$$\begin{aligned}\eta^{\mu\nu}\mathsf{P}^\sigma - \eta^{\mu\sigma}\mathsf{P}^\nu + C^{\mu,\nu\sigma}\mathsf{Z} &= \eta^{\mu\nu}\mathsf{P}^\sigma - \eta^{\mu\sigma}\mathsf{P}^\nu + (\eta^{\mu\nu}\xi^\sigma - \eta^{\mu\sigma}\xi^\nu)\mathsf{Z} \\ &= \eta^{\mu\nu}\tilde{\mathsf{P}}^\sigma - \eta^{\mu\sigma}\tilde{\mathsf{P}}^\nu.\end{aligned} \tag{2.82}$$

To prove eq. (2.81), consider the Jacobi identity

$$[\mathsf{J}^{\mu\nu}, [\mathsf{P}^\sigma, \mathsf{J}^{\lambda\rho}]] + [\mathsf{P}^\sigma, [\mathsf{J}^{\lambda\rho}, \mathsf{J}^{\mu\nu}]] - [\mathsf{J}^{\lambda\rho}, [\mathsf{P}^\sigma, \mathsf{J}^{\mu\nu}]] = 0. \tag{2.83}$$

The terms here are given by

$$\begin{aligned}[\mathsf{J}^{\mu\nu}, [\mathsf{P}^\sigma, \mathsf{J}^{\lambda\rho}]] &= (\eta^{\sigma\rho}\eta^{\lambda\nu} - \eta^{\sigma\lambda}\eta^{\rho\nu})\mathsf{P}^\mu + (\eta^{\sigma\lambda}\eta^{\rho\mu} - \eta^{\sigma\rho}\eta^{\lambda\mu})\mathsf{P}^\nu \\ &\quad + (\eta^{\sigma\lambda}C^{\rho,\mu\nu} - \eta^{\sigma\rho}C^{\lambda,\mu\nu})\mathsf{Z}, \\ [\mathsf{J}^{\lambda\rho}, [\mathsf{P}^\sigma, \mathsf{J}^{\mu\nu}]] &= (\eta^{\sigma\nu}\eta^{\mu\rho} - \eta^{\sigma\mu}\eta^{\nu\rho})\mathsf{P}^\lambda + (\eta^{\sigma\mu}\eta^{\nu\lambda} - \eta^{\sigma\nu}\eta^{\mu\lambda})\mathsf{P}^\rho \\ &\quad + (\eta^{\sigma\mu}C^{\nu,\lambda\rho} - \eta^{\sigma\nu}C^{\mu,\lambda\rho})\mathsf{Z}, \\ [\mathsf{P}^\sigma, [\mathsf{J}^{\lambda\rho}, \mathsf{J}^{\mu\nu}]] &= (\eta^{\rho\nu}\eta^{\sigma\lambda} - \eta^{\lambda\nu}\eta^{\sigma\rho})\mathsf{P}^\mu + (\eta^{\lambda\mu}\eta^{\sigma\rho} - \eta^{\rho\mu}\eta^{\sigma\lambda})\mathsf{P}^\nu \\ &\quad + (\eta^{\lambda\nu}\eta^{\sigma\mu} - \eta^{\lambda\mu}\eta^{\sigma\nu})\mathsf{P}^\rho + (\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\mu})\mathsf{P}^\lambda \\ &\quad + (\eta^{\lambda\mu}C^{\sigma,\rho\nu} - \eta^{\lambda\nu}C^{\sigma,\rho\mu} + \eta^{\rho\nu}C^{\sigma,\lambda\mu} - \eta^{\rho\mu}C^{\sigma,\lambda\nu})\mathsf{Z},\end{aligned} \tag{2.84}$$

so the Jacobi identity eq. (2.83) reads

$$\begin{aligned}&\eta^{\sigma\lambda}C^{\rho,\mu\nu} - \eta^{\sigma\rho}C^{\lambda,\mu\nu} - \eta^{\sigma\mu}C^{\nu,\lambda\rho} + \eta^{\sigma\nu}C^{\mu,\lambda\rho} \\ &\quad + \eta^{\lambda\mu}C^{\sigma,\rho\nu} - \eta^{\lambda\nu}C^{\sigma,\rho\mu} + \eta^{\rho\nu}C^{\sigma,\lambda\mu} - \eta^{\rho\mu}C^{\sigma,\lambda\nu} = 0.\end{aligned} \tag{2.85}$$

Contracting with $\eta_{\mu\sigma}$ yields $dC^{\nu,\lambda\rho} - \eta^{\lambda\nu}C^{\sigma,\rho}{}_\sigma + \eta^{\rho\nu}C^{\sigma,\lambda}{}_\sigma = 0$. (Here we are using standard conventions regarding raised and lowered indices. A review is in §5.A.) Rearranging and renaming dummy variables ($\nu \rightsquigarrow \mu$, $\lambda \rightsquigarrow \nu$, $\sigma \rightsquigarrow \lambda$, $\rho \rightsquigarrow \sigma$), this last equation becomes

$$C^{\mu,\nu\sigma} = d^{-1}(\eta^{\mu\nu}C^{\lambda,\sigma}{}_\lambda - \eta^{\mu\sigma}C^{\lambda,\nu}{}_\lambda) \tag{2.86}$$

This says that eq. (2.81) holds for $\xi^\sigma = d^{-1}C^{\lambda,\sigma}{}_\lambda$.

- (**Eliminating** $C^{\mu\nu,\sigma\lambda}$.) This is equivalent to showing that the cohomology $\mathrm{H}^2(\mathfrak{o}(1+d))$ of the Lorentz Lie algebra is trivial. We have separated this as its own lemma, Lemma 2.19.

□

LEMMA 2.19. For any $d \geq 1$, $\mathrm{H}^2(\mathfrak{o}(1+d)) = \{0\}$. ■

REMARK 2.20. *Whitehead's second lemma* [Jac79] says that $\mathrm{H}^2(\mathfrak{g})$ is trivial whenever $\mathfrak{g}$ is a semisimple Lie algebra. Since the Lorentz Lie algebra $\mathfrak{o}(D)$ is semisimple for $D \geq 3$ (in fact simple, with the one exception $D = 4$), when $d \geq 2$, Lemma 2.19 is a special case of this. The Lie algebra $\mathfrak{o}(2)$ relevant to the $d = 1$ case is abelian and therefore not considered semisimple, but this case is trivial regardless.

Note that $\mathfrak{p}(1,d)$ is *not* semisimple, because of the abelian subalgebra of translations. So, Whitehead's lemma does not apply, and indeed we saw that $\mathfrak{p}(1,1)$ has nontrivial H^2. ■

Proof of Lemma 2.19. The only two-dimensional Lie algebra with nontrivial center is the abelian one, hence a trivial central extension of $\mathfrak{o}(2) \cong \mathbb{R}$. This gives the $d=1$ case of the lemma. For the rest of the proof, suppose $d \geq 2$.

Let $\mathbb{R} \hookrightarrow \mathfrak{e} \twoheadrightarrow \mathfrak{o}(1+d)$ denote a central extension of $\mathfrak{o}(1+d)$, $\mathsf{Z} \in \mathfrak{e}$ denote a central charge, and $\mathsf{J}^{\mu\nu}$ denote arbitrarily chosen elements of the preimages of the identically named generators of $\mathfrak{o}(1+d)$. The Lie bracket of $\mathfrak{e}$ then takes the form

$$i[\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}] = \eta^{\mu\lambda}\mathsf{J}^{\nu\sigma} - \eta^{\mu\sigma}\mathsf{J}^{\nu\lambda} + \eta^{\nu\sigma}\mathsf{J}^{\mu\lambda} - \eta^{\nu\lambda}\mathsf{J}^{\mu\sigma} + C^{\mu\nu,\sigma\lambda}\mathsf{Z}, \tag{2.87}$$

for some structure constants $C^{\mu\nu,\sigma\lambda}$. The game is to show that we can choose $D^{\mu\nu} = D^{-\nu\mu} \in \mathbb{C}$ such that, if we define $\tilde{\mathsf{J}}^{\mu\nu} = \mathsf{J}^{\mu\nu} + D^{\mu\nu}\mathsf{Z}$, then eq. (2.87) can be written

$$i[\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{J}}^{\sigma\lambda}] = \eta^{\mu\lambda}\tilde{\mathsf{J}}^{\nu\sigma} - \eta^{\mu\sigma}\tilde{\mathsf{J}}^{\nu\lambda} + \eta^{\nu\sigma}\tilde{\mathsf{J}}^{\mu\lambda} - \eta^{\nu\lambda}\tilde{\mathsf{J}}^{\mu\sigma}, \tag{2.88}$$

which says that $\tilde{\mathsf{J}}^{\mu\nu}$ satisfy the usual Lorentz algebra. However, we only have $d(d-1)/2$ different $D^{\mu\nu}$'s, and many more $C^{\mu\nu,\sigma\lambda}$'s that we want to eliminate, so it had better be the case that the latter are severely restricted. By $\mathsf{J}^{\mu\nu} = -\mathsf{J}^{\nu\mu}$ and the anti-symmetry of $\mathfrak{e}$'s Lie bracket, $C^{\mu\nu,\sigma\lambda}$ switches sign under interchanging the two Lorentz indices on either side of the comma, or under interchanging the two sides of the comma.

Owing to the centrality of Z, we have $[\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}] = [\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{J}}^{\sigma\lambda}]$, so what we want to do is choose $D^{\mu\nu}$ such that the right-hand sides of eq. (2.87), eq. (2.88) agree. This means

$$C^{\mu\nu,\sigma\lambda} = \eta^{\mu\lambda}D^{\nu\sigma} - \eta^{\mu\sigma}D^{\nu\lambda} + \eta^{\nu\sigma}D^{\mu\lambda} - \eta^{\nu\lambda}D^{\mu\sigma}. \tag{2.89}$$

So, we win if the $C^{\mu\nu,\sigma\lambda}$ can be shown to have this form.

The restriction comes from the Jacobi identity

$$[\mathsf{J}^{\mu\nu}, [\mathsf{J}^{\sigma\lambda}, \mathsf{J}^{\rho\tau}]] + [\mathsf{J}^{\sigma\lambda}, [\mathsf{J}^{\rho\tau}, \mathsf{J}^{\mu\nu}]] + [\mathsf{J}^{\rho\tau}, [\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}]] = 0. \tag{2.90}$$

Note that this is $\mathfrak{e}$'s Lie bracket. Let $\mathscr{J} = \operatorname{span}_{\mathbb{C}}\{\mathsf{J}^{\mu\nu} : 0 \leq \mu < \nu \leq d\} \subseteq \mathfrak{e}$ denote the span of the $\mathsf{J}^{\mu\nu}$'s. Define $J_{\mathrm{I}}, J_{\mathrm{II}}, J_{\mathrm{III}} \in \mathscr{J}$ and $\xi_{\mathrm{I}}, \xi_{\mathrm{II}}, \xi_{\mathrm{III}} \in \mathbb{C}$ by

$$\begin{aligned}
[\mathsf{J}^{\mu\nu}, [\mathsf{J}^{\sigma\lambda}, \mathsf{J}^{\rho\tau}]] &= J_{\mathrm{I}} + \xi_{\mathrm{I}}\mathsf{Z} \\
[\mathsf{J}^{\sigma\lambda}, [\mathsf{J}^{\rho\tau}, \mathsf{J}^{\mu\nu}]] &= J_{\mathrm{II}} + \xi_{\mathrm{II}}\mathsf{Z} \\
[\mathsf{J}^{\rho\tau}, [\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}]] &= J_{\mathrm{III}} + \xi_{\mathrm{III}}\mathsf{Z}.
\end{aligned} \tag{2.91}$$

The Jacobi identity for $\mathfrak{so}(1+d)$ guarantees that $J_{\mathrm{I}} + J_{\mathrm{II}} + J_{\mathrm{III}} = 0$, so eq. (2.90) (which was the Jacobi identity for $\mathfrak{e}$) says $\xi_{\mathrm{I}} + \xi_{\mathrm{II}} + \xi_{\mathrm{III}} = 0$. Let us compute what these ξ's are:

$$\begin{aligned}
[\mathsf{J}^{\mu\nu}, [\mathsf{J}^{\sigma\lambda}, \mathsf{J}^{\rho\tau}]] &= -i[\mathsf{J}^{\mu\nu}, \eta^{\rho\lambda}\mathsf{J}^{\tau\sigma} - \eta^{\rho\sigma}\mathsf{J}^{\tau\lambda} + \eta^{\tau\sigma}\mathsf{J}^{\rho\lambda} - \eta^{\tau\lambda}\mathsf{J}^{\rho\sigma}] \\
&= -(\eta^{\rho\lambda}C^{\mu\nu,\tau\sigma} - \eta^{\rho\sigma}C^{\mu\nu,\tau\lambda} + \eta^{\tau\sigma}C^{\mu\nu,\rho\lambda} - \eta^{\tau\lambda}C^{\mu\nu,\rho\sigma})\mathsf{Z} \bmod \mathscr{J}, \\
[\mathsf{J}^{\sigma\lambda}, [\mathsf{J}^{\rho\tau}, \mathsf{J}^{\mu\nu}]] &= -i[\mathsf{J}^{\sigma\lambda}, \eta^{\mu\tau}\mathsf{J}^{\nu\rho} - \eta^{\mu\rho}\mathsf{J}^{\nu\tau} + \eta^{\nu\rho}\mathsf{J}^{\mu\tau} - \eta^{\nu\tau}\mathsf{J}^{\mu\rho}] \\
&= -(\eta^{\mu\tau}C^{\sigma\lambda,\nu\rho} - \eta^{\mu\rho}C^{\sigma\lambda,\nu\tau} + \eta^{\nu\rho}C^{\sigma\lambda,\mu\tau} - \eta^{\nu\tau}C^{\sigma\lambda,\mu\rho})\mathsf{Z} \bmod \mathscr{J}, \\
[\mathsf{J}^{\rho\tau}, [\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}]] &= -i[\mathsf{J}^{\rho\tau}, \eta^{\sigma\nu}\mathsf{J}^{\lambda\mu} - \eta^{\sigma\mu}\mathsf{J}^{\lambda\nu} + \eta^{\lambda\mu}\mathsf{J}^{\sigma\nu} - \eta^{\lambda\nu}\mathsf{J}^{\sigma\mu}] \\
&= -(\eta^{\sigma\nu}C^{\rho\tau,\lambda\mu} - \eta^{\sigma\mu}C^{\rho\tau,\lambda\nu} + \eta^{\lambda\mu}C^{\rho\tau,\sigma\nu} - \eta^{\lambda\nu}C^{\rho\tau,\sigma\mu})\mathsf{Z} \bmod \mathscr{J}.
\end{aligned} \tag{2.92}$$

That is,

$$\begin{aligned}
-\xi_{\mathrm{I}} &= \eta^{\rho\lambda}C^{\mu\nu,\tau\sigma} - \eta^{\rho\sigma}C^{\mu\nu,\tau\lambda} + \eta^{\tau\sigma}C^{\mu\nu,\rho\lambda} - \eta^{\tau\lambda}C^{\mu\nu,\rho\sigma}, \\
-\xi_{\mathrm{II}} &= \eta^{\mu\tau}C^{\sigma\lambda,\nu\rho} - \eta^{\mu\rho}C^{\sigma\lambda,\nu\tau} + \eta^{\nu\rho}C^{\sigma\lambda,\mu\tau} - \eta^{\nu\tau}C^{\sigma\lambda,\mu\rho}, \\
-\xi_{\mathrm{III}} &= \eta^{\sigma\nu}C^{\rho\tau,\lambda\mu} - \eta^{\sigma\mu}C^{\rho\tau,\lambda\nu} + \eta^{\lambda\mu}C^{\rho\tau,\sigma\nu} - \eta^{\lambda\nu}C^{\rho\tau,\sigma\mu}.
\end{aligned} \tag{2.93}$$

So, the Jacobi identity $\xi_{\mathrm{I}} + \xi_{\mathrm{II}} + \xi_{\mathrm{III}} = 0$ says

$$\eta^{\rho\lambda}C^{\mu\nu,\tau\sigma} - \eta^{\rho\sigma}C^{\mu\nu,\tau\lambda} + \eta^{\tau\sigma}C^{\mu\nu,\rho\lambda} - \eta^{\tau\lambda}C^{\mu\nu,\rho\sigma} + \eta^{\mu\tau}C^{\sigma\lambda,\nu\rho} - \eta^{\mu\rho}C^{\sigma\lambda,\nu\tau} + \eta^{\nu\rho}C^{\sigma\lambda,\mu\tau} - \eta^{\nu\tau}C^{\sigma\lambda,\mu\rho} + \eta^{\sigma\nu}C^{\rho\tau,\lambda\mu} - \eta^{\sigma\mu}C^{\rho\tau,\lambda\nu} + \eta^{\lambda\mu}C^{\rho\tau,\sigma\nu} - \eta^{\lambda\nu}C^{\rho\tau,\sigma\mu} = 0. \quad (2.94)$$

Contracting with $\eta_{\mu\sigma}$ gives

$$-(d-1)C^{\rho\tau,\lambda\nu} = \eta^{\tau\lambda}C_{\sigma}{}^{\nu,\rho\sigma} + \eta^{\nu\tau}C_{\sigma}{}^{\lambda,\sigma\rho} - \eta^{\rho\lambda}C_{\sigma}{}^{\nu,\tau\sigma} - \eta^{\nu\rho}C_{\sigma}{}^{\lambda,\sigma\tau}, \quad (2.95)$$

having used the various anti-symmetries of $C^{\bullet}$ under interchanging various Lorentz indices. Renaming dummy variables ($\rho \rightsquigarrow \mu$, $\tau \rightsquigarrow \nu$, $\lambda \rightsquigarrow \sigma$, $\nu \rightsquigarrow \lambda$, $\sigma \rightsquigarrow \rho$) and rearranging, we end up with

$$(d-1)C^{\mu\nu,\sigma\lambda} = \eta^{\mu\lambda}C_{\rho}{}^{\sigma,\rho\nu} - \eta^{\mu\sigma}C_{\rho}{}^{\lambda,\rho\nu} + \eta^{\nu\sigma}C_{\rho}{}^{\lambda,\rho\mu} - \eta^{\nu\lambda}C_{\rho}{}^{\sigma,\rho\mu}. \quad (2.96)$$

So, eq. (2.89) holds with $D^{\nu\sigma} = (d-1)^{-1}C_{\rho}{}^{\sigma,\rho\nu} = -(d-1)^{-1}C_{\rho}{}^{\nu,\rho\sigma}$. □

Exercises and problems

EXERCISE 2.1: Show that the only one-dimensional continuous representation of SU(2) is the trivial one.

Hint: see the proof of Proposition 3.8.

EXERCISE 2.2: Show that $\mathrm{SO}(1,3) \cong \mathrm{PSL}(2,\mathbb{C})$.

EXERCISE 2.3: Let $\mathfrak{t} = \mathbb{R}^2$ denote the abelian two-dimensional Lie algebra. Show that $\mathrm{H}^2(\mathfrak{t};\mathbb{R}) \cong \mathbb{R}$.

EXERCISE 2.4: Prove eq. (2.66) ($\mathrm{H}^2(\mathfrak{p}_{\mathbb{C}};\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}^2(\mathfrak{p};\mathbb{R})$).

PROBLEM 2.1: Prove the claim in Remark 2.2: there does not exist a continuous map $\mu : \mathrm{SO}(3) \to \mathrm{U}(2)$ such that, for all $U \in \mathrm{SU}(2)$, $\mu(\pi(U))$ differs from U by a phase.

Hint: show that such an existence would imply a homeomorphism $\mathrm{U}(2) \cong \mathrm{U}(1) \times \mathrm{SO}(3)$. This contradicts

$$\begin{aligned} \pi_1(\mathrm{U}(2)) &\cong \mathbb{Z} \\ \pi_1(\mathrm{U}(1) \times \mathrm{SO}(3)) &\cong \pi_1(\mathrm{U}(1)) \times \pi(\mathrm{SO}(3)) \\ &\cong \mathbb{Z} \times \mathbb{Z}_2, \end{aligned}$$

since $\mathbb{Z}_2 \not\cong \mathbb{Z} \times \mathbb{Z}_2$.

PROBLEM 2.2: This problem discusses a few isomorphisms similar to $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$ that hold for other numbers of spatial dimensions besides the physical $d = 3$ case.

(a) Show that $\mathrm{Spin}(1,2) \cong \mathrm{SL}(2,\mathbb{R})$.

(b) Show that $\mathrm{Spin}(1,4)$ is isomorphic to the group

$$\mathrm{Sp}(1,1) = \left\{ A \in M_2(\mathbb{H}) : A^{\dagger} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad (2.97)$$

consisting of 2-by-2 quaternionic "Lorentz matrices."

(c) (Optional.) Show that the group $\mathrm{Spin}(1,5)$ is isomorphic to the group

$$\mathrm{SL}(2,\mathbb{H}) = \{ M \in M_2(\mathbb{H}) : \det_{\mathrm{D}}(M) = 1 \} \quad (2.98)$$

of 2-by-2 quaternionic unimodular "matrices." Note: the correct notion of determinant for quaternionic matrices is the Dieudonné determinant, $\det_{\mathrm{D}}(M)$. This is the same thing as $\sqrt{\det(M_{\mathbb{C}})}$, where $M_{\mathbb{C}}$ is the 4-by-4 complex matrix representing M.

(d) (Optional.) Combining the results of (a), (c), the following pattern appears: for the three division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the isomorphism $\mathrm{Spin}(1, 1 + \dim \mathbb{K}) \cong \mathrm{SL}(2,\mathbb{K})$ holds, where $\dim \mathbb{K} \in \{1,2,4\}$ is the dimension of $\mathbb{K}$ as a real vector space. What does this suggest about $\mathrm{Spin}(1,9)$?

PROBLEM 2.3: This problem continues Example 2.6.

(a) Consider the following elements of $\mathfrak{h}_{\mathbb{C}}$:

$$\mathsf{x} = -i \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathsf{p} = -i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathsf{Z} = -i \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.99}$$

Show that these satisfy the "canonical commutation relation"

$$i[\mathsf{x}, \mathsf{p}] = \mathsf{Z}. \tag{2.100}$$

(b) Check that the Schrödinger representation is in fact a representation of the Heisenberg group.

(c) Show that in the Schrödinger representation,

$$i[\mathsf{x}, \mathsf{p}] = \hbar. \tag{2.101}$$

Why does this imply that the projective representation $\rho_\hbar : (\mathbb{R}^2, +) \to \mathrm{PU}(\mathcal{H})$ cannot be lifted to a unitary representation of $(\mathbb{R}^2, +)$?

PROBLEM 2.4: (a) Formulate a category relQM of relativistic quantum mechanical systems, and show that, if $d \geq 2$, then this category is equivalent to the category of (continuous) unitary representations of $\mathrm{P}^*(1, d)$.

(b) (Optional.) Formulate and prove a similar statement for $d = 1$.

PROBLEM 2.5: The *celestial sphere* $\mathcal{C}\mathbb{S}^2 = \{\Gamma_{\mathbf{y}} : \mathbf{y} \in \mathbb{S}^2\}$ is the set of null lines $\Gamma_{\mathbf{y}} = \{(t, t\mathbf{y}) : t \in \mathbb{R}\}$ in $\mathbb{R}^{1,3}$. This is naturally given a smooth manifold structure, identifying it with $\mathbb{S}^2$ via the map $\Gamma_{\mathbf{y}} \mapsto \mathbf{y}$.

(a) Show that any restricted Lorentz transformation induces a conformal transformation of the celestial sphere.

Hint: $\mathbb{S}^2 \cong \mathbb{C}P^1$, with conformal transformations being those on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ induced by Möbius transformations

$$z \mapsto \frac{az + b}{cz + d} \quad a, b, c, d \in \mathbb{C} \text{ s.t. } ad - bc \neq 0. \tag{2.102}$$

(b) (Optional.) Show that this yields a surjective map $\mathrm{SO}(1, 3) \to \mathrm{PSL}(2, \mathbb{C})$.

CHAPTER 3

The spectrum

In the previous lecture, we covered how every projective unitary representation of the restricted Poincaré group lifts to an ordinary unitary representation of the universal cover $\mathrm{P}^*(1,d)$.[1] This lecture begins our analysis of the lift. We concentrate on the subgroup

$$(\mathbb{R}^{1,d},+) \subset \mathrm{P}^*(1,d) \tag{3.1}$$

of translations, whose representation theory we understand. The decomposition of a unitary representation of this group into irreps is an abstract version of Fourier analysis. The *spectrum*

$$\sigma \subseteq \mathbb{R}^{1,d} \tag{3.2}$$

is the (closed) set keeping track of which irreps show up. A rigorous definition appears below; roughly, it is the support of the Fourier transforms of states in our space. This is an invariant of the representation, meaning that the spectra of two unitarily equivalent representations agree.

On an intuitive level, the spectrum consists of the possible "energy-momentum" vectors $p=(E,\mathbf{p})$ of states of our system. In classical-mechanical special relativity, a physical system of mass $m>0$ and velocity $\mathbf{v}\in\mathbb{B}^d$ has momentum

$$\mathbf{p} = \gamma m \mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1-\|\mathbf{v}\|^2}}. \tag{3.3}$$

The energy E is

$$\begin{aligned} E = \gamma m = \frac{m}{\sqrt{1-\|\mathbf{v}\|^2}} &\approx \underbrace{m}_{\text{Rest energy}} + \overbrace{\frac{m\|\mathbf{v}\|^2}{2}}^{\text{Newtonian kinetic energy}} + \cdots \\ &= \sqrt{\|\mathbf{p}\|^2+m^2}. \end{aligned} \tag{3.4}$$

These are collected into the $(1+d)$-vector $p(\mathbf{v})=(E,\mathbf{p})$. If our relativistic quantum mechanical system is modeling the quantum analogue of the classical system described above, then $p(\mathbf{v})$ should be in the spectrum σ, for any $\mathbf{v}\in\mathbb{B}^d$.

Throughout this lecture, we use $\rho : \mathrm{P}^*(1,d)\to \mathrm{U}(\mathcal{H})$ to denote our unitary representation of $\mathrm{P}^*(1,d)$ and $\varrho = \rho|_{\mathbb{R}^{1,d}}$ to denote the restriction of ρ to the subgroup of translations.

1. The technical challenge

Given a unitary representation of a group G, the simplest hope for decomposing the representation into irreps would be that the representation decomposes as a direct sum of its irreducible subspaces.[2] Unitary representations of finite groups and more generally compact Lie groups do decompose in this way, but non-compact groups are more subtle.

To see why, consider the representation of the abelian group $(\mathbb{R},+)$ on $L^2(\mathbb{R})$ acting via translation: $(T_a\psi)(x)=\psi(x+a)$. Since the irreps of abelian groups are one-dimensional, this infinite-dimensional representation is certainly reducible. An irrep $\mathcal{X}\subset L^2(\mathbb{R})$ would be the span of

[1] As long as $d\geq 2$.

[2] Or primary subspaces.

a nonzero element $f \in L^2(\mathbb{R})$, that, when translated by any amount, yields a multiple of f: for each $a \in \mathbb{R}$,

$$f(x+a) \in \mathbb{C} f(x). \tag{3.5}$$

Some functions $f : \mathbb{R} \to \mathbb{C}$ do have this property, for instance $f(x) = e^{\alpha x}$ for $\alpha \in \mathbb{C}$, but such f never lie in $L^2(\mathbb{R})$. Consequently, $L^2(\mathbb{R})$ does not contain *even one* irrep of $(\mathbb{R}, +)$. It is not the sum of its irreducible subspaces. [Exercise 3.1]

The Fourier transform allows us to write, for any $f \in L^2(\mathbb{R})$,

$$f(x) = \int_{-\infty}^{\infty} e^{ix\xi} \hat{f}(\xi)\,\mathrm{d}\xi, \tag{3.6}$$

for some $\hat{f} \in L^2(\mathbb{R})$.[3] This *looks* like a decomposition of f into components in irreps. Indeed, the function $\chi_\xi(x) = e^{ix\xi}$, when translated by $a \in \mathbb{R}$ units, changes by a factor of $e^{ia\xi}$:

$$\chi_\xi(x+a) = e^{ia\xi}\chi_\xi(x). \tag{3.7}$$

Now, χ_ξ is not in $L^2(\mathbb{R})$, but we can consider $\operatorname{span}_{\mathbb{C}} \chi_\xi$ its own Hilbert space (isomorphic to $\mathbb{C}$), hosting the one-dimensional irrep $\mathbb{R} \ni a \mapsto e^{ia\xi}$ of $(\mathbb{R}, +)$. So, the (inverse) Fourier transform acts like a decomposition of f into an *integral* of irreducible components χ_ξ that fail to lie in $\mathcal{H}$. The coefficient of χ_ξ is $\hat{f}(\xi) \in \mathbb{C}$, which is defined for almost all $\xi \in \mathbb{R}$. The moral is:

We need to use a direct integral to carry out the decomposition into irreps, instead of a direct sum.

Note that the example of $G = (\mathbb{R}, +)$ is of direct relevance to $G = \mathrm{P}^*(1,d)$, because $\mathrm{P}^*(1,d)$ contains the former as a subgroup.

2. Abstract plane waves

Consider the group $(\mathbb{R}^{1,d}, +)$. Because it is abelian, its irreps are all one-dimensional. For each p,

$$\chi_p : \mathbb{R}^{1,d} \ni x \mapsto e^{i\langle x,p\rangle} \in \mathbb{C} \tag{3.8}$$

is a one-dimensional irrep of $(\mathbb{R}^{1,d}, +)$, and every irrep has this form.

As far as the previous sentence is concerned, the bilinear form $\langle -,- \rangle : (\mathbb{R}^{1+d})^2 \to \mathbb{R}$ is arbitrary, as long as it is non-degenerate. Different choices lead to different parametrizations of the set of irreps by p. However, one choice is particularly natural: the *Lorentzian* "inner product"

$$\langle x,p \rangle = x \cdot p = -x^0 p_0 + \mathbf{x} \cdot \mathbf{p}. \tag{3.9}$$

For this choice of $\langle -,- \rangle$,

$$\chi_p(x) = e^{ix\cdot p} \tag{3.10}$$

is described as a plane wave with "momentum" $p = (p_0, \mathbf{p})$. The temporal component $p_0 \in \mathbb{R}$ is the temporal frequency of the wave, while $\mathbf{p} \in \mathbb{R}^d$ is the velocity. The quantity $\|\mathbf{p}\|$ is the spatial frequency. Plane waves are standing waves, not wavepackets, so the wave never "goes anywhere," in the sense that the amplitude is constant in time. Nevertheless, it makes sense to talk about the wave's velocity, speed, direction of travel $\mathbf{p}/\|\mathbf{p}\| \in \mathbb{S}^{d-1}$ (unless $\mathbf{p} = \mathbf{0}$), etc.

What makes the Lorentzian inner product natural is:

PROPOSITION 3.1. *For any $\Lambda \in \mathrm{O}(1,d)$, we have $\chi_p \circ \Lambda^{-1} = \chi_{\Lambda p}$.* ■

[3] Recall that, though the integral

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \frac{\mathrm{d}x}{2\pi}$$

might not make literal sense for all $f \in L^2(\mathbb{R})$, it makes sense for f lying in some dense subset $\mathcal{D} \subset L^2(\mathbb{R})$, and that is good enough; the Fourier transform $\mathcal{F} : \mathcal{D} \ni f \mapsto \hat{f}$, defined initially only for $f \in \mathcal{D}$, is bounded $L^2 \to L^2$ (say by Parseval–Plancherel), and therefore extends boundedly to all $f \in L^2$.

In words, acting (via precomposition with Λ^{-1}) on a plane wave with momentum p results in a plane wave with momentum Λp.

PROOF. $\chi_p \circ \Lambda^{-1}(x) = e^{i(\Lambda^{-1}x)\cdot p} = e^{ix\cdot(\Lambda p)} = \chi_{\Lambda p}(x)$. □

By an *abstract plane wave* with momentum p, we mean a unitary representation of $(\mathbb{R}^{1,d}, +)$ consisting of a direct sum of countably many copies of the same χ_p. These model polarized plane waves.

EXAMPLE 3.2 (Light). A light wave with frequency $\omega > 0$ traveling in the direction $\theta \in \mathbb{S}^2$ is modeled by a function $A_{\bullet,\theta} \in C^\infty(\mathbb{R}^{1,3}; \mathbb{C})$ of the form

$$A_{\varepsilon,\theta}(x) = \varepsilon e^{-i\omega t + i\omega \mathbf{x}\cdot\theta}, \tag{3.11}$$

where $\varepsilon \in \mathbb{C}^3$ satisfies $\varepsilon \perp \theta$ (meaning that the vectors $\Re\varepsilon, \Im\varepsilon \in \mathbb{R}^3$ are both orthogonal to θ). So, $A_{\varepsilon,\theta}(x) = \varepsilon e^{i\langle x,p\rangle}$ for $p = (\omega, \omega\theta)$. The vector ε is known as the *polarization* of the plane wave. It specifies the direction orthogonal to the wavevector θ in which the electric field is oscillating. The magnetic field is oscillating in the remaining orthogonal direction. Because the space

$$\mathcal{H}[p] = \{A_{\varepsilon,\theta} : \varepsilon \perp \theta\} \tag{3.12}$$

of light waves traveling in the fixed direction θ is two-dimensional, one says that light admits two distinct polarizations. The group $(\mathbb{R}^{1,3}, +)$ of spacetime translations acts on the space $\mathcal{H}_\theta$ in the obvious way. This representation is equivalent to $\chi_p \oplus \chi_p$. We have two copies of χ_p, one for each "polarization." ■

EXAMPLE 3.3 (Dirac plane waves). In Dirac's theory of the electron, an electron traveling with velocity exactly $\mathbf{p} \in \mathbb{R}^3$ is described by a plane wave

$$u_{\phi,\mathbf{p}}(x) = e^{-it\sqrt{m_e^2 + \|\mathbf{p}\|^2} + i\mathbf{x}\cdot\mathbf{p}} \begin{pmatrix} \phi \\ \frac{\sigma\cdot\mathbf{p}}{\omega + m_e}\phi \end{pmatrix}, \quad \phi \in \mathbb{C}^2, \tag{3.13}$$

where $m_e > 0$ is the mass of the electron (in natural units), $\omega = \sqrt{m_e^2 + \|\mathbf{p}\|^2}$. This has energy-momentum $p = (\omega, \mathbf{p})$. (When thinking about an electron, you probably have in mind a *wavepacket*, not a plane wave. But because we are specifying the velocity exactly, the Heisenberg uncertainty principle forces the particle to be de-localized, like a plane wave, rather than a wavepacket.) Here, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices.

A "spin up" electron is $u_{(1,0),\mathbf{p}}$, and a "spin down" electron is $u_{(0,1),\mathbf{p}}$. The group $(\mathbb{R}^{1,3}, +)$ of spacetime translations acts on

$$\mathcal{H}[p] = \{u_{\phi,\mathbf{p}} : \phi \in \mathbb{C}^2\} \tag{3.14}$$

in the obvious way. This representation is equivalent to $\chi_p \oplus \chi_p$. We have two copies of χ_p, corresponding to the two degrees-of-freedom present in the pair $\phi \in \mathbb{C}^2$. One for the spin-up electron, and one for the spin-down electron. ■

EXAMPLE 3.4 (Phonons). Phonons represent oscillations of a crystal lattice, or massive analogues of the photon. The plane waves have the form

$$A_{\varepsilon,\mathbf{p}}(x) = \varepsilon e^{-it\sqrt{m^2 + \|\mathbf{p}\|^2}} e^{i\mathbf{x}\cdot\mathbf{p}}, \tag{3.15}$$

where $\varepsilon \in \mathbb{C}^3$ is *not* required to be orthogonal to $\mathbf{p} \in \mathbb{R}^3$. Thus, compared to the photon, we have an extra polarization, $\varepsilon \parallel \mathbf{p}$, whose interpretation is that it describes a pressure wave, in which the density of the crystal is oscillating along the wave's direction of propagation. Imagine the atoms in the crystal oscillating back and forth. The interpretation of the polarizations $\varepsilon \perp \mathbf{p}$ is that they represent transverse waves, in which the crystal lattice is oscillating orthogonally to the direction in which the wave is propagating. ■

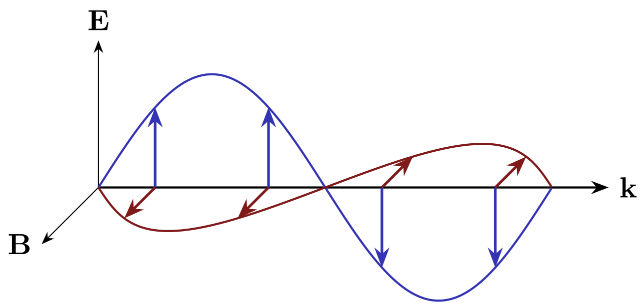


FIGURE 3.1. A plane wave of the electromagnetic field, showing the electric and magnetic fields oscillating in orthogonal directions, both in the plane $\mathbf{k}^\perp$ perpendicular to the wavevector $\mathbf{k}$. Not depicted is the wave's variation in time.

3. Definition of the spectrum

3.1. The spectrum: incorrect method. We would like to decompose $\varrho = \rho|_{(\mathbb{R}^{1,d},+)}$ into isotypic components, i.e. generalized plane waves. Each generalized plane wave is described by a (unique) $p \in \mathbb{R}^{1,d}$. Define

$$\mathcal{H}[p] = \{\psi \in \mathcal{H} \text{ s.t. } \varrho(-)\psi = \chi_p(-)\psi\}. \tag{3.16}$$

This consists of all of the generalized plane waves *in* $\mathcal{H}$ with energy-momentum exactly p.

As a preliminary attempt at defining the spectrum, let

$$\sigma_0 = \{p \in \mathbb{R}^{1,d} \text{ s.t. } \mathcal{H}[p] \neq \{0\}\}. \tag{3.17}$$

This will not quite work, but it is instructive to push ahead.

PROPOSITION 3.5. *If* $\Lambda \in \mathrm{Spin}(1,d)$ *and* $\psi \in \mathcal{H}[p]$*, then* $\rho(\Lambda)\psi \in \mathcal{H}[\Lambda p]$. ■

PROOF. Using $T_a T_\Lambda = T_\Lambda T_{\Lambda^{-1}a}$,

$$\begin{aligned}\varrho(a)\rho(\Lambda)\psi = \rho(T_a\Lambda)\psi = \rho(\Lambda(\Lambda^{-1}T_a\Lambda))\psi = \rho(\Lambda T_{\Lambda^{-1}a})\psi = \rho(\Lambda)\varrho(\Lambda^{-1}a)\psi \\ = e^{i\langle p,\Lambda^{-1}a\rangle}\rho(\Lambda)\psi = e^{i\langle \Lambda p,a\rangle}\rho(\Lambda)\psi.\end{aligned} \tag{3.18}$$

□

So, σ_0 is closed under the action of the restricted Lorentz group:

$$p \in \sigma_0 \Longrightarrow \Lambda p \in \sigma_0 \text{ for all } \Lambda \in \mathrm{SO}(1,d). \tag{3.19}$$

By the same logic, if $S \subset \mathbb{R}^{1,d}$ is closed under the action of the restricted Lorentz group, then the subspace

$$\overline{\bigoplus_{p\in S} \mathcal{H}[p]} \subseteq \mathcal{H} \tag{3.20}$$

consisting of superpositions of finitely many of the generalized plane waves with momentum $p \in S$ is a subspace closed under the action of $\mathrm{P}^*(1,d)$.

If ρ is irreducible, this subspace can only be $\{\mathbf{0}\}$ or all of $\mathcal{H}$:

$$\overline{\bigoplus_{p\in S} \mathcal{H}[p]} = \begin{cases}\{0\},\\ \mathcal{H}.\end{cases} \tag{3.21}$$

(If we are only assuming that ρ is primary, the reasoning is similar.) It follows that either $\sigma_0 = \varnothing$ or it consists of the closure of precisely one orbit O of the Lorentz group.

Here's the major technical snag: $\sigma_0 \subseteq \{0\}$. No irreps exist as actual subspaces of $\mathcal{H}$, with the one possible exception $p = 0$:

$$\mathcal{H}[p] = \{0\} \text{ for all } p \neq 0, \tag{3.22}$$

exactly as in the example of the Fourier transform that we discussed earlier. Indeed, if $q \neq p$, then $\mathcal{H}[p]$ and $\mathcal{H}[q]$ must be orthogonal; this is the usual computation that eigenvectors of a self-adjoint operator with different eigenvalues must be orthogonal. Because every Lorentz orbit $\subset \mathbb{R}^{1,d}$ besides the singleton $\{0\}$ consists of uncountably many different points, $\mathcal{H}[p] \neq \{0\}$ would imply that $\mathcal{H}$ is not separable.

While $\mathcal{H}$ will be decomposable into irreps, it will be via a direct integral, not direct sum. So, σ_0 is *not* a good definition of the spectrum. This does not mean that the preceding discussion was without purpose. It will apply, with the technical details modified, once we have a rigorous definition of σ. The equality $\sigma = \sigma_0$ may be false, but it is false for technical reasons rather than moral ones, and so it does not lead us far astray.

3.2. The spectrum: correct method. To define σ, we can use the following version of the spectral theorem. Recall that a projection-valued Borel measure is a map $\Pi : \mathrm{Borel}(\mathbb{R}^{1,d}) \to \mathcal{B}(\mathcal{H})$ assigning to each Borel subset $E \subseteq \mathbb{R}^{1,d}$ an orthogonal projection $\Pi(E)$, satisfying some axioms analogous to those a measure is required to satisfy. The spectral theorem for locally compact abelian groups [Fol95, Thm. 4.45] ("SNAG theorem"), like $(\mathbb{R}^{1,+}, +)$, tells us that there exists a projection-valued Borel measure Π such that

$$\varrho(x) = \int_{\mathbb{R}^{1,d}} e^{ix\cdot p}\,\mathrm{d}\Pi(p). \tag{3.23}$$

This expression can be interpreted in the following way: for $\phi, \psi \in \mathcal{H}$, $\mathrm{Borel}(\mathbb{R}^{1,d}) \ni E \mapsto \langle \phi, \Pi(E)\psi\rangle \in \mathbb{C}$ defines an ordinary complex measure (with finite mass) $\mu_{\phi,\psi} : \mathrm{Borel}(\mathbb{R}^{1,d}) \to \mathbb{C}$, and this satisfies

$$\langle \phi, \varrho(x)\psi\rangle = \int_{\mathbb{R}^{1,d}} e^{ix\cdot p}\,\mathrm{d}\mu_{\phi,\psi}(p). \tag{3.24}$$

Part of the spectral theorem is the *functional calculus*: for any $f \in L^\infty \cap C^0(\mathbb{R}^{1,d})$, we can define an operator $\Pi(f) \in \mathcal{B}(\mathcal{H})$ by

$$\begin{aligned} \Pi(f) &= \int_{\mathbb{R}^{1,d}} f(p)\,\mathrm{d}\Pi(p) \\ \langle \phi, \Pi(f)\psi\rangle &= \int_{\mathbb{R}^{1,d}} f(p)\,\mathrm{d}\mu_{\phi,\psi}(p). \end{aligned} \tag{3.25}$$

For example, $\Pi(e^{ix\cdot\bullet}) = \varrho(x)$. The operator $\Pi(f)$ depends linearly on f, and, for all other $g \in L^\infty \cap C^0$,

$$\Pi(fg) = \Pi(f)\Pi(g), \tag{3.26}$$

as suggested by the formal idempotency identity

$$\mathrm{d}\Pi(p)\,\mathrm{d}\Pi(q) = \delta^{1+d}(p-q)\,\mathrm{d}\Pi(p) \tag{3.27}$$

that ought to hold on account of Π being *projection*-valued. Moreover, for $A \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned} [A, \varrho(x)] = 0 \text{ for all } x \in \mathbb{R}^{1,d} &\iff [A, \Pi(f)] = 0 \text{ for all } f \in L^\infty \cap C^0 \\ &\iff [A, \Pi(E)] = 0 \text{ for all Borel } E. \end{aligned} \tag{3.28}$$

The spectrum σ is defined – correctly, now – as the support of the measure Π. That is, the complement $\sigma^{\complement} = \mathbb{R}^{1,d}\backslash\sigma$ is defined by

$$\sigma^{\complement} = \{p \in \mathbb{R}^{1,d} : \exists \text{ open } U \ni p \text{ s.t. } \Pi(U) = 0\}. \tag{3.29}$$

Note that σ is nonempty, because $\Pi(\mathcal{H})$ is the identity operator on $\mathcal{H}$ (via a standard argument using countable additivity). [Exercise 3.6]

Lemma 3.6. $\rho(a,\Lambda)^{-1}\Pi(E)\rho(a,\Lambda) = \Pi(\Lambda^{-1}E)$ for all Borel sets $E \subset \mathbb{R}^{1,d}$. ■

Proof. It suffices to show that

$$\langle \rho(a,\Lambda)\phi, \Pi(E)\rho(a,\Lambda)\psi\rangle = \langle \phi, \Pi(\Lambda^{-1}E)\psi\rangle \tag{3.30}$$

for all $\phi, \psi \in \mathcal{H}$ and $E \in \mathrm{Borel}(\mathbb{R}^{1,d})$. Phrased in terms of spectral measures, this is $\mu_{\rho(a,\Lambda)\phi,\rho(a,\Lambda)\psi} = \mu_{\phi,\psi} \circ \Lambda^{-1}$. It suffices to check that their Fourier transforms

$$\begin{aligned}\int_{\mathbb{R}^{1,d}} e^{ix\cdot p}\,\mathrm{d}\mu_{\rho(a,\Lambda)\phi,\rho(a,\Lambda)\psi}(p) &\overset{\text{eq. (3.24)}}{=} \langle \rho(a,\Lambda)\phi, \varrho(x)\rho(a,\Lambda)\psi\rangle \\ &= \langle \phi, \rho(a,\Lambda)^{-1}\varrho(x)\rho(a,\Lambda)\psi\rangle\end{aligned} \tag{3.31}$$

$$\begin{aligned}\int_{\mathbb{R}^{1,d}} e^{ix\cdot p}\,\mathrm{d}\mu_{\phi,\psi}(\Lambda^{-1}p) \overset{\det\Lambda=1}{=} \int_{\mathbb{R}^{1,d}} e^{ix\cdot(\Lambda p)}\,\mathrm{d}\mu_{\phi,\psi}(p) &= \int_{\mathbb{R}^{1,d}} e^{i(\Lambda^{-1}x)\cdot p}\,\mathrm{d}\mu_{\phi,\psi}p \\ &\overset{\text{eq. (3.24)}}{=} \langle \phi, \varrho(\Lambda^{-1}x)\psi\rangle\end{aligned} \tag{3.32}$$

agree. This holds because $\rho(a,\Lambda)^{-1}\varrho(x)\rho(a,\Lambda) = \varrho(\Lambda^{-1}x)$. □

Proposition 3.7. *The spectrum σ is Lorentz-closed.* ■

Proof. We prove the contrapositive: $\Lambda p \in \sigma(P)^{\complement} \Rightarrow p \in \sigma(P)^{\complement}$.

Suppose that $\Lambda p \in \sigma(P)^{\complement}$, so that there exists an open neighborhood $U \ni \Lambda p$ such that $\Pi(U) = 0$. Lemma 3.6 gives

$$\Pi(\Lambda^{-1}U) = \rho(0,\Lambda^{-1})\Pi(U)\rho(0,\Lambda) = 0. \tag{3.33}$$

Since $\Lambda^{-1}U$ is an open neighborhood of p, we can conclude that $p \in \sigma(P)^{\complement}$. □

Note that

$$\mathcal{H}[p] = \Pi(\{p\})\mathcal{H} \tag{3.34}$$

The fact that this is usually trivial means $\Pi(\{p\}) = 0$. This is consistent with $\Pi(\sigma) = \mathrm{id}_{\mathcal{H}}$, because σ (if not Ω) has uncountably many different p's in it, and measures are only *countably* additive.

4. Orbits of the Lorentz group

In the previous section, we proved that the spectrum $\sigma \subset \mathbb{R}^{1,d}$ of a unitary representation of the Poincaré group is closed under the action of the Lorentz group on $\mathbb{R}^{1,d}$. Consequently, σ is a union of orbits $O \subset \mathbb{R}^{1,d}$ of the Lorentz group. Since $d \geq 2$, these orbits are: [Exercise 3.3]

- the origin $\mho = \{0\}$,
- the forward/backward "light cones" $V_\pm = \{(E,\mathbf{p}) \in \mathbb{R}^{1,d}\backslash\{0\} : \pm E = \|\mathbf{p}\|\}$ (which in this context are defined to exclude the origin $p = 0$ and are therefore not closed),
- the *mass shell*

$$X_{m,\pm} = \left\{(E,\mathbf{p}) \in \mathbb{R}^{1,d} : E = \pm\sqrt{m^2 + \|\mathbf{p}\|^2}\right\}, \tag{3.35}$$

  labeled by a parameter $m > 0$ (the "mass"),
- the one-component hyperboloid $Y_\Gamma = \{(E,\mathbf{p}) \in \mathbb{R}^{1,d} : \|\mathbf{p}\| = \sqrt{\Gamma^2 + E^2}\}$.

We can consider $V_\pm$ as $X_{0,\pm}$.

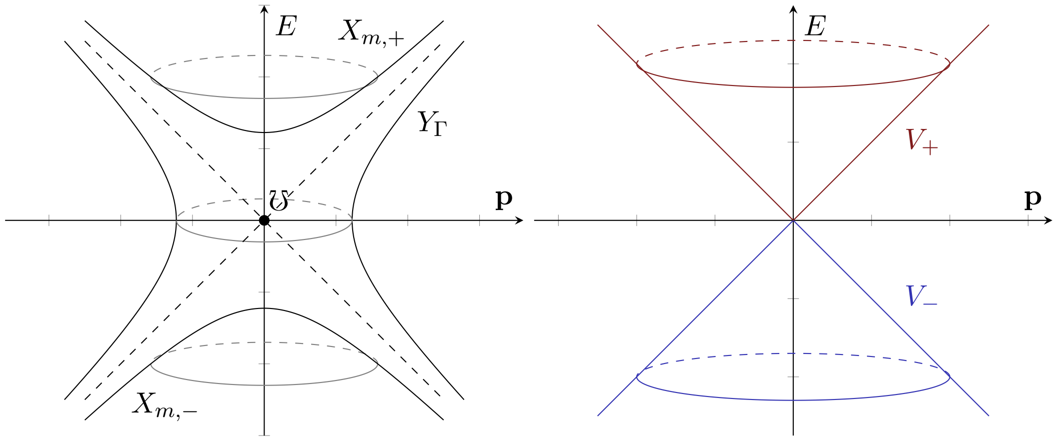


FIGURE 3.2. The Lorentz orbits $\mho$, $X_{m,\pm}$, Y_Γ, V_+, V_-, plotted in $\mathbb{R}^{1,d}_p$, $p = (E, \mathbf{p})$, for particular m, Γ.

5. Vacua

An important theoretical role is played by the subspace $\mathcal{H}[0] = \Pi(\{0\})$ consisting of translation-invariant states. These are called *vacuum states*. Intuitively, the vacuum is homogeneous, so it should remain unchanged under the application of any translation.

Because all Lorentz transformations $\Lambda \in \widetilde{\mathrm{SO}(1,d)}$ map $\mathcal{H}[0]$ to itself, this space hosts a unitary representation of the (universal) Lorentz group. Vacuum states are typically isotropic in addition to homogeneous. That is, they remain unchanged under the application of any Lorentz transformation. Recalling that states are complex lines in $\mathcal{H}$, a Lorentz-invariant vacuum state is the same thing as a one-dimensional representation of the (universal) Lorentz group inside $\mathcal{H}[0]$. Of course, the trivial representation is an example.

PROPOSITION 3.8. *For $d \geq 2$, the only (continuous) one-dimensional representation of $\widetilde{\mathrm{SO}(1,d)}$ is the trivial one.* ■

PROOF. Any (continuous) one-dimensional representation of this connected group induces one of the Lie algebra $\mathfrak{o}(1,d)$. It suffices to show that this induced representation is zero, as it then follows by exponentiating that the original representation is trivial (by Lie's second theorem). Complexifying, we have a one-dimensional representation of $\mathfrak{o}(1+d)_\mathbb{C}$.

Fact: no such representation exists if $d \geq 2$, except for the zero representation.

Given this fact, we are done. □

One way of proving the fact stated above is to use semisimplicity. For $d \geq 2$, the Lie algebra $\mathfrak{o} = \mathfrak{o}(1+d)_\mathbb{C}$ is semisimple, a consequence of which is that $\mathfrak{o}$ is "perfect;" this means

$$[\mathfrak{o}, \mathfrak{o}] = \mathfrak{o}. \tag{3.36}$$

The left-hand side is the subspace of $\mathfrak{o}$ spanned by the commutators of the various elements of $\mathfrak{o}$. In any one-dimensional representation of a Lie algebra $\mathfrak{g}$, the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ must be mapped to 0, since scalars commute. So if the commutator subalgebra is everything, no nonzero one-dimensional representations exist.

The key identity eq. (3.36) can be proven directly from the Lie algebra

$$i[\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}] = \eta^{\mu\lambda}\mathsf{J}^{\nu\sigma} - \eta^{\mu\sigma}\mathsf{J}^{\nu\lambda} + \eta^{\nu\sigma}\mathsf{J}^{\mu\lambda} - \eta^{\nu\lambda}\mathsf{J}^{\mu\sigma}. \tag{3.37}$$

Here $\mathsf{J}^{\mu\nu} = -\mathsf{J}^{\nu\mu}$ are the generators of $\mathfrak{o}$. For any distinct values of ν, σ, choose a third distinct value (which exists because $d \geq 2$) and plug it in for μ, λ to get

$$i[\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}] = \pm \mathsf{J}^{\nu\sigma}. \tag{3.38}$$

Thus, $\mathsf{J}^{\nu\sigma} \in [\mathfrak{o}, \mathfrak{o}]$.

REMARK: Actually, there are no nontrivial finite-dimensional unitary representations of $\widetilde{\mathrm{SO}(1,d)}$, but this requires a different argument. [*]

6. Wigner's concept of particle

Wigner identifies particle species with irreducible (continuous, as always) unitary representations of $\mathrm{P}^*(1,3)$. We will use "irrep" as an abbreviation. Note that this includes unitarity. Because the Poincaré group is noncompact, unitarizability of representations is not automatic, and non-unitarizable representations do exist.

The irreducibility requirement in Wigner's definition is sometimes conveyed as the particle being elementary. However, composite particles can be modeled as elementary if their internal structure can be ignored. Baryons are bound states of quarks held together by a matrix of gluons; they have a rich internal structure, but that structure is not visible to low-energy experiments, so, for the purpose of such experiments, they can be treated as elementary. Within *high energy* particle colliders, like the LHC, baryons cannot be treated as elementary — the whole purpose of the LHC is to break protons apart.

Besides irreducibility, we include in Wigner's definition the following postulates:

- (Stability.) $\sigma \subseteq \{(E, \mathbf{p}) : E \geq 0\}$.
- (Localizability.) $\sigma \neq \{0\}$.

The first of these says that the particle has non-negative energy. The second ends up being equivalent to the particle not having exactly zero energy. We can summarize their combination by saying that the particle is "positive-energy." This does *not* mean that $0 \notin \sigma$. Indeed, $0 \in \sigma$ happens for massless particles.

In isolation, a negative-energy particle would behave sensibly. After all, the difference between positive/negative energy amounts to a sign convention, like $+i$ vs. $-i$. However, once a coupling has been turned on between particles and some external field, then the *relative* signs matter. For example, if we have chosen sign conventions such that the electromagnetic field has positive energy, then an atomic system whose energy spectrum is unbounded below would be expected to radiate away an infinite amount. This is the sort of runaway instability that the stability axiom is supposed to prevent.

What $\sigma = \{0\}$ would mean is that translations act trivially. Vacua are examples. There is nothing physically wrong with vacua — they exist in any QFT — but these sorts of delocalized entities do not deserve to be called "particles." Barring these is the purpose of the localizability requirement, and the rationale for its name.

7. The typical spectrum of QFTs

The spectrum of quantum field theories takes two typical forms.

(1) The first form describes a theory with a *mass gap* with mass $m > 0$. The spectrum σ takes the form

$$\sigma = \{0\} \cup X_{m,+} \cup X^+_{2m,+}, \tag{3.39}$$

where

$$X^+_{2m,+} = X_{m,+} + X_{m,+} = \{p \in \mathbb{R}^{1,d} : p^0 > 0 \text{ and } p^2 \leq -4m^2\}. \tag{3.40}$$

The origin $0 \in \sigma$ is point spectrum, meaning that it is associated with the eigenspace $\mathcal{H}[0] \subset \mathcal{H}$ of states which are left invariant under all translations. Typically, $\mathcal{H}[0] = \mathbb{C}\Omega$ is the span of a single vacuum state Ω, which is invariant under the whole Poincaré group.

The "mass shells" $X_{m,+}$ describe stable massive particles.

The region $X^+_{2m,+}$ consists of spectrum which is continuous with respect to the Lebesgue measure $\mathrm{d}^{1+d}p$ (except possibly for some embedded mass shells $X_{M,+}$, $M \geq 2m$). These describe multi-particle configurations.

Note: it is possible to have additional mass shells $X_{m',+} \subset \sigma$ for $m' \in (m, 2m)$ in between m and the two-particle threshold.

(2) The second form describes a theory with massless particles. As $m \to 0^+$, the mass shell $X_{m,+}$ converges to the (closed) lightcone

$$\bar{V}_+ = \{p \in \mathbb{R}^{1,d} : p^0 \geq 0 \text{ and } p^2 = 0\}, \tag{3.41}$$

and the two-particle region $X^+_{2m,+}$ converges to

$$\bar{V}^+_+ = \{p \in \mathbb{R}^{1,d} : p^0 \geq 0 \text{ and } p^2 \leq 0\}. \tag{3.42}$$

In this case, the spectrum is just $\sigma = \bar{V}^+_+$. Usually, one expects that σ° consists entirely of continuous spectrum. It is possible that there exist some exceptional mass shells $X_{m,+}$ describing stable particles.

If QED exists as a well-defined theory, it is expected that the spectrum has a logarithmic singularity as

$$p \to X_{m_\mathrm{e},+} \text{ from above}, \tag{3.43}$$

i.e. as $p^2 \to -m_\mathrm{e}^2$, where $m_\mathrm{e} > 0$ is the mass of the electron. In this sort of situation, where the spectrum is continuous with respect to the Lebesgue measure $\mathrm{d}^{1+d}p$, with an L^1_{loc} density, but the density blows up at $X_{m,+}$, one says that an *infraparticle* is present. In QED, electrons are expected to be inseparable from a dressing of low-energy (a.k.a. "soft") photons, resulting in a smearing of the spectrum upwards in energy.

A. The Lebesgue decomposition ($\star$)

Let $\mathcal{O}$ denote the set of all Lorentz orbits $O \subset \mathbb{R}^{1,d}$ such that $\Pi(O) \neq 0$. For each $O \in \mathcal{O}$, let

$$\mathcal{H}[O] = \mathrm{range}(\Pi(O)) \tag{3.44}$$

denote the range of $\Pi(O)$. Each of these is closed under the action of the Poincaré group.

If O, O' are distinct Lorentz orbits, then $\mathcal{H}[O] \perp \mathcal{H}[O']$, by the generalization to infinite-dimensions of the usual computation that eigenvectors of a Hermitian matrix with distinct eigenvalues are orthogonal.

Now consider the subspaces

$$\mathcal{H}_{\mathrm{pp}} = \overline{\bigoplus_{O \in \mathcal{O}}} \mathcal{H}[O], \quad \mathcal{H}_\mathrm{c} = \mathcal{H}^\perp_{\mathrm{pp}}. \tag{3.45}$$

By construction, $\mathcal{H} = \mathcal{H}_{\mathrm{pp}} \oplus \mathcal{H}_\mathrm{c}$. Since $\mathcal{H}_{\mathrm{pp}}$ is closed under the action of the Poincaré group (and our representation is unitary), $\mathcal{H}_\mathrm{c}$ is also closed under that action.

Proposition 3.9 (Lebesgue decomposition theorem). *There exists a decomposition $\mathcal{H}_\mathrm{c} = \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{ac}}$ into orthogonal subspaces, each of which is closed under the action of the Poincaré group, such that*

- *the spectral projector $\Pi_{\mathrm{sc}} = \Pi|_{\mathcal{H}_{\mathrm{sc}}}$ associated with $\mathcal{H}_{\mathrm{sc}}$ is singular with respect to the Lebesgue measure on $\mathbb{R}^{1,d}$,*
- *for any $\phi, \psi \in \mathcal{H}_{\mathrm{ac}}$, the spectral measure $\mu_{\phi,\psi} = \langle \phi, \Pi(-)\psi \rangle$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{1,d}$.*

■

Proof. [*] □

In standard quantum field theory, the singular continuous spectrum is assumed to be trivial. Physical examples with singular continuous spectrum are somewhat exotic and usually associated with chaotic or quasi-periodic systems (like quasicrystals).

B. Differentiating a Lie group representation

Let G denote an arbitrary Lie group, and let $\mathfrak{g} = T_e G$ denote its Lie algebra. Our goal here is to prove that any continuous unitary representation

$$\rho : G \to \mathrm{U}(\mathcal{H}) \tag{3.46}$$

of G on a separable Hilbert space $\mathcal{H}$ induces a representation ϱ of $\mathfrak{g}_{\mathbb{C}}$ (and ultimately its universal enveloping algebra) via (usually unbounded) operators, defined on some common dense domain $\mathcal{D} \subseteq \mathcal{H}$.

The prototype for this is *Stone's theorem*:

Example 3.10. Suppose that $G = (\mathbb{R}, +)$. A strongly-continuous unitary representation $\rho : G \to \mathrm{U}(\mathcal{H})$ is the same thing as a strongly-continuous one-parameter family of unitary operators

$$\{U_t\}_{t\in\mathbb{R}} \subset \mathrm{U}(\mathcal{H}), \tag{3.47}$$

meaning that

- $U_0 = I$,
- $U_t U_s = U_{t+s}$ for all $t, s \in \mathbb{R}$,
- $\lim_{t\to t_0} U_t\psi = U_{t_0}\psi$ for all $t_0 \in \mathbb{R}$ and $\psi \in \mathcal{H}$ (strong continuity).

Stone's theorem [Sto32] says that any such family has the form $U_t = e^{itA}$ for some (possibly unbounded) self-adjoint operator $A : \mathcal{D}(A) \to \mathcal{H}$ with domain

$$\mathcal{D}(A) = \{\psi \in \mathcal{H} : \lim_{t\to 0} t^{-1}(U_t - I)\psi \text{ exists}\}. \tag{3.48}$$

Specifically, $A\psi = -i \lim_{t\to 0} t^{-1}(U_t - I)\psi$ for $\psi \in \mathcal{D}$. ■

If $\rho : G \to \mathrm{U}(\mathcal{H})$ is a strongly-continuous unitary representation of a Lie group G, and if $j : (\mathbb{R}, +) \to G$ is a non-trivial smooth homomorphism, then we can form $\rho \circ j$. By Stone's theorem, there exists a self-adjoint operator $A : \mathcal{D}(A) \to \mathcal{H}$ such that

$$\rho \circ j(t) = e^{itA} \tag{3.49}$$

for all $t \in \mathbb{R}$. Any element of X of the Lie algebra $\mathfrak{g}$ generates a one-parameter group via exponentiation: $j(t) = e^{tX}$. So, we can define $\varrho(X) = iA$. This is anti-self-adjoint and satisfies

$$\rho(e^{tX}) = e^{t\varrho(X)}. \tag{3.50}$$

One would like to say that ϱ is a representation of $\mathfrak{g}$, but now domain issues, stemming from the possible unboundedness of the generator A, rear their head. The problem is that the domain $\mathcal{D}(A)$ in Stone's theorem depends on A. Consequently, it is not clear that we can compose two elements $\varrho(X), \varrho(Y)$. This leaves the interpretation of their commutator unclear.

Example 3.11. Let $G = (\mathbb{R}^d, +)$, $\mathcal{H} = L^2(\mathbb{R}^d)$, and G act on $\mathcal{H}$ via translation. Then, $\mathfrak{g} = \mathbb{R}^d$, and

$$\varrho(X) = X^j \partial_j = \partial_X \tag{3.51}$$

for any $X \in \mathbb{R}^d$, where X^j is the jth component of X. So, $\varrho(X)$ is a differential operator and therefore unbounded (except if $X = 0$). The domain $\mathcal{D}(X)$ on which ∂_X is anti-self-adjoint is the set of $u \in L^2(\mathbb{R}^d)$ such that $\partial_X u$, which makes sense as a distribution, lies in $L^2(\mathbb{R}^d)$. ■

EXAMPLE 3.12. Let $G = \mathrm{SO}(2)$, $\mathcal{H} = L^2(\mathbb{R}^2_{x,y})$, and G act on $\mathcal{H}$ via rotations around the origin. Then, $\mathfrak{g}$ has a single generator, L, satisfying

$$\varrho(L) = \partial_\theta = -y\partial_x + x\partial_y, \tag{3.52}$$

where ∂_θ is the usual angular partial derivative in polar coordinates (r, θ), i.e. $r\hat{\boldsymbol{\theta}}$. As in the previous example, $\varrho(L)$ is an unbounded operator. ■

Of course, no domain issues are present in the finite-dimensional setting:

EXAMPLE 3.13. Let $\mathcal{H} = \mathbb{C}^N$ for $N \in \mathbb{N}$, so that $\mathrm{U}(\mathcal{H}) = \mathrm{U}(N)$. It turns out that any continuous finite-dimensional unitary representation of any Lie group must be smooth, but rather than prove this, let us just assume that $\rho : G \to \mathrm{U}(N)$ is some smooth unitary representation. Differentiating then yields a map

$$D\rho : TG \to T\,\mathrm{U}(N). \tag{3.53}$$

Restricting to the fibers over the identity element $e \in G$ gives a linear map $\mathfrak{g} \to \mathfrak{u}(N)$, denoted $D\rho(e)$. Because ρ is a homomorphism, $D\rho(e)$ must preserve the Lie bracket, so is a homomorphism of Lie algebras. ■

All domain issues are resolved by:

PROPOSITION 3.14 (Gårding [Gr47]). *There exists a dense domain $\mathcal{D} \subseteq \mathcal{H}$ such that:*

(i) for each $X \in \mathfrak{g}$, the operator $\varrho(X)$ contains $\mathcal{D}$ in its domain and maps $\mathcal{D}$ back to itself,
(ii) $\varrho(X)$ is essentially anti-self-adjoint when restricted to $\mathcal{D}$,
(iii) $\rho(g)\mathcal{D} = \mathcal{D}$, for all $g \in G$.
(iv) The map $\varrho : \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ is a map of Lie algebras, from $\mathfrak{g}$ to the algebra of linear endomorphisms of $\mathcal{D}$.

■

We will provide the proof below. In the two examples above, one can take $\mathcal{D} = C^\infty_{\mathrm{c}}$, the domain of smooth compactly supported functions.

The proof uses a *smoothing procedure*: for $\psi \in \mathcal{H}$ and $\chi \in C^\infty_{\mathrm{c}}(G)$, let

$$\operatorname{avg}_\chi \psi = \int_G \chi(g)\rho(g)\psi \, \mathrm{d}\mu(g). \tag{3.54}$$

Here, μ denotes a fixed left-invariant Haar measure $\mu : \mathrm{Borel}(G) \to [0, \infty]$, so that

$$\int_G \chi(hg) \, \mathrm{d}\mu(g) = \int_G \chi(g) \, \mathrm{d}\mu(g) \tag{3.55}$$

for any $\chi \in L^1(G, \mu)$ and $h \in G$.

REMARK 3.15 (The Bochner/Pettis-integral). The integrand on the right-hand side of eq. (3.54) is $\mathcal{H}$-valued, so an iota of care is required in interpreting it. The simplest way of doing so is via duality: consider the map $\Lambda : \mathcal{H} \to \mathbb{C}$ given by

$$\Lambda(\phi) = \int_G \chi(g)\langle \rho(g)\psi, \phi\rangle \, \mathrm{d}\mu(g). \tag{3.56}$$

The map $g \mapsto \langle \rho(g)\psi, \phi\rangle$ is continuous (since we are assuming that ρ is strongly-continuous), so the integral on the right-hand side of eq. (3.56) is a well-defined, ordinary numerical integral on G. Evidently, Λ is linear, and, using the unitarity of ρ, the estimate

$$|\Lambda(\phi)| \leq \|\chi\|_{L^1}\|\psi\|\|\phi\| \tag{3.57}$$

holds. So, $\Lambda \in \mathcal{H}^*$, and the Riesz representation theorem guarantees the existence of a (unique) vector $\operatorname{avg}_\chi \psi \in \mathcal{H}$ such that $\Lambda(\phi) = \langle \operatorname{avg}_\chi \psi, \phi\rangle$ for all $\phi \in \mathcal{H}$. ■

In order to understand why the map

$$\operatorname{avg}_\chi : \psi \mapsto \operatorname{avg}_\chi \psi \tag{3.58}$$

is described as smoothing, consider the example above, where $G = (\mathbb{R}^d, +)$ and $\mathcal{H} = L^2(\mathbb{R}^d)$. Then, averaging is convolution:

$$\operatorname{avg}_\chi \psi(x) = \int_{\mathbb{R}^d} \chi(a)(\rho(a)\psi)(x)\, \mathrm{d}^d a = \int_{\mathbb{R}^d} \chi(a)\psi(x+a)\, \mathrm{d}^d a = \tilde{\chi} * \psi(x), \tag{3.59}$$

where $\tilde{\chi}(x) = \chi(-x)$. Convolution smooths, since derivatives fall on χ:

$$L(\tilde{\chi} * \psi) = (L\tilde{\chi}) * \psi \tag{3.60}$$

for any constant-coefficient differential operator L. So, in this example, $\operatorname{avg}_\chi \psi$ lands in the set

$$H^\infty(\mathbb{R}^d) = \bigcap_{m \in \mathbb{N}} H^m(\mathbb{R}^d) \subset L^2 \tag{3.61}$$

of functions all of whose derivatives (including zeroth derivatives, second derivatives, third derivatives, etc.) lie in L^2 as well. We saw previously that, in this example, the Stone generators $A = -i\partial_t e^{tX}|_{t=0}$ are constant-coefficient vector fields, such as ∂_{x_j}, $j = 1, \ldots, d$. These do in fact act on $H^\infty(\mathbb{R}^d)$, and they map $H^\infty(\mathbb{R}^d)$ to itself.

This suggests: in the proof of Proposition 3.14, the subset $\mathcal{D} \subset \mathcal{H}$ will be defined by

$$\mathcal{D} = \{\operatorname{avg}_\chi \psi : \chi \in C_c^\infty(G), \psi \in \mathcal{H}\}. \tag{3.62}$$

This is manifestly invariant under the application of any $\rho(g)$, $g \in G$.

Now let us prove the theorem.

Proof of Proposition 3.14. (I) *Claim: $\mathcal{D}$ is dense.*

Because ρ is strongly-continuous, if $\{\chi_j\}_{j=1}^\infty \subseteq C_c^\infty(G)$ are chosen so that $\chi_j\, \mathrm{d}\mu \to \delta$ weakly, where δ is a Dirac-δ function at the identity element of G, then

$$\operatorname{avg}_{\chi_j} \psi \to \psi \tag{3.63}$$

in $\mathcal{H}$. So, $\mathcal{D}$ is a dense subspace of $\mathcal{H}$.

(II) *Claim: if $X \in \mathfrak{g}$ and $\phi = \operatorname{avg}_\chi \psi \in \mathcal{D}$, then*

$$\lim_{t \to 0^+} \frac{\rho(e^{tX})\phi - \phi}{t} = -\operatorname{avg}_{\mathcal{L}_{\tilde{X}}\chi} \psi \in \mathcal{D}, \tag{3.64}$$

where $\mathcal{L}_{\tilde{X}}\chi$ is the Lie derivative of χ by the right-invariant vector field $\tilde{X}$ with $\tilde{X}(e) = X$. So, not only is $\mathcal{D}$ a subspace of the domain described in Stone's theorem, but the generator A in that theorem maps $\mathcal{D}$ back to itself. Indeed,

$$\frac{\rho(e^{tX})\phi - \phi}{t} = \frac{1}{t}\int_G \chi(g)(\rho(e^{tX}g) - \rho(g))\psi\, \mathrm{d}\mu(g), \tag{3.65}$$

and using the left- invariance of the Haar measure,

$$\int_G \chi(g)(\rho(e^{tX}g) - \rho(g))\psi\, \mathrm{d}\mu(g) = \int_G (\chi(e^{-tX}g) - \chi(g))\rho(g)\psi\, \mathrm{d}\mu(g). \tag{3.66}$$

We can write

$$\chi(e^{-tX}g) = \chi(g) - t\mathcal{L}_{\tilde{X}}\chi(g) + O_{L^1}(t^2), \tag{3.67}$$

where the error term $O_{L^1}(t^2)$ has $L^1(G,\mu)$-norm bounded above by Ct^2, for some $C > 0$. So,

$$\frac{\rho(e^{tX})\phi - \phi}{t} = -\int_G \mathcal{L}_{\tilde{X}}\chi(g)\rho(g)\psi\, \mathrm{d}\mu(g) + O(t) = -\operatorname{avg}_{\mathcal{L}_{\tilde{X}}\chi} \psi + O(t). \tag{3.68}$$

(III) *Claim: The Hermitian generators $A = -i\varrho(X)$ are essentially self-adjoint on $\mathcal{D}$*, i.e. $\mathcal{D}$ is a core for A.

We already know that A is self-adjoint on the domain $\mathcal{D}(A)$ from Stone's theorem. Moreover, $\mathcal{D}$ is invariant under G. A general theorem from functional analysis [RS80, Thm. VIII.11] then guarantees that $\mathcal{D}$ is a core.

(IV) *Claim: ϱ is a map of Lie algebras.* We want to show that if $X, Y \in \mathfrak{g}$, then

$$(\varrho(X)\varrho(Y) - \varrho(Y)\varrho(X))\phi = \varrho([X,Y])\phi, \tag{3.69}$$

for all $\phi \in \mathcal{D}$. Let $\phi = \operatorname{avg}_\chi \psi$. Then,

$$\begin{aligned} \varrho(X)\varrho(Y)\phi &= -\varrho(X)\operatorname{avg}_{\mathcal{L}_{\tilde{Y}}\chi}\psi = \operatorname{avg}_{\mathcal{L}_{\tilde{X}}\mathcal{L}_{\tilde{Y}}\chi}\psi \\ \varrho(Y)\varrho(X)\phi &= -\varrho(X)\operatorname{avg}_{\mathcal{L}_{\tilde{X}}\chi}\psi = \operatorname{avg}_{\mathcal{L}_{\tilde{Y}}\mathcal{L}_{\tilde{X}}\chi}\psi. \end{aligned} \tag{3.70}$$

So, $(\varrho(X)\varrho(Y) - \varrho(Y)\varrho(X))\phi$ is $\operatorname{avg}_{[\mathcal{L}_{\tilde{X}},\mathcal{L}_{\tilde{Y}}]\chi}\psi$. Since

$$[\mathcal{L}_{\tilde{X}}, \mathcal{L}_{\tilde{Y}}] = \mathcal{L}_{[\tilde{X},\tilde{Y}]} = \mathcal{L}_{-\widetilde{[X,Y]}} = -\mathcal{L}_{\widetilde{[X,Y]}}, \tag{3.71}$$

this is

$$-\operatorname{avg}_{\mathcal{L}_{\widetilde{[X,Y]}}\chi}\psi = \varrho([X,Y])\phi. \tag{3.72}$$

The sign in eq. (3.71) is due to the fact that the Lie bracket of right-invariant vector fields on G corresponds to minus the Lie bracket on $\mathfrak{g}$.

□

In representation theory, three different sorts of representations act as building blocks. A (not-necessarily unitary) representation $\rho : G \to \mathrm{GL}(\mathcal{H}) = \{\text{invertible } A \in \mathcal{B}(\mathcal{H})\}$ of a group G on a Hilbert space $\mathcal{H}$ is said to be:

- *irreducible* if the only G-closed subspaces $\mathcal{X} \subseteq \mathcal{H}$ are $\mathcal{X} = \{\mathbf{0}\}, \mathcal{H}$,
- *indecomposable* if, whenever $\mathcal{X}, \mathcal{Y}$ are G-closed subspaces such that $\mathcal{H} = \mathcal{X} \oplus \mathcal{Y}$, then one of $\mathcal{X}, \mathcal{Y}$ is $\{\mathbf{0}\}$, in which case the other is $\mathcal{H}$,
- *primary* (a.k.a. factorial) if the centralizer

$$\mathcal{C} = \{A \in \mathcal{B}(\mathcal{H}) : [A, \rho(g)] = 0 \text{ for all } g \in G\} \tag{3.73}$$

has a center consisting only of scalar multiples of the identity.

Here, a subspace $\mathcal{X} \subseteq \mathcal{H}$ is described as "G-closed" (or "ρ-closed") if it is closed and $\rho(g)\mathcal{X} \subseteq \mathcal{X}$ for all $g \in G$.

For finite-dimensional representations:

$$\begin{tikzcd} \text{irreducible} \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "\text{Schur}"'] & \text{indecomposable} \\ & \text{primary.} \end{tikzcd}$$

None of the other implications hold in general. Fortunately, more can be said when ρ is a *unitary* representation. Firstly, Schur's lemma holds, even if $\mathcal{H}$ is infinite-dimensional (Lemma 3.18). Secondly:

Proposition 3.16. *When ρ is a (continuous) unitary representation, irreducibility and indecomposability are equivalent.* ■

Proof. If $\mathcal{X}$ is a proper G-closed subspace, so is $\mathcal{X}^\perp$, owing to

$$\begin{aligned} v \in \mathcal{X}^\perp &\iff \langle v, w\rangle = 0 \text{ for all } w \in \mathcal{X} \iff \langle \rho(g)v, \rho(g)w\rangle = 0 \text{ for all } w \in \mathcal{X}, g \in G \\ &\iff \langle \rho(g)v, w\rangle = 0 \text{ for all } w \in \mathcal{X}, g \in G \Longrightarrow \rho(g)v \in \mathcal{X}^\perp. \end{aligned} \tag{3.74}$$

So $\mathcal{H} = \mathcal{X} \oplus \mathcal{X}^\perp$ is a decomposition of $\mathcal{H}$ into proper G-closed subspaces. □

Any unitary representation of a Lie group decomposes as a direct integral of irreducible representations [Fol95, Thm. 7.38]. The irreps, rather than being subrepresentations, will be found by a limiting procedure. We will take a hands-on attitude towards direct integrals in these notes, preferring to avoid any general theory.

The *uniqueness* of the direct integral decomposition is subtle for general Lie groups [Fol95, §7]. What is always unique is the decomposition into primary components. Fortunately, for all of the groups we work with in these notes, all primary decompositions decompose canonically as a direct sum of countably many copies of a single irrep. Such groups are called "Type I." Finite groups, locally compact abelian groups, and compact groups are all Type I. Wigner showed in [Wig39] that $\mathrm{P}^*(1,3)$ is Type I (and the same proof works for $\mathrm{P}^*(1,d)$).

Indecomposability does not imply irreducibility or primality in general:

Example 3.17. Consider the two-dimensional non-unitary representation $\rho : (\mathbb{R},+) \to \mathrm{GL}(2)$ given by

$$\rho(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \tag{3.75}$$

This is not irreducible, because the span of $\mathbf{e} = (1,0)$ is an invariant subspace, but it is indecomposable. Indeed, the span of $\mathbf{e}$ is the only ρ-closed proper subspace, and none of its linear-algebraic complements are closed under the group action.

In this example, $\mathcal{C}$ is the image of ρ. Since this contains non-diagonalizable matrices, ρ is not primary. ■

Schur's lemma holds for unitary representations on Hilbert spaces:

Lemma 3.18. For any group G, if $\rho : G \to \mathrm{U}(\mathcal{H})$ is irreducible, then $\mathcal{C}(\rho) = \{c\,\mathrm{id}_\mathcal{H} : c \in \mathbb{C}\}$. ■

[Exercise 3.5] Proof. If $A \in \mathcal{C}$, then $A^\dagger \in \mathcal{C}$ as well. Consequently, $B = (A + A^\dagger)/2$ and $C = (A - A^\dagger)/(2i)$ are both in $\mathcal{C}$. If both of these are scalar multiples of the identity, then the same applies to $A = B + iC$. Note that B, C are self-adjoint. So, it suffices to prove every self-adjoint element of $\mathcal{C}$ is a scalar multiple of the identity I.

Let $O \in \mathcal{C}$ be self-adjoint, and let Π denote its spectral projectors. *Claim: its spectrum $\sigma(O)$ consists of only one element,* hence $O \propto I$. Suppose, to the contrary, that $|\sigma(O)| \geq 2$. Then, let $U \subset \mathbb{R}$ denote an open set containing part of $\sigma(O)$ whose closure does not contain all of $\sigma(O)$. Then, the ranges

$$\mathcal{X} = \Pi(U)\mathcal{H}, \quad \mathcal{Y} = \Pi(\mathbb{R}\backslash U)\mathcal{H} \tag{3.76}$$

are orthogonal and satisfy $\mathcal{X} \oplus \mathcal{Y} = \mathcal{H}$. Both of these must be proper (by the definition of U). □

If ρ is irreducible, then $\rho \oplus \rho$, $\rho \oplus \rho \oplus \rho$, and so on are examples of primary representations that are not irreducible. Indeed, the direct sum of d copies of ρ is equivalent to $\rho \otimes (\mathbf{1}^{\oplus d})$, whose centralizer is

$$\mathcal{C}(\rho \otimes (\mathbf{1}^{\oplus d})) = \{c\,\mathrm{id}_\mathcal{H} \otimes A : A \in \mathbb{C}^{d\times d}\}, \tag{3.77}$$

and the center of this consists only of scalar multiples of the identity. There are more exotic possibilities, but only for somewhat wild groups like a free group on multiple generators.

Lemma 3.19. A representation of an abelian group is primary if and only if it is a direct sum of countably many instances of the same irreducible representation. ■

Proof. Suppose that $\rho : A \to \mathrm{GL}(\mathcal{H})$ is a primary representation of an abelian group A. Then, $\rho(a)$ is in the center of the centralizer $\mathcal{C}(\rho)$, for each $a \in A$. So, primacy implies that $\rho(a) = \chi(a)\,\mathrm{id}_\mathcal{H}$, for some $\chi(a) \in \mathbb{C}$. Thus, given any nonzero $\phi \in \mathcal{H}$, the span $\mathbb{C}\phi$ is a sub-representation and, since it is one-dimensional, irreducible. So, any orthonormal basis of $\mathcal{H}$ gives a decomposition of ρ into copies of the character χ. □

Exercises and problems

EXERCISE 3.1: (a) Suppose that $f \in L^2(\mathbb{R})$ has the property that $\forall a \in \mathbb{R}$, $\exists c \in \mathbb{C}$ s.t. $f(x+a) = cf(x)$. Show that $f = 0$.

(b) Use the Fourier transform to show that the only tempered distributions having the property in the previous part are of the form $f(x) \propto e^{i\xi x}$, for $\xi \in \mathbb{R}$.

(c) Show that $\chi_\xi : x \mapsto e^{ix\xi}$ are all of the irreps of $(\mathbb{R}, +)$.

EXERCISE 3.2: Let $\rho : \mathrm{P}^*(1,d) \to \mathrm{U}(\mathcal{H})$ be a (continuous) unitary representation of $\mathrm{P}^*(1,d)$. Suppose that $\mathcal{X}$ is a Poincaré-closed (norm-closed) subspace of $\mathcal{H}$. Thus, $\mathcal{X}$ constitutes a subrepresentation. Show that the spectrum of this subrepresentation is a subset of the spectrum of the whole representation.

EXERCISE 3.3: (a) Prove that, if $d \geq 2$, the orbits $O \subset \mathbb{R}^{1,d}$ of the restricted Lorentz group are $\Omega, V_\pm, X_{m,\pm}, Y_\Gamma$, for $m, \Gamma > 0$.

(b) What about $d = 1$?

EXERCISE 3.4: Show that the (complexified) Poincaré algebra $\mathfrak{p}_\mathbb{C} = \mathfrak{p}(1,d)_\mathbb{C}$ satisfies $[\mathfrak{p},\mathfrak{p}] = \mathfrak{p}$, thus proving that it has no nonzero one-dimensional representations.

EXERCISE 3.5: Consider the setup of Lemma 3.18. Prove the claim that $A \in \mathcal{C} \Rightarrow A^\dagger \in \mathcal{C}$. *Hint:* this uses unitarity.

EXERCISE 3.6: Use the definition of the spectrum σ, eq. (3.29), to prove that $\sigma \neq \varnothing$.

CHAPTER 4

Wigner's classification of particles

In the previous lecture, we presented Wigner's definition of a relativistic particle: a positive-energy irreducible (continuous) unitary representation – or "irrep" for short – of the universal Poincaré group

$$\mathrm{P}^*(1,d) = \mathbb{R}^{1,d} \rtimes \underbrace{\widetilde{\mathrm{SO}(1,d)}}_{=\mathrm{Spin}(1,d) \text{ when } d\geq 3}. \tag{4.1}$$

This lecture is devoted to Wigner's classification of such irreps, a result contained in the seminal paper [Wig39]. Put more colorfully, this lecture is devoted to the classification of particles in terms of their kinematics.

In the physical $d=3$ case (and ignoring one irrep not known to have physical relevance):

For each $m \geq 0$ and

$$s \in \begin{cases} 2^{-1}\mathbb{N} & (m>0), \\ 2^{-1}\mathbb{Z} & (m=0), \end{cases} \tag{4.2}$$

we have one irrep, $\pi_{m,s}$. The parameter m is the "mass," which identifies the spectrum $\sigma = X_{m,+}$, and s is the "spin" (if $m>0$) or "helicity" (if $m=0$), which will have to do with the action of rotations.

This classification has two parts:

(i) the construction of the irreps $\pi_\bullet : \mathrm{P}^*(1,d) \to \mathrm{U}(\mathcal{H}_\bullet)$, and
(ii) a proof that every irrep is unitarily equivalent to (exactly) one of them.

These notes will go through both. Beginning with an irrep of $\mathrm{P}^*(1,d)$, we will attempt to understand its structure. This guides us to the definition of $\pi_\bullet$.

Particle	Representation	# of internal d.o.f.
Higgs bosons	$\pi_{m,0}$	1
quarks, leptons	$\pi_{m,1/2}$	2
photons, gluons	$\pi_{0,-1} \oplus \pi_{0,1}$	2
$\mathrm{W}^\pm$, Z bosons	$\pi_{m,1}$	3
pions	$\pi_{m,0}$	1
ρ, ω mesons	$\pi_{m,1}$	3
Δ, Ω^- baryons	$\pi_{m,3/2}$	4
sterile neutrinos	$\pi_{0,1/2}$	1
gravitinos	$\pi_{0,-3/2} \oplus \pi_{0,3/2}$	2
gravitons	$\pi_{0,-2} \oplus \pi_{0,2}$	2

Table 4.1. Some elementary and composite particles and their respective $\pi_\bullet$'s.

Warning: The usage of similar notation belies the differences between $\pi_{0,\pm s}$ and $\pi_{m,s}$, $m>0$. Whereas a particle described by the latter representation has $2s+1$ internal degrees-of-freedom, a particle described by the former has only a single internal degree-of-freedom. Much of our intuition about massive particles comes from their "non-relativistic limit." Massless particles always travel the speed of light and admit no non-relativistic limit.

We will provide below a similar classification for any number $d \geq 4$ of spatial dimensions. Massive particles are classified by their mass $m > 0$ and an irrep of $\mathrm{Spin}(d)$, and massless particles (barring one unphysical possibility) are classified by irreps of $\mathrm{Spin}(d-1)$. A simpler statement holds when $d = 2$. However, we will not enumerate these irreps explicitly, except in the physical $d = 3$ case. For that, see [BB21].

Throughout this section, $d \geq 3$.

1. The method of induced representations: summary

The strategy of the classification involves exploiting the semidirect product structure of P^*. The key is to understand how irreps of P^* are assembled from irreps of the subgroup $(\mathbb{R}^{1,d}, +)$ of translations and irreps of a certain (m-dependent) subgroup $L \subset \mathrm{Spin}(1,d)$ known as the "little group." This method is known by several names — the one we will use is the *method of induced representations*.

Suppose we are given an irreducible representation

$$\rho : \mathrm{P}^*(1,d) \to \mathrm{U}(\mathcal{H}). \tag{4.3}$$

The steps in the analysis of ρ are:

(I) Restrict ρ to the abelian subgroup consisting of translations, giving a representation

$$\begin{aligned} \varrho &: (\mathbb{R}^{1,d}, +) \to \mathrm{U}(\mathcal{H}) \\ &: a \mapsto \rho(T_a) \end{aligned} \tag{4.4}$$

of $(\mathbb{R}^{1,d}, +)$. We discussed in the last lecture how to decompose ϱ into irreps via the spectral measure. The support of that measure is the spectrum $\sigma \subset \mathbb{R}^{1,d}$.

It will be shown that, as a consequence of Lorentz covariance, σ consists of the closure of precisely one orbit O of the restricted Lorentz group.

(II) Pick any "reference momentum" $p_{\circledast} \in \mathbb{R}^{1,d}$ in the relevant one of $O = \Omega, V_\pm, X_{m,\pm}, Y_\Gamma$. Then, the *little group* $L = L[p_{\circledast}]$ is defined to be the subgroup of $\mathrm{Spin}(1,d)$ which stabilizes $p_{\circledast}$; that is,

$$L[p_{\circledast}] = \{\Lambda \in \mathrm{Spin}(1,d) \text{ s.t. } \Lambda p_{\circledast} = p_{\circledast}\}. \tag{4.5}$$

[Exercise 4.1] This depends on $p_{\circledast}$, but only as a set; different choices of $p_{\circledast} \in O$ lead to conjugate, and therefore isomorphic, subgroups of $\mathrm{Spin}(1,d)$. The possible isomorphism classes are listed in Table 4.2.

It will be shown that the space of generalized plane waves with momentum exactly $p_{\circledast}$ hosts a representation of the little group:

$$\varsigma : L[p_{\circledast}] \to \mathrm{U}(\mathcal{V}). \tag{4.6}$$

(We are sweeping under the rug some major technical headache involved in making the definition of ς precise, but the gist of the foregoing discussion is correct. See §2.2 for the rigorous definition.) Different choices of reference momentum $p_{\circledast}$ lead to unitarily equivalent ς.

It will then be shown that ρ is unitarily equivalent to a representation, denoted $\pi_\bullet$, consisting of superpositions of plane waves with momenta $p \in O$ with $\dim \mathcal{V}$ "spin indices," acted on via ς. Mathematically,

$$\pi_\bullet = \mathrm{ind}_{\mathbb{R}^{1,d} \rtimes L}^{\mathbb{R}^{1,d} \rtimes \mathrm{Spin}(1,d)} (\varrho \otimes \varsigma) \tag{4.7}$$

Orbit O	Eg. $p_{\divideontimes} \in O$	Little group	Description	Example
$\Omega = \{0\}$	$(0,0,0,0)$	$\mathrm{Spin}(1,d)$		vacuum
$V_\pm$	$(\pm 1,0,0,1)$	$\mathrm{E}^*(d-1)$	Massless	photon
$X_{m,\pm}$, $m>0$	$(\pm m,0,0,0)$	$\mathrm{Spin}(d)$	Massive	electron
Y_Γ, $\Gamma>0$	$(0,0,0,\Gamma)$	$\mathrm{Spin}(1,d-1)$	Tachyonic	string tachyon

TABLE 4.2. The six sorts of orbits O of the Lorentz group in $d \geq 2$ spatial dimensions, representative momenta $p_{\divideontimes}$ (in the physical $d=3$ case, for simplicity), and the isomorphism classes of the corresponding little groups $L[p_{\divideontimes}]$. Here, $\mathrm{E}^*(j)$ is the double cover of the isometry group of $j \geq 2$ dimensional space; this is the universal cover if $j \geq 3$.

is the induced representation constructed from ς. The subscript '$\bullet$' will contain a label indicating ς as well as $O \in \{\Omega, V_\pm, X_{m,\pm}, Y_\Gamma\}$. The irreducibility of $\pi_\bullet$, and thus of the original representation ρ, will be equivalent to the irreducibility of ς.

In summary, to each irrep ρ is associated two pieces of data, which characterize it:

- a Lorentz-orbit $O \subset \mathbb{R}^{1,d}$, whose closure $\bar{O}$ is the spectrum σ, describing which representations of the normal subgroup $\mathbb{R}^{1,d} \lhd \mathrm{P}^*(1,d)$ are present,
- an irrep ς of a certain subgroup $L \subseteq \mathrm{Spin}(1,d)$, describing the representation's "spin degrees-of-freedom."

Conversely, given these two pieces of data, O and ς, an irrep $\pi_\bullet$ can be constructed from them — the Lorentz orbit associated to $\pi_\bullet$ is O, and the little group representation is ς.

This reduces the classification of irreps of $\mathrm{P}^*(1,d)$ to the classification of irreps of the various L. For example, when $O = X_{m,+}$ for $m>0$, the little group is the group of rotations, or really the double cover

$$\mathrm{Spin}(d) \twoheadrightarrow \mathrm{SO}(d). \tag{4.8}$$

This is a compact simple Lie group, so its representation theory is relatively easy. In the physical $d=3$ case, the little group is

$$\mathrm{Spin}(3) = \mathrm{SU}(2), \tag{4.9}$$

and we know the irreps of $\mathrm{SU}(2)$; we have one, $\mathbf{j}$ for each dimension $j \in \mathbb{N}$, the spin $s = 2^{-1}(j-1)$ representation.

This completes our summary.

The reader may prefer to try their hand at the method of induced representations for semidirect products of finite groups. No new algebraic ideas are required to handle $\mathrm{P}^*(1,d)$, only analytic technicalities owing to the existence of the noncompact abelian subgroup of translations, some of which we have already discussed. On the other hand, if the reader just wants to see the definition of $\pi_\bullet$ without having to work through its motivation, they can skip directly to §2.5.

To cut down on the amount of notational casework, we will sometimes use "$X_{0,\pm}$" to mean $V_\pm$.

2. The method of induced representations: details

Let $\rho : \mathrm{P}^*(1,d) \to \mathrm{U}(\mathcal{H})$ denote a (continuous) unitary representation of $\mathrm{P}^*(1,d)$. We will not assume just yet that ρ is irreducible, but we will assume that it is primary.

2.1. Regarding the spectrum.

PROPOSITION 4.1. *Assuming that ρ is irreducible, σ consists of the closure of a single orbit of the Lorentz group.* ■

PROOF. We already know that σ consists of a (nonempty) union of orbits Ω, V_-, V_+, $X_{m,+}$, $X_{m,-}$, Y_Γ of the Lorentz group.

Suppose that $E \in \mathrm{Borel}(\mathbb{R}^{1,d})$ is Lorentz-closed. This means that $E = \Lambda E$, so Lemma 3.6 says

$$\Pi(E)\rho(a,\Lambda) = \rho(a,\Lambda)\Pi(E). \tag{4.10}$$

Thus, $\mathrm{range}(\Pi(E)) \subseteq \mathcal{H}$ is a Poincaré-closed subspace of $\mathcal{H}$. It will be proper if $\Pi(E) \neq 0, I$. Since we are assuming that ρ is irreducible, $\Pi(E)$ must be 0 or I. This applies to all of the operators $\Pi(\bullet)$ appearing in the rest of the proof.

Suppose that O_j are two distinct Lorentz orbits $O_j \subseteq \mathbb{R}^{1,d}$, such that neither of $\bar{O}_0, \bar{O}_1$ is a subset of the other (this just rules out the case where $O_0 = \Omega$ and $O_1 = V_\pm$, or vice versa). Unless $O_0 = V_-$ and $O_1 = V_+$, or vice versa, there exist disjoint Lorentz-closed open neighborhoods
[Exercise 4.2] $U_j \supset \bar{O}_j$. Because

$$I = \Pi(\mathbb{R}^{1,d}) = \Pi(U_0) + \Pi(U_1) + \Pi(\mathbb{R}^{1,d}\backslash(U_0 \cup U_1)), \tag{4.11}$$

at most one of $\Pi(U_0), \Pi(U_1)$ can be nonzero. Consequently, at least one of O_0 or O_1 is not a subset of σ.

The previous paragraph shows that

$$\sigma \in \{\Omega, \bar{V}_\pm, V_- \cup V_+ \cup \Omega, X_{m,\pm}, Y_\Gamma : m, \Gamma > 0\}. \tag{4.12}$$

The last remaining thing to do is rule out the unwanted possibility $\sigma = V_- \cup V_+ \cup \Omega$. In this case, $\Pi(\mathbb{R}^{1,d}\backslash(\bar{V}_- \cup \bar{V}_+)) = 0$. If it were also the case that $\Pi(V_+) = 0$, then we would have

$$\Pi(\mathbb{R}^{1,d}\backslash\bar{V}_-) = \Pi(\mathbb{R}^{1,d}\backslash(\bar{V}_- \cup \bar{V}_+)) + \Pi(V_+) = 0. \tag{4.13}$$

But $\mathbb{R}^{1,d}\backslash\bar{V}_-$ is an open set containing all of V_+, so this would imply $V_+ \cap \sigma = \varnothing$, contradicting our assumption.

So, $\Pi(V_+) \neq 0$. For the same reason, $\Pi(V_-) \neq 0$.

If it were the case that $\Pi(V_+) = I$, then we would have

$$\Pi(\mathbb{R}^{1,d}\backslash\bar{V}_-) = \Pi(\mathbb{R}^{1,d}\backslash(\bar{V}_- \cup \bar{V}_+)) + \Pi(V_+) = I. \tag{4.14}$$

But because $I = \Pi(\mathbb{R}^{1,d}) = \Pi(\mathbb{R}^{1,d}\backslash\bar{V}_-) + \Pi(V_-) + \Pi(\Omega)$, this would imply $\Pi(V_-) = 0$, which has already been ruled out. □

REMARK: The conclusion of the previous proposition holds under the weaker assumption that ρ is
[Problem 4.1] primary.

2.2. The abstract Fourier transform. We have discussed the technical obstruction to decomposing wavefunctions into generalized plane waves — it requires a direct integral, analogous to the Fourier transform, as the required generalized plane waves do not sit in our Hilbert space $\mathcal{H}$. For each $\phi, \psi \in \mathcal{H}$, we have a spectral measure $\mu_{\phi,\psi} : \mathrm{Borel}(\mathbb{R}^{1,d}) \to \mathbb{C}$, such that $\mu_{\phi,\psi}(E) = \langle \phi, \Pi(E)\psi \rangle$ for all Borel subsets $E \subseteq \mathbb{R}^{1,d}$. The support of $\mu_{\phi,\psi}$ is a subset of that of Π, which is the spectrum σ. When ρ is primary, and thus $\sigma = \bar{O}$ for a Lorentz orbit O, this means that $\mu_{\phi,\psi}$ is an ordinary complex measure supported on $\bar{O}$.

There is a natural choice of Lorentz-invariant measure $\mu : \mathrm{Borel}(\mathbb{R}^{1,d}) \to [0,\infty)$ supported on O. This comes from the homeomorphism

$$O \cong \mathrm{SO}(1,d)/L_*, \tag{4.15}$$

where $L_* = L_*(p_⁎)$ is the stabilizer of $p_⁎$ in the restricted Lorentz group $\mathrm{SO}(1,d)$. The Haar measure on $\mathrm{SO}(1,d)$ descends via this quotient to O. We will see explicit formulas later.

The following technical result is proven in §A:

PROPOSITION 4.2. *There exists a dense subspace $\mathcal{D} \subset \mathcal{H}$, closed under the action of* $\mathrm{P}^*(1,d)$, *such that, for all $\phi, \psi \in \mathcal{D}$, the spectral measure $\mu_{\phi,\psi}$ has a continuous density*

$$\frac{\mathrm{d}\mu_{\phi,\psi}}{\mathrm{d}\mu} \in C^0(O) \cap L^1(O,\mu) \tag{4.16}$$

on O. That is, $\mu_{\phi,\psi}$ is absolutely continuous with respect to μ, and its Radon–Nikodym derivative is continuous. ■

PROOF IDEA. The subspace $\mathcal{D}$ will be the Gårding domain (eq. (3.62)). This means that elements of $\mathcal{D}$ arise via "smoothing" elements $\psi \in \mathcal{H}$ by applying a Poincaré transformation $\rho(T)$ and then averaging over T (pairing against a bump function). The averaging over translations doesn't do much here; the important thing is the averaging over Lorentz transformations. Because Lorentz transformations act transitively on O, this averaging has an effect similar to convolution, smoothing things out over O. It is essential here that $\mu_{\phi,\psi}$ is supported on O, because we do not get any smoothing effect across O. □

REMARK 4.3. The Radon–Nikodym derivative is characterized by:

$$\mu_{\phi,\psi}(E) = \int_E \frac{\mathrm{d}\mu_{\phi,\psi}}{\mathrm{d}\mu}(p)\,\mathrm{d}\mu(p) \tag{4.17}$$

for all Borel $E \subseteq \mathbb{R}^{1,d}$. The following direct construction may be employed. Fix $p \in \mathbb{R}^{1,d}$, and consider a sequence $\{\chi_n\}_{n=1}^{\infty} \subset C^0_{\mathrm{c}}(\mathbb{R}^{1,d})$

- $\chi_n \geq 0$, $\|\chi_n\|_{L^1(O,\mu)} = 1$, where μ is a Lorentz invariant measure on O (see below),
- $\operatorname{supp}\chi_n$ shrinks to $\{p\}$ as $n \to \infty$.

Then, the value of $\mathrm{d}\mu_{\phi,\psi}/\,\mathrm{d}\mu$ at p is $\lim_{n\to\infty}\langle\psi, \Pi(\chi_n)\phi\rangle$. ■

Consider

$$\langle\!\langle \phi, \psi; p \rangle\!\rangle \stackrel{\text{def}}{=} \frac{\mathrm{d}\mu_{\phi,\psi}}{\mathrm{d}\mu}(p) \tag{4.18}$$

for $\psi, \phi \in \mathcal{D}$. This is a sesquilinear, positive semi-definite form on $\mathcal{D}$. Semi-definiteness means

$$\langle\!\langle \psi, \psi; p \rangle\!\rangle \geq 0 \tag{4.19}$$

(which holds because $\mu_{\psi,\psi}$ is an ordinary nonnegative measure). This form is *not* definite – and therefore not an inner product – so cannot be used to complete $\mathcal{D}$ to a Hilbert space. This is because if the spectral measure $\mu_{\phi,\psi}$ is supported away from p, then $\langle\!\langle \psi, \phi; p \rangle\!\rangle = 0$. This is a feature, not a bug, because we want this to be measuring something like the density of the wavefunctions' "Fourier transforms" near p. If these Fourier transforms are vanishing near p, we should get 0.

Fortunately, semi-definite forms are almost as good as inner products, and there exists a standard way of getting a Hilbert space from them. This relies on Cauchy–Schwarz,

$$|\langle\!\langle \phi, \psi; p \rangle\!\rangle| \leq \sqrt{\langle\!\langle \phi, \phi; p \rangle\!\rangle} \cdot \sqrt{\langle\!\langle \psi, \psi; p \rangle\!\rangle}, \tag{4.20}$$

which only requires *semi*-definiteness, as the standard proof shows. Let

$$\mathcal{N}[p] = \{\psi \in \mathcal{D}, \langle\!\langle \psi, \psi; p \rangle\!\rangle = 0\} \tag{4.21}$$

denote the subset of *degenerate* vectors. An immediate consequence of Cauchy–Schwarz is that $\langle\!\langle \psi, \phi; p \rangle\!\rangle = 0$ whenever at least one of $\phi, \psi \in \mathcal{D}$ lie in $\mathcal{N}$. It follows that $\mathcal{N}[p]$ is a linear subspace of $\mathcal{D}$. We can therefore form the quotient space

$$\mathcal{Q}[p] = \mathcal{D}/\mathcal{N}[p]. \tag{4.22}$$

Note that this could be the trivial (zero-dimensional) vector space $\{0\}$, if $\mathcal{N}[p] = \mathcal{D}$. This will be the case if p is not in the spectrum σ of the representation.

The form $\langle\!\langle -,-;p\rangle\!\rangle$ descends to an actual inner product on $\mathcal{Q}[p]$, which we denote with the same symbols:

$$\langle\!\langle \psi \bmod \mathcal{N}[p], \phi \bmod \mathcal{N}[p]; p\rangle\!\rangle = \langle\!\langle \psi,\phi;p\rangle\!\rangle. \tag{4.23}$$

The key point is that this sesquilinear form is positive *definite*, not semidefinite. Finally, let $\mathcal{H}[p]$ denote the completion of $\mathcal{Q}[p]$ under $\langle\!\langle -,-;p\rangle\!\rangle$. This is a different Hilbert space than what we called $\mathcal{H}[p]$ in §3.2, but, unlike our first candidate, it will usually be nonzero. Going forward, this is what we mean by "$\mathcal{H}[p]$," not the original attempt.

We have a canonical map $\mathcal{D}\to\mathcal{H}[p]$, namely $\psi\mapsto\psi \bmod \mathcal{N}[p]$. We will denote

$$\mathcal{F}\psi(p) = \psi \bmod \mathcal{N}[p]. \tag{4.24}$$

The notation "$\mathcal{F}$" is meant to evoke the Fourier transform. For $\psi\in\mathcal{D}$,

$$\int_O \|\mathcal{F}\psi(p)\|^2_{\mathcal{H}[p]}\,\mathrm{d}\mu(p) = \int_O \underbrace{\langle\!\langle \psi,\psi;p\rangle\!\rangle}_{\mathrm{d}\mu_{\psi,\psi}/\,\mathrm{d}\mu}\,\mathrm{d}\mu(p) = \int_O \mathrm{d}\mu_{\psi,\psi}(p) = \|\psi\|^2. \tag{4.25}$$

So, $\mathcal{F}$ is unitary, in a sense.

2.3. The little group representation.

PROPOSITION 4.4. *If $T=(a,\Lambda)\in \mathrm{P}^*$, then $\rho(T):\mathcal{N}[p]\to\mathcal{N}[\Lambda p]$. Consequently, $\rho(T)$ induces a unitary map*

$$\rho(T)[p]:\mathcal{H}[p]\to\mathcal{H}[\Lambda p], \tag{4.26}$$

as expected. ■

PROOF. There are two non-trivial things to check: (i) that $\rho(T):\mathcal{N}[p]\to\mathcal{N}[\Lambda p]$, and (ii) that the induced map eq. (4.26), which is well-defined if we know (i), is unitary. Both claims follow from

$$\langle\!\langle \psi,\phi;p\rangle\!\rangle = \langle\!\langle \rho(T)\psi,\rho(T)\phi;\Lambda p\rangle\!\rangle, \tag{4.27}$$

, which is a consequence of the Lorentz-covariance of the spectral measures $\mu_{\phi,\psi}$. □

A corollary of this is that all $\mathcal{H}[p]$'s, $p\in O$, are isomorphic.

The maps $\rho(T)[p]$ satisfy the following "groupoid" property:

PROPOSITION 4.5. $\rho(T_2T_1)[p] = \rho(T_2)[\Lambda_1 p]\circ\rho(T_1)[p]$. ■

PROOF. Immediate from above. □

PROPOSITION 4.6. $\rho(T_a)[p] = e^{i\langle a,p\rangle}$. ■

PROOF. Follows from $\mathrm{d}\mu_{\phi,\rho(T_a)\psi}/\,\mathrm{d}\mu(p) = e^{i\langle a,p\rangle}\,\mathrm{d}\mu_{\phi,\psi}/\,\mathrm{d}\mu(p)$. □

COROLLARY. Fix $p_*\in O$, and let $L=L[p_*]$ be the stabilizer of p_* in $\mathrm{Spin}(1,d)$. Then,

$$\rho(-)[p_*]:\mathbb{R}^{1,d}\rtimes L\to \mathrm{U}(\mathcal{H}[p_*]) \tag{4.28}$$

is a unitary representation of $\mathbb{R}^{1,d}\rtimes L$, with all of the translation operators T_a acting via multiplication by $e^{i\langle a,p\rangle}$. ■□

Restricting $\rho(-)[p]$ to the subgroup $(\mathbb{R}^{1,d},+)$ of translations, the result is a direct sum of countably many copies of χ_p. We have a generalized plane wave with momentum exactly p, constructed from our original representation, but not as a subspace thereof.

Let $\varsigma: L[p_*]\to\mathrm{U}(\mathcal{H}[p_*])$ be the restriction of this unitary representation to Lorentz transformations:

$$\varsigma(\Lambda) = \rho(\Lambda)[p_*]. \tag{4.29}$$

This is the "little group representation" that will help us in classifying ρ.

PROPOSITION 4.7. *Let $T=(a,\Lambda)$. Then, $\rho(T)[p]((\mathcal{F}\psi)(p)) = (\mathcal{F}(\rho(T)\psi))(\Lambda p)$.* ■

PROOF. $\rho(T)[p]((\mathcal{F}\psi)(p)) = \rho(T)[p](\psi \bmod \mathcal{N}[p]) = (\rho(T)\psi \bmod \mathcal{N}[\Lambda p]) = (\rho(T)\psi)(\Lambda p) = (\mathcal{F}(\rho(T)\psi))(\Lambda p)$. □

Moving ahead, we will drop the "$[p]$" in $\rho(T)[p]$ and use the same notation to refer to the representation on $\mathcal{H}$ as the one induced on $\mathcal{H}[p]$.

2.4. The group law. Next, we identify $\mathcal{F}\psi$ with a function on O valued in some *fixed* vector space.

REMARK 4.8. This can be couched in geometric terms. The various Hilbert spaces $\mathcal{H}[p], p \in O$ can be aggregated into a "bundle" $\mathcal{H}[\bullet] \to O$ over O, whose "fiber" over a point $p \in O$ is the Hilbert space $\mathcal{H}[p]$. Since

$$\mathcal{F}\psi(p) \in \mathcal{H}[p], \tag{4.30}$$

$\mathcal{F}\psi$ is a "section" of the bundle. What we want to do is trivialize the bundle, identifying it with $O \times \mathcal{V}$ for some vector space $\mathcal{V}$. ■

Fix a reference momentum $p_\circledast \in O$, and abbreviate $\mathcal{H}[p_\circledast] = \mathcal{V}$. Suppose that we have chosen, for each $p \in O$, a "standard boost" $D[p] \in \mathrm{Spin}(1,d)$ with

$$D[p]p_\circledast = p. \tag{4.31}$$

($D[p_\circledast] = I_{d+1}$.) Then, $\rho(D[p]^{-1}) : \mathcal{H}[p] \to \mathcal{V}$ is some unitary map, for each $p \in O$. We assume that $D[p]$ has been chosen "sufficiently nicely" on p. It suffices for $D[p]$ to depend Borel measurably on p and be continuous on an open set $U_0 \ni p_\circledast$ whose complement has measure zero.[1] (This can always be done.) From $\mathcal{F}\psi$, we can form a map $O \to \mathcal{V}$ by

$$p \mapsto \rho(D[p]^{-1})\mathcal{F}\psi(p). \tag{4.32}$$

Call this function $U\psi$. It follows immediately from eq. (4.25) that $U\psi \in L^2(O,\mu;\mathcal{V})$, with

$$U : \mathcal{H} \to L^2(O,\mu,\mathcal{V}) \tag{4.33}$$

a partial isometry onto its image. In fact, this map is surjective.

PROPOSITION 4.9. *U is onto $L^2(O,\mu,\mathcal{V})$.* ■

PROOF. Let $\mathcal{X}$ denote the image of U and $\mathcal{Y}$ its orthogonal complement. By Lorentz-smoothing, the subspace $C^0(O;\mathcal{V}) \cap \mathcal{Y}$ is dense in $\mathcal{Y}$. Suppose that Ψ is in this intersection. Suppose that $\Psi(p_\circledast) \neq 0$. We can find a set of vectors $\psi_1, \psi_2, \dots \in \mathcal{D}$ (not necessarily independent) such that the $\psi_j \bmod \mathcal{N}[p_\circledast]$ span (this is the finite span) a dense subspace of $\mathcal{V} = \mathcal{H}[p_\circledast]$. So, there is an element of this span, Σ, such that $\langle (\mathcal{F}\Sigma)(p_\circledast), \Psi(p_\circledast)\rangle_{\mathcal{V}} > 0$. By continuity, there exists a $\varepsilon > 0$ and a neighborhood $U \ni p_\circledast$ such that $\Re\langle \rho(D[p]^{-1})\mathcal{F}\Sigma(p), \Psi(p)\rangle_{\mathcal{V}} > \varepsilon$ for all $p \in U$. Now consider $\Pi(\chi)\Sigma$ for $\chi \in C^0_{\mathrm{c}}(U)$ with $\chi(p_\circledast) = 1$ and $\chi(p) \geq 0$ everywhere. Then, $\mathcal{F}(\Pi(\chi)\Sigma)(p) = \chi(p)\mathcal{F}\Sigma(p)$, so

$$\langle \rho(D[p]^{-1})\mathcal{F}\Pi(\chi)\Sigma, \Psi\rangle_{L^2(O,\mu;\mathcal{V})} = \int_O \chi(p)\langle \rho(D[p]^{-1})\mathcal{F}\Sigma(p), \Sigma(p)\rangle_{\mathcal{V}} \,\mathrm{d}\mu(p) \tag{4.34}$$

has positive real part. But this contradicts $\Psi \in \mathcal{X}^\perp$.

What we have shown is that $\Psi \in \mathcal{Y} \Rightarrow \Psi(p_\circledast) = 0$. But there is nothing special about $p_\circledast$ (in the open set U_0 mentioned above), so $\Psi = 0$ identically. Thus, $\mathcal{X}$ is all of $L^2(O,\mu;\mathcal{V})$. □

We can therefore "transport" our original representation ρ on $\mathcal{H}$ to one on the image of U (which will end up being all of $L^2(O,\mu,\mathcal{V})$):

$$\mathrm{P}^*(1,d) \ni T \mapsto U\rho(T)U^{-1} \in \mathrm{U}(L^2(O,\mu;\mathcal{V})). \tag{4.35}$$

[1] We will talk as though $D[p]$ is defined for all $p \in O$, though we actually only need $D[p]$ to be defined for almost all $p \in O$.

Note that $L^2(O,\mu;\mathcal{V})$ carries no *obvious* representation of $\mathrm{P}^*(1,d)$, only the (usually proper) little group (including spacetime translations). Thus, we have no guess for what $U\rho(\Lambda)U^{-1}$ had ought to be. But we can just work it out:

PROPOSITION 4.10. *For any* $\Psi \in L^2(O,\mu;\mathcal{V})$ *and* $T = (a,\Lambda) \in \mathrm{P}^*$,

$$(U\rho(T)U^{-1}\Psi)(p) = e^{i\langle a,p\rangle}\rho(D[p]^{-1}\Lambda D[\Lambda^{-1}p])\Psi(\Lambda^{-1}p). \tag{4.36}$$

■

PROOF. It suffices to prove the proposition for Ψ of the form $\Psi = U\psi$ for $\psi \in \mathcal{D}$, since these are dense in $L^2(O,\mu;\mathcal{V})$ (by above). For such Ψ, our goal is to prove

$$(U\rho(T)\psi)(p) = e^{i\langle a,p\rangle}\rho(D[p]^{-1}\Lambda D[\Lambda^{-1}p])((U\psi)(\Lambda^{-1}p)). \tag{4.37}$$

Now we compute the left-hand side:

$$\begin{aligned}(U\rho(T)\psi)(p) = \rho(D[p]^{-1})\mathcal{F}(\rho(T)\psi)(p) &\overset{\text{(Prop. 4.7)}}{=} \rho(D[p]^{-1})\rho(T)\mathcal{F}\psi(\Lambda^{-1}p)\\ &= \rho(D[p]^{-1}T)\mathcal{F}\psi(\Lambda^{-1}p)\\ &= \rho(D[p]^{-1}TD[\Lambda^{-1}p])\rho(D[\Lambda^{-1}p]^{-1})\mathcal{F}\psi(\Lambda^{-1}p)\\ &= \rho(D[p]^{-1}TD[\Lambda^{-1}p])((U\psi)(\Lambda^{-1}p)).\end{aligned} \tag{4.38}$$

Now write $T = T_a\Lambda = \Lambda T_{\Lambda^{-1}a}$. The operator above is

$$\begin{aligned}\rho(D[p]^{-1}TD[\Lambda^{-1}p]) &= \rho(D[p]^{-1}\Lambda T_{\Lambda^{-1}a}D[\Lambda^{-1}p])\\ &= \rho(D[p]^{-1}\Lambda D[\Lambda^{-1}p]T_{D[\Lambda^{-1}p]^{-1}\Lambda^{-1}a})\\ &= \rho(D[p]^{-1}\Lambda D[\Lambda^{-1}p])\rho(T_{D[\Lambda^{-1}p]^{-1}\Lambda^{-1}a}).\end{aligned} \tag{4.39}$$

In eq. (4.38), this is applied to $(U\psi)(\Lambda^{-1}p)$, which is an element of $\mathcal{H}[p_{\circledast}]$. The effect of applying a translation $\rho(T_b)$ to $(U\psi)(\Lambda^{-1}p)$ is therefore to multiply by $e^{i\langle b,p_{\circledast}\rangle}$. Thus,

$$\begin{aligned}\rho(T_{D[\Lambda^{-1}p]^{-1}\Lambda^{-1}a})((U\psi)(\Lambda^{-1}p)) &= e^{i\langle D[\Lambda^{-1}p]^{-1}\Lambda^{-1}a,p_{\circledast}\rangle}(U\psi)(\Lambda^{-1}p)\\ &= e^{i\langle \Lambda^{-1}a,D[\Lambda^{-1}p]p_{\circledast}\rangle}(U\psi)(\Lambda^{-1}p)\\ &= e^{i\langle \Lambda^{-1}a,\Lambda^{-1}p\rangle}(U\psi)(\Lambda^{-1}p) = e^{i\langle a,p\rangle}(U\psi)(\Lambda^{-1}p).\end{aligned} \tag{4.40}$$

Putting everything together, we have proven eq. (4.37). □

This looks god-awful, but let us press ahead. We can simplify the final formula somewhat by defining

$$W(\Lambda,p) = D[p]^{-1}\Lambda D[\Lambda^{-1}p], \tag{4.41}$$

in terms of which the formula reads

$$U\rho(T)U^{-1}\Psi(p) = e^{i\langle a,p\rangle}\rho(W(\Lambda,p))[p_{\circledast}]\Psi(\Lambda^{-1}p). \tag{4.42}$$

We have restored the '$[p_{\circledast}]$' that we have been notationally suppressing so as to emphasize the fact that $\Psi(\Lambda^{-1}p)$ lies in $\mathcal{V} = \mathcal{H}[p_{\circledast}]$, and thus, strictly speaking, Lorentz transformations act on it via the representation $\rho(-)[p_{\circledast}]$ induced on $\mathcal{V}$ by ρ and not by ρ itself.

The Lorentz transformation $W(\Lambda,p)$ lies in the little group:

PROPOSITION 4.11. *For any* $\Lambda \in \mathrm{Spin}(1,d)$*, we have* $W(\Lambda,p) \in L$. ■

PROOF. Straightforwardly,

$$W(\Lambda,p)p_{\circledast} = D[p]^{-1}\Lambda D[\Lambda^{-1}p]p_{\circledast} = D[p]^{-1}\Lambda\Lambda^{-1}p = D[p]^{-1}p = p_{\circledast}. \tag{4.43}$$

□

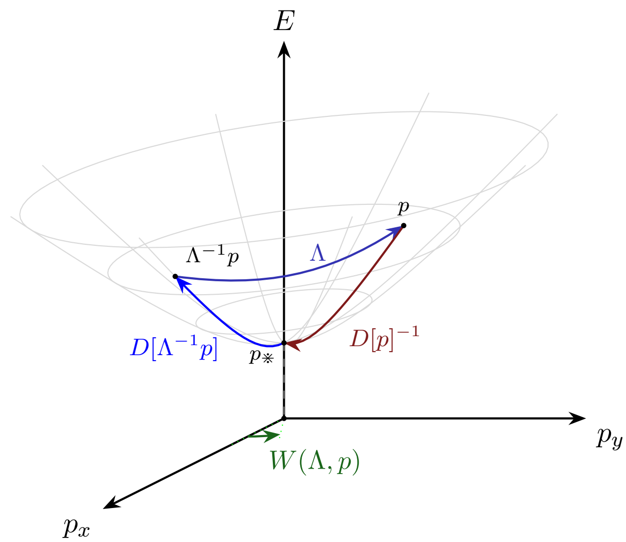


FIGURE 4.1. The Wigner rotation (massive case).

The Lorentz "matrix"[2] $W(\Lambda,p)$ is called a *Wigner rotation.* It depends on p_* through $D[p]$, but this dependence is left implicit.

The use of the term "rotation" is motivated by the $O = X_{m,+}$ case. When $p_* = (m, \mathbf{0})$ for some $m > 0$, then the stabilizer L is naturally $\mathrm{Spin}(d)$. Thus,

$$W(\Lambda,p) \in \mathrm{Spin}(d) \tag{4.44}$$

as well. So, $W(\Lambda,p)$ is "literally" a rotation, except it may be spinorial. In the massless case, $m = 0$, the little group will be seen to consist of some rotations and some Lorentz transformations known as null rotations. They are not literally rotations of either space or spacetime, but the terminology is standard.

Because the Wigner rotation $W = W(\Lambda,p)$ lies in the little group, eq. (4.42) can be rewritten compactly as:

$$\boxed{U\rho(T)U^{-1}\Psi(p) = e^{i\langle a,p\rangle}\varsigma(W)\Psi(\Lambda^{-1}p).} \tag{4.45}$$

Note that the little group representation ς has arisen. To summarize:

THEOREM. *Given the setup above, the primary representation ρ must be unitarily equivalent to a representation on $L^2(O,\mu;\mathcal{V})$ of the form*

$$(T\Psi)(p) = e^{i\langle a,p\rangle}\varsigma(W)\Psi(\Lambda^{-1}p) \tag{4.46}$$

for $T = (a,\Lambda)$. The intertwining map is the U above. ■

So, it suffices to restrict attention to representations of this form. Note: we have not proven that there actually *exists* a representation with spectrum $\sigma = \bar{O}$ and little group representation ς. We deduced that any such representation must be equivalent to one of a particular form, but it could be the case that eq. (4.46) fails to define a valid representation, in which case we would have reached a contradiction. That would mean that no representation exists with the prescribed spectrum and little group representation. The first thing we will do in the next subsection is prove that the formula above actually does define a valid representation.

[2] Using scare quotes because $W(\Lambda,p)$ is technically an element of $\widetilde{\mathrm{SO}(1,d)}$, not $\mathrm{SO}(1,d)$ itself.

2.5. Facts about induced representations. To repeat the definition:

Let

- $O \in \{\Omega, V_\pm, X_{m,\pm}, Y_\Gamma : m, \Gamma > 0\}$ and $L \subset \mathrm{Spin}(1,d)$ be the corresponding little group, and
- $\varsigma : L \to \mathrm{U}(\mathcal{V})$ denote a unitary representation of L on some Hilbert space $\mathcal{V}$.

Assume that a standard boost $D[p] : p_\circledast \mapsto p$ has been chosen so as to depend Borel measurably on $p \in O$. Consider the Hilbert space

$$\mathcal{H} = L^2(O, \mu; \mathcal{V}). \tag{4.47}$$

For each $a \in \mathbb{R}^{1,d}$ and $\Lambda \in \mathrm{Spin}(1,d)$, consider the operator $\pi(a,\Lambda) \in \mathrm{U}(\mathcal{H})$ on $\mathcal{H}$ defined by

$$(\pi(a,\Lambda)\Psi)(p) = e^{i\langle a,p\rangle}\varsigma(W(\Lambda,p))\Psi(\Lambda^{-1}p), \tag{4.48}$$

where $W(\Lambda,p) = D[p]^{-1}\Lambda D[\Lambda^{-1}p]$.

PROPOSITION 4.12. *Then, $(a,\Lambda) \mapsto \pi(a,\Lambda)$ defines a continuous unitary representation of* $\mathrm{P}^*(1,d)$ ■

PROOF. Evidently, $\pi(a,\Lambda)$ is unitary, and depends continuously on a, Λ with respect to the strong-operator topology. That this defines a representation of P^* is the following calculation:

$$\begin{aligned}(\pi(a,\Lambda)\pi(a',\Lambda')\Psi)(p) &= e^{i\langle a,p\rangle}\varsigma(W(\Lambda,p))(\pi(a',\Lambda')\Psi)(\Lambda^{-1}p) \\ &= e^{i\langle a,p\rangle}\varsigma(W(\Lambda,p))e^{i\langle a',\Lambda^{-1}p\rangle}\varsigma(W(\Lambda',\Lambda^{-1}p))\Psi((\Lambda\Lambda')^{-1}p) \\ &= e^{i\langle a+\Lambda a',p\rangle}\varsigma(W(\Lambda,p)W(\Lambda',\Lambda^{-1}p))\Psi((\Lambda\Lambda')^{-1}p).\end{aligned} \tag{4.49}$$

Note that $W(\Lambda,p)W(\Lambda',\Lambda^{-1}p) = D[p]^{-1}\Lambda\Lambda' D[(\Lambda\Lambda')^{-1}p] = W(\Lambda\Lambda',p)$. So, the above is

$$\begin{aligned}&= e^{i\langle a,+\Lambda a',p\rangle}\varsigma(W(\Lambda\Lambda',p))\Psi((\Lambda\Lambda')^{-1}p) \\ &= \pi(a+\Lambda a',\Lambda\Lambda')\Psi(p).\end{aligned} \tag{4.50}$$

□

REMARK: If ς descends to a representation of $L_* \subseteq \mathrm{O}(1,d)$, then π descends to a representation of $\mathrm{P}(1,d)$.

The representation defined in the previous theorem is denoted

$$\pi = \mathrm{Ind}_{\mathbb{R}^{1,d}\rtimes L}^{\mathbb{R}^{1,d}\rtimes \mathrm{Spin}(1,d)}(\chi\otimes\varsigma), \tag{4.51}$$

where $\chi\otimes\varsigma : \mathbb{R}^{1,d}\rtimes L \to \mathrm{U}(\mathcal{V})$ is given by $(x,A) \mapsto e^{i\langle x,p\rangle}\varsigma(A)$. See [Fol08], [Tal22] for alternative expositions.

Having succeeded in defining the representation π, one should check that it has the desired
[Exercise 4.3] spectrum $\sigma = \bar{O}$ and little group representation ς. This is left as an exercise.

PROPOSITION 4.13. *π is irreducible if and only if ς is.* ■

PROOF. If ς is reducible, so that there exists some ς-closed subspace $\mathcal{W} \subseteq \mathcal{V}$, then $L^2(O,\mu;\mathcal{W})$ is evidently a π-closed subspace of $\mathcal{H} = L^2(O,\mu;\mathcal{V})$.

Conversely, suppose that π is reducible, so that $\mathcal{H} = \mathcal{X}\oplus\mathcal{Y}$ for proper subrepresentations $\mathcal{X},\mathcal{Y}$. These have to have the same spectrum. The space $\mathcal{H}[p_\circledast] \cong \mathcal{V}$ splits as

$$\mathcal{H}[p_\circledast] \cong \mathcal{X}[p_\circledast] \oplus \mathcal{Y}[p_\circledast], \tag{4.52}$$

each summand in which is ς-closed. We know that each $\mathcal{Z} = \mathcal{X},\mathcal{Y}$ is unitarily equivalent to a subspace of $L^2(O,\mu;\mathcal{Z}[p_\circledast])$, so $\mathcal{Z}[p_\circledast]$ must be a proper subspace of $\mathcal{H}[p_\circledast]$. We conclude that $\mathcal{H}[p_\circledast]$, and hence $\mathcal{V}$, is reducible as a ς-representation. □

REMARK: Similarly, π is primary if and only if ς is. [Problem 4.2]

REMARK: The definition of eq. (4.48) does not depend on the choice of standard boosts $D[p] \in \mathrm{Spin}(1,d)$, whose only defining properties (besides depending continuously on p) were that $D[p]p_{\divideontimes} = p$. Call an alternative $\tilde{D}[p]$, leading to the alternative Wigner rotation $\tilde{W}(\Lambda, p)$. Then, we can rewrite eq. (4.48) as

$$\begin{aligned}(U(a,\Lambda)\Psi)(p) &= e^{i\langle a,p\rangle} \\ &\times \varsigma(D[p]^{-1}\tilde{D}[p])\varsigma(\underbrace{\tilde{D}[p]^{-1}\Lambda\tilde{D}[\Lambda^{-1}p]}_{\tilde{W}(\Lambda,p)})\varsigma(\tilde{D}[\Lambda^{-1}p]^{-1}D[\Lambda^{-1}p])\Psi(\Lambda^{-1}p).\end{aligned} \tag{4.53}$$

Note that this makes sense because

$$D[p]^{-1}\tilde{D}[p], \tilde{D}[\Lambda^{-1}p]^{-1}D[\Lambda^{-1}p] : p_{\divideontimes} \mapsto p_{\divideontimes} \tag{4.54}$$

and are therefore in the little group. So, if we define $\tilde{\Psi}(p) = \varsigma(\tilde{D}[p]^{-1}D[p])\Psi(p)$, then the unitary map $\Psi \mapsto \tilde{\Psi}$ intertwines the original representation with that constructed using $\tilde{D}[p]$.

So, every pair of a Lorentz orbit O and little group irrep ς is realized by some irrep of $\mathrm{P}^*(1,d)$. Combining this with previous propositions, we have a complete classification of irreps: *for each Lorentz orbit O and corresponding little group irrep ς, we have* (modulo unitary equivalence) *exactly one irrep of* $\mathrm{P}^*(1,d)$ *with those two pieces of data*, and it is the induced representation

$$\mathrm{Ind}_{\mathbb{R}^{1,d}\rtimes L}^{\mathbb{R}^{1,d}\rtimes \mathrm{Spin}(1,d)}(\chi\otimes\varsigma) \tag{4.55}$$

defined above.

2.6. An alternative construction. The homeomorphism $O \cong \mathrm{Spin}(1,d)/L$ suggests attempting to build the representation in question as consisting of $\mathcal{V}$-valued functions on $\mathrm{Spin}(1,d)$ satisfying some kind of L-invariance that allows us to pass to functions on the quotient. The correct notion is "equivariance" — a function $\Psi : \mathrm{Spin}(1,d) \to \mathcal{V}$ is called *L-equivariant* if

$$\Psi(\Lambda A^{-1}) = \varsigma(A)\Psi(\Lambda) \tag{4.56}$$

for all $\Lambda \in \mathrm{Spin}(1,d)$ (or almost all Λ) and $A \in L$.

Given a function $\Psi : \mathrm{Spin}(1,d) \to \mathcal{V}$, a natural action of $\Lambda \in \mathrm{Spin}(1,d)$ on Ψ is

$$(\Lambda\Psi)(\Lambda_0) = \Psi(\Lambda^{-1}\Lambda_0). \tag{4.57}$$

This suggests considering the big Hilbert space $\mathcal{H}_{\mathrm{big}} = L^2(\mathrm{Spin}(1,d);\mathcal{V})$. For any Λ, the map $\Psi \mapsto \Lambda\Psi$ is unitary by virtue of the invariance of the Haar measure. So, we get a manifest continuous unitary representation of $\mathrm{Spin}(1,d)$. For any $p_{\divideontimes}$, we can extend this to a representation $\tilde{U}$ of the full Poincaré group by writing

$$(\tilde{U}(a,\Lambda)\Psi)(\Lambda_0) = e^{i\langle a,\Lambda_0 p_{\divideontimes}\rangle}\Psi(\Lambda^{-1}\Lambda_0). \tag{4.58}$$

Clearly, $\tilde{U}(a,\Lambda)$ is unitary and depends continuously on a, Λ.

PROPOSITION 4.14. *$\tilde{U}$ is a representation of* P^*. ■

PROOF. Calculate

$$\begin{aligned}(\tilde{U}(a,\Lambda)\tilde{U}(a',\Lambda')\Psi)(\Lambda_0) &= e^{i\langle a,\Lambda_0 p_{\divideontimes}\rangle}(\tilde{U}(a',\Lambda')\Psi)(\Lambda^{-1}\Lambda_0) \\ &= e^{i\langle a,\Lambda_0 p_{\divideontimes}\rangle}e^{i\langle a',\Lambda^{-1}\Lambda_0 p_{\divideontimes}\rangle}(\Psi((\Lambda\Lambda')^{-1}\Lambda_0)) \\ &= e^{i\langle a+\Lambda a',\Lambda_0 p_{\divideontimes}\rangle}(\Psi((\Lambda\Lambda')^{-1}\Lambda_0)) = (\tilde{U}(a+\Lambda a',\Lambda\Lambda')\Psi)(\Lambda_0).\end{aligned} \tag{4.59}$$

□

Note that we have not yet assumed that L is the stabilizer of $p_{\circledast}$.

Let $\mathcal{H}_{\text{small}}$ denote the subspace of $\mathcal{H}_{\text{big}}$ consisting of L-equivariant functions. It is clear from the definition of equivariance (because the A^{-1} in ΛA^{-1} is on the *right*) that

$$\Psi \ L\text{-equivariant} \Longrightarrow (\Lambda\Psi) = \Psi \circ \Lambda^{-1} \ L\text{-equivariant}, \tag{4.60}$$

so $\mathcal{H}_{\text{small}}$ is a P^*-subrepresentation of $\mathcal{H}_{\text{big}}$.

PROPOSITION 4.15. *This subrepresentation is unitarily equivalent to the one defined above.* ■

PROOF. See [Tal22, §9.8]. □

3. The massive case

Consider the Lorentz orbit $O = X_{m,+}$, for $m > 0$. Recall that this is

$$X_{m,+} = \{(\omega, \mathbf{p}) \in \mathbb{R}^{1,d} : \omega = \sqrt{m^2 + \|\mathbf{p}\|^2}\}, \tag{4.61}$$

one sheet of a hyperboloid. The most natural choice of reference momentum $p_{\circledast} \in X_{m,+}$ is $p_{\circledast} = (m, \mathbf{0}) \in X_{m,+}$. In this section, we discuss the (spinorial) Poincaré-irreps $\pi_{\bullet}$ whose spectrum is O. By the classification theorem above, we get exactly one irrep from each irrep ς of the little group

$$L = \{\Lambda \in \mathrm{Spin}(1, d) : \Lambda p_{\circledast} = p_{\circledast}\}, \tag{4.62}$$

the stabilizer of $p_{\circledast}$ in $\mathrm{Spin}(1, d)$. In addition, we will write down explicitly the Lorentz-invariant measure μ on $X_{m,+}$ and the standard boosts $D[p]$. These are required to make explicit the description of $\pi_{\bullet}$.

3.1. Little group and its representations.

Recall that $\mathrm{Spin}(d)$ denotes the universal cover of $\mathrm{SO}(d)$, for $d \geq 2$.

PROPOSITION 4.16. *$L \cong \mathrm{Spin}(d)$ if $d \geq 3$, and $L \cong (\mathbb{R}, +)$ if $d = 2$.* ■

PROOF. The stabilizer of $p_{\circledast}$ in the restricted Lorentz group, L_*, is $\mathrm{SO}(d)$. The group L is the pre-image of L_* under the universal cover. □

So, the method of induced representations has reduced the classification of irreps of the spinorial Poincaré group to that of irreps of the Lie group $\mathrm{Spin}(d)$, which is compact, unless $d = 2$. The irreps of $\mathrm{Spin}(d)$ are well-understood.

The physically-relevant $d = 3$ case, in which $\mathrm{Spin}(3) = \mathrm{SU}(2)$ can be exposited explicitly. For each $j \in \mathbb{N}^+$, we have exactly one j-dimensional irrep, labeled $\mathbf{j}$, and referred to as the "spin-s", $s = (j-1)/2$, irrep. In particle physics, one rarely needs the irreps $\mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \ldots$. The low-dimensional examples $\mathbf{1}, \mathbf{2}, \mathbf{3}$ can all be described simply:

- $\mathbf{1}$ is the trivial representation.
- $\mathbf{2}$ is the defining representation of $\mathrm{SU}(2)$.
- $\mathbf{3}$ is the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

(If j is odd, then $\mathbf{j}$ has a real structure. Otherwise, if j is even, it has a quaternionic structure.)

Explicit formulas for general j are given by the *Wigner D-matrices*.

So, we have, for each $s \in 2^{-1}\mathbb{N}$, an irrep $\pi_{m,s}$ of the spinorial Poincaré group $\mathrm{P}^*(1, d)$. The rest of this section is devoted to describing this representation more explicitly.

3.2. Invariant measure. Up to a multiplicative constant, the Lorentz-invariant measure $\mu = \mu_{X_{m,+}}$ on $\mathbb{R}^{1,d}$ supported on $X_{m,+}$ is given by

$$\mathrm{d}\mu(p) = \Theta(p_0)\delta(p^2+m^2)\,\mathrm{d}^{1+d}p, \tag{4.63}$$

where $\Theta(\omega) = 1_{\omega>0}$ is a Heaviside step function. The δ-function $\delta(p^2+m^2)$ is either defined as or, if defined in some other way, is proven to be equal to

$$\delta\Big(p_0 \mp \sqrt{m^2+\|\mathbf{p}\|^2}\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}p_0}p^2\Big|_{p^2=m^2}\Big)^{-1} = \frac{\delta\big(p_0-\sqrt{m^2+\|\mathbf{p}\|^2}\big)}{2(m^2+\|\mathbf{p}\|^2)^{1/2}}. \tag{4.64}$$

So, we have the following concrete formula for μ: if $f \in L^1(X_{m,+},\mu)$, then

$$\int_{X_{m,+}} f(p)\,\mathrm{d}\mu(p) = \int_{\mathbb{R}^3} \frac{f\big((m^2+\|\mathbf{p}\|^2)^{1/2},\mathbf{p}\big)\,\mathrm{d}^d\mathbf{p}}{2(m^2+\|\mathbf{p}\|^2)^{1/2}}. \tag{4.65}$$

The manipulations of δ-functions above are rigorously justifiable, but the reader may prefer to take eq. (4.65) as the *definition* of μ.

Why is μ Lorentz-invariant? If we know how to make sense of eq. (4.63), then this is more or less obvious, since the terms

$$\mathrm{sign}(p_0), p^2,\ \mathrm{d}^{1+d}p \tag{4.66}$$

out of which $\mathrm{d}\mu = \Theta(p_0)\delta(p^2+m^2)\,\mathrm{d}^{1+d}p$ is built are individually invariant. (The invariance of the Lebesgue measure $\mathrm{d}^{1+d}p$ on $\mathbb{R}^{1+d}$ is due to special Lorentz matrices having unit determinant.) However, we would like to verify Lorentz-invariance without needing any general theory of δ-functions. It should be possible to deduce this directly from the formula/alternative definition eq. (4.65). Indeed:

PROPOSITION 4.17. *For any orthochronous, orthochorous Lorentz matrix Λ and $f \in L^1(X_{m,+},\mu)$,*

$$\int_{X_{m,+}} f(p)\,\mathrm{d}\mu(p) = \int_{X_{m,+}} f(\Lambda p)\,\mathrm{d}\mu(p). \tag{4.67}$$

That is, the measure μ is Lorentz-invariant. ■

PROOF. Since the definition eq. (4.65) is manifestly *rotation* invariant, and since rotations together with boosts in a single direction generate the whole Lorentz group, it suffices to consider the boost

$$\Lambda = \begin{pmatrix} c & s & 0 \\ s & c & 0 \\ 0 & 0 & I_{d-1}\end{pmatrix} \tag{4.68}$$

for $c=\cosh(\beta)$, $s=\sinh(\beta)$, $\beta\in\mathbb{R}$. Then, by definition,

$$\int_{X_{m,+}} f(\Lambda p)\,\mathrm{d}\mu(p) = \int_{\mathbb{R}^d} f\big(cp^0(\mathbf{p})+sp^1, sp^0(\mathbf{p})+cp^1,p^2,\cdots\big)\frac{\mathrm{d}^d\mathbf{p}}{2(m^2+\|\mathbf{p}\|^2)^{1/2}}, \tag{4.69}$$

where $p^0(\mathbf{p}) = \pm(m^2+\|\mathbf{p}\|^2)^{1/2}$. Let $q^1 = sp^0(\mathbf{p})+cp^1$, $q^2=p^2,\cdots$. Then,

$$\begin{aligned} cp^0(\mathbf{p})+sp^1 &= (m^2+\|\mathbf{q}\|^2)^{1/2}, \\ \mathrm{d}^d\mathbf{p} &= (-sq^1(m^2+\|\mathbf{q}\|^2)^{-1/2}+c)\,\mathrm{d}^d\mathbf{q}, \\ p^0(\mathbf{p}) &= c(m^2+\|\mathbf{q}\|^2)^{1/2}-sq^1. \end{aligned} \tag{4.70}$$

These give

$$\frac{\mathrm{d}^d\mathbf{p}}{(m^2+\|\mathbf{p}\|^2)^{1/2}} = \frac{\mathrm{d}^d\mathbf{q}}{(m^2+\|\mathbf{q}\|^2)^{1/2}}, \tag{4.71}$$

and consequently

$$\int_{X_{m,+}} f(\Lambda p)\,\mathrm{d}\mu(p) = \int_{\mathbb{R}^d} \frac{f\big((m^2+\|\mathbf{q}\|^2)^{1/2},\mathbf{q}\big)\,\mathrm{d}^d\mathbf{q}}{2(m^2+\|\mathbf{q}\|^2)^{1/2}} = \int_{X_{m,+}} f(q)\,\mathrm{d}\mu(q). \tag{4.72}$$

□

3.3. Standard boosts and Wigner rotations. Given some other $p \in O$, the standard boost $D[p]$ is supposed to take p_{*} to $p = ((m^2+\|\mathbf{p}\|^2)^{1/2},\mathbf{p})$, $\mathbf{p} \neq 0$. The standard boost $\Lambda(-\mathbf{v})$ has this property for $\mathbf{v} = (m^2+\|\mathbf{p}\|^2)^{-1/2}\mathbf{p}$.

Note that this depends continuously on $\mathbf{p}$, as we required.

We now compute the Wigner rotation $W(\Lambda,p) = D[p]^{-1}\Lambda D[\Lambda^{-1}p]$. We consider two cases, (i) Λ a rotation, (ii) Λ a pure boost.

(i) When $\Lambda = R$ is a rotation, then $D[\Lambda^{-1}p] = \Lambda(-R^{-1}\mathbf{v}) = R^{-1}\Lambda(-\mathbf{v})R$, so the Wigner rotation is
$$W(R,p) = (\Lambda(-\mathbf{v}))^{-1}R(R^{-1}\Lambda(-\mathbf{v})R) = \Lambda(\mathbf{v})\Lambda(-\mathbf{v})R = R, \tag{4.73}$$
just r itself.

(ii) When $\Lambda = \Lambda(\mathbf{w})$ is a pure boost, $D[p]^{-1}\Lambda = \Lambda(\mathbf{v})\Lambda(\mathbf{w}) = W\Lambda(\mathbf{v}\oplus\mathbf{w})$, by the velocity addition formula Problem 1.3, where W is the same matrix called the Wigner rotation there.

Above, we actually wanted $D[p]$ to be an element of $\mathrm{Spin}(1,d)$, not $\mathrm{SO}(1,d)$, but this is easily amended, as the map $p \mapsto \Lambda(-\mathbf{v})$ can be lifted to $\mathrm{Spin}(1,d)$.

4. The massless case

4.1. Invariant measure. The construction of the invariant measure $\mu = \mu_{V_\pm}$ is the same in the massless case as the massive case:

$$\mathrm{d}\mu(p) = \Theta(p_0)\delta(p^2)\,\mathrm{d}^{1+d}p, \tag{4.74}$$

except it is a bit more difficult to make rigorous sense of $\delta(p^2)$, as compared to $\delta(p^2+m^2)$, since the zero set of p^2 is not smooth. It is a bicone, which fails to be a submanifold of $\mathbb{R}^{1,d}$ at the origin, whereas the zero set of p^2-m^2 is a two-sheeted hyperboloid. Making sense of the product $\Theta(p_0)\delta(p^2)$ is also not automatic, because the singular supports of the individual factors $\Theta(p_0)$, $\delta(p^2)$ are not disjoint.

So, we just take eq. (4.65), with $m=0$, as our definition: μ is the Borel measure on $V_\pm$ such that

$$\int_{V_+} f(p)\,\mathrm{d}\mu(p) = \int_{\mathbb{R}^d} f(\|\mathbf{p}\|,\mathbf{p})\frac{\mathrm{d}^d\mathbf{p}}{2\|\mathbf{p}\|} \tag{4.75}$$

for all $f \in C^0_{\mathrm{c}}(\mathbb{R}^{1,d})$. The same computation as in the massive case shows that μ, defined in this way, is Lorentz-invariant.

4.2. Little group. As our reference momentum $p_{*} \in O$, we can take $p_{*} = (1,1,0,\dots)$.

Unlike in the massive case, the stabilizer L_* of p_{*} in $\mathrm{SO}(1,d)$ requires some work to compute. The obvious members of L_* are those Lorentz transformations

$$I_2 \oplus R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}, \quad R \in \mathrm{SO}(d-1) \tag{4.76}$$

which fix the t- and x^1-axes and rotate the remaining variables. Perhaps surprisingly, these are not the only elements of L_* (excluding the $d=1$ case, as we are assuming that $d \geq 2$). A natural candidate to try would be a boost along the x-axis, but this doesn't work — no nontrivial boost along the x-axis preserves p_*. So, any remaining elements of the little group would have to mix x^1 with the other spatial components $x^2, x^3, \cdots$ in some interesting way.

EXAMPLE 4.18 ($d = 2$ case). The most general 3-by-3 real-valued matrix fixing $p_* = (+1, 1, 0)$ is

$$\Lambda = \begin{pmatrix} a & 1-a & b \\ c & 1-c & d \\ e & -e & f \end{pmatrix}, \tag{4.77}$$

where $a, b, c, d, e, f \in \mathbb{R}$. The question is: for which values of these parameters is Λ a proper, orthochronous Lorentz matrix? For each $w \in \mathbb{R}^3$, we get a linear constraint from equating $\langle w, p_* \rangle$ with $\langle \Lambda w, \Lambda p_* \rangle = \langle \Lambda w, p_* \rangle$.

- Taking $w = (1, 0, 0)$, in which case $\langle w, p_* \rangle = -1$, we get the constraint $c = a - 1$.
- Taking $w = (0, 1, 0)$ yields the same constraint.
- Taking $w = (0, 0, 1)$ yields the constraint $d = b$.

So,

$$\Lambda = \begin{pmatrix} a & 1-a & b \\ a-1 & 2-a & b \\ e & -e & f \end{pmatrix}. \tag{4.78}$$

For each $w, v \in \mathbb{R}^3$, we get a quadratic constraint from equating $\langle w, v \rangle$ and $\langle \Lambda w, \Lambda v \rangle$.

- Taking $w, v = (0, 0, 1)$, we get $f = \pm 1$.
- Then, taking $w = (1, 0, 0)$ and $v = (0, 0, 1)$, we get $e = \pm b$. The sign here is the same as that of f.
- Finally, taking $w, v = (1, 0, 0)$, we get $a = 1 + e^2/2$.

We could continue in this way, but we would not get any more constraints. Anyways, we only have a single degree-of-freedom e, left, in addition to an undetermined sign:

$$\Lambda = \begin{pmatrix} 1+\zeta & -\zeta & \pm e \\ \zeta & 1-\zeta & \pm e \\ e & -e & \pm 1 \end{pmatrix} \tag{4.79}$$

for $\zeta = e^2/2$. This turns out to be a Lorentz matrix (which is why trying different pairs w, v of vectors above fails to yield new constraints), as some straightforward algebra reveals. [Exercise 4.5]

We have not done anything to guarantee that Λ is proper or orthochronous. Apparently, orthochronicity is automatic, since $\Lambda^0_0 = 1 + \zeta > 0$. Propriety must depend on the sign $\pm$, since the two possible choices lead to opposite determinants. If $e = 0$, then we get a positive determinant if the sign is $+$. Considering the whole continuous family of Lorentz matrices $\Lambda(e)$, the sign of the determinant cannot change (since Lorentz matrices have determinant ± 1), so the previous conclusion must hold for all e. ■

The discussion above generalizes:

PROPOSITION 4.19. *Let $d \geq 2$. For each $\mathbf{e} \in \mathbb{R}^d$ and $R \in \mathrm{SO}(d-1)$, let*

$$N(\mathbf{e}, R) = \begin{pmatrix} 1+\zeta & -\zeta & \mathbf{e}^\intercal R \\ \zeta & 1-\zeta & \mathbf{e}^\intercal R \\ \mathbf{e} & -\mathbf{e} & R \end{pmatrix}, \tag{4.80}$$

where $\zeta = \|\mathbf{e}\|^2/2$. Then, $L_*(p_*) = \{N(\mathbf{e}, R) : \mathbf{e} \in \mathbb{R}^d, R \in \mathrm{SO}(d-1)\}$. ■

The matrix $N(\mathbf{e}, I_{d-1})$ is what is known as a *null rotation*.

PROOF. The claim is that if Λ is a proper, orthochronous Lorentz matrix such that $\Lambda p_* = p_*$, then there exist $\mathbf{e} \in \mathbb{R}^d$ and $R \in \mathrm{SO}(d-1)$ such that $\Lambda = N(\mathbf{e}, R)$.

The computation for general $d \geq 2$ is almost identical to the $d = 2$ case discussed above, so let us just explain where the R comes from. After going through the linear constraints on the entries of Λ imposed by $\langle \Lambda \bullet, p_{\circledast} \rangle = \langle \bullet, p_{\circledast} \rangle$, we get that

$$\Lambda = \begin{pmatrix} a & 1-a & \mathbf{b}^\intercal \\ a-1 & 2-a & \mathbf{b}^\intercal \\ \mathbf{e} & -\mathbf{e} & R \end{pmatrix}. \tag{4.81}$$

for some $a \in \mathbb{R}$, $\mathbf{b}, \mathbf{e} \in \mathbb{R}^{d-1}$, $R \in \mathbb{R}^{(d-1)\times(d-1)}$. Equating $w^2 = (\Lambda w)^2$ for $w = (1, 0, 0, \dots)$ yields $a = 1 + \zeta$ for $\zeta = \|\mathbf{e}\|^2/2$. Now taking $w = (0, 0, \mathbf{z})$ for $\mathbf{z} \in \mathbb{R}^{d-1}$ and equating $(\Lambda w)^2 = w^2$, we get that $\|R\mathbf{z}\|^2 = \|\mathbf{z}\|^2$. So, $R \in \mathrm{O}(d-1)$. To see that

$$\mathbf{b} = R^\intercal \mathbf{e} = R^{-1}\mathbf{e}, \tag{4.82}$$

equate $0 = \langle w, v \rangle = \langle \Lambda w, \Lambda v \rangle$ for that same w and $v = (1, 0, \dots)$. The Lorentz product $\langle \Lambda w, \Lambda v \rangle$ is given explicitly by

$$\langle \Lambda w, \Lambda v \rangle = -\mathbf{b}\cdot\mathbf{z} + \mathbf{e}\cdot R\mathbf{z}. \tag{4.83}$$

So, $\mathbf{b}\cdot\mathbf{z} = \mathbf{e}\cdot R\mathbf{z} = (R^\intercal \mathbf{e})\cdot\mathbf{z}$. Since this holds for all $\mathbf{z} \in \mathbb{R}^{d-1}$, we must have $\mathbf{b} = R^\intercal \mathbf{e}$.

To see that $N(\mathbf{e}, R)$ is a Lorentz matrix, we can use $N(\mathbf{e}, R) = N(\mathbf{e}, I_{d-1})N(\mathbf{0}, R)$. The second factor, $N(\mathbf{0}, R)$, is a spatial rotation and therefore a Lorentz matrix, and to show that $N(\mathbf{e}, I_{d-1})$ is a Lorentz matrix, it suffices (via a coordinate change) to consider $\mathbf{e} = (e, 0, \dots)$. Then,

$$N(\mathbf{e}, I_{d-1}) = \begin{pmatrix} a & 1-a & e & \mathbf{0}^\intercal \\ a-1 & 2-a & e & \mathbf{0}^\intercal \\ e & -e & 1 & \mathbf{0}^\intercal \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{d-2} \end{pmatrix}. \tag{4.84}$$

The check that $N(\mathbf{e}, I_{d-1})$ is a Lorentz matrix therefore reduces to the $d = 2$ case above.

Finally, note that Λ is automatically orthochronous, and it is proper if and only if $R \in \mathrm{SO}(d-1)$. Indeed, $N(\mathbf{e}, I_{d-1})$ is proper, and $N(\mathbf{0}, R)$ is proper if and only if $R \in \mathrm{SO}(d-1)$. So, $N(\mathbf{e}, R)$ is proper and orthochronous if and only if R is orientation-preserving. □

For the rest of the section, $d \geq 3$.

The spinorial little group

$$L = \pi^{-1}(L_*) \tag{4.85}$$

is then the pre-image of all of the null rotations $N(\mathbf{e}, R)$ under the cover $\pi : \mathrm{Spin}(1, d) \twoheadrightarrow \mathrm{SO}(1, d)$.

Example 4.20. For concrete computations in the $d = 3$ case, the choice $p_{\circledast} = (1, 0, 0, 1)$ is convenient with the matrix

$$I_2 + \sigma_3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.86}$$

and $L(p_{\circledast}) \subseteq \mathrm{SL}(2, \mathbb{C})$ is the set of $S \in \mathrm{SL}(2, \mathbb{C})$ such that $S(I_2 + \sigma_3) = (I_2 + \sigma_3)S^{-1\dagger}$. Concretely, for $S \in \mathrm{SL}(2, \mathbb{C})$,

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow S^{-1\dagger} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}, \tag{4.87}$$

so the condition for S to be in $L[p_{\circledast}]$ becomes

$$\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} d^* & -c^* \\ 0 & 0 \end{pmatrix}. \tag{4.88}$$

Thus, we have $c = 0$, $d = a^*$, and b is unconstrained. Note that this implies $\det S = |a|^2$, so if S is to lie in $\mathrm{SL}(2, \mathbb{C})$, then we must have $|a| = 1$. We conclude that

$$L[p_{\circledast}] = \left\{ \begin{pmatrix} a & b \\ 0 & a^* \end{pmatrix} : a, b \in \mathbb{C},\ |a| = 1 \right\}. \tag{4.89}$$

■

Let $\mathrm{E}(j) = \mathbb{R}^j \rtimes \mathrm{SO}(j)$ denote the group of orientation-preserving isometries of j-dimensional Euclidean space, and let

$$\mathrm{E}^*(j) = \mathbb{R}^j \rtimes \mathrm{Spin}(j) \tag{4.90}$$

denote its double cover. (We will also use $\pi : \mathrm{E}^*(j) \twoheadrightarrow \mathrm{E}(j)$ to denote the covering map.)

PROPOSITION 4.21. *The little groups L_*, L are isomorphic to $\mathrm{E}(d-1)$, $\mathrm{E}^*(d-1)$, respectively. More precisely, letting $A(\mathbf{e}, R) \in \mathrm{E}(d-1)$ denote the isometry $\mathbf{x} \mapsto \mathbf{e} + R\mathbf{x}$, consider the bijection $\iota : A(\mathbf{e}, R) \mapsto N(\mathbf{e}, R)$. Then, ι is an isomorphism, and there exists an isomorphism $\iota^* : \mathrm{E}^*(d-1) \to L(p_\circledast)$ such that $\pi \circ \iota^* = \iota \circ \pi$.* ■

PROOF. Since $A(\mathbf{e}, R)A(\mathbf{e}', R) = A(\mathbf{e} + R\mathbf{e}', RR')$, for ι to be a homomorphism (and thus an isomorphism) means that

$$N(\mathbf{e}, R)N(\mathbf{e}', R') = N(\mathbf{e} + R\mathbf{e}', RR'). \tag{4.91}$$

This identity is straightforwardly checked directly, just multiplying the two matrices $N(\mathbf{e}, R), N(\mathbf{e}', R')$ together. □

4.3. Representations of little groups. The classification of massless irreps of $\mathrm{P}^*(1, d)$ has been reduced to the classification of irreps of the cover

$$\mathrm{E}^*(d-1) = \mathbb{R}^{d-1} \rtimes \mathrm{Spin}(d-1) \tag{4.92}$$

of the group $\mathrm{E}(d-1)$ of Euclidean isometries. This is because the stabilizer in $\mathrm{Spin}(1, d)$ of a nonzero null vector $k \in V_+$ happened to be isomorphic to $\mathrm{E}^*(d-1)$. This isomorphism identified Euclidean translations with null rotations of spacetime and Euclidean rotations with rotations in space around the spatial component of k.

Like the Poincaré group, the group $\mathrm{E}^*(d-1)$ is a semidirect product of $\mathbb{R}^{d-1}$ and another (semisimple) Lie group, so its irreps can be classified using the method of induced representations. Since the analysis is almost identical to that of the Poincaré group, we can just state the upshot. The irreps are classified by two pieces of data: an orbit $o \subseteq \mathbb{R}^d$ of the group $\mathrm{SO}(d-1)$ of Euclidean rotations, and an irrep of the stabilizer

$$\ell(k_\circledast) = \{R \in \mathrm{Spin}(d-1) : Rk_\circledast = k_\circledast\}, \quad k_\circledast \in o, \tag{4.93}$$

called the *short little group* in this context. The orbit o is the spectrum of the restriction of the representation to Euclidean translations – i.e. to null rotations.

Unlike in the Lorentzian setting, where we had six sorts of possible orbits, in the Euclidean setting there are only two:

- the sphere $r\mathbb{S} = \{q \in \mathbb{R}^{d-1} : \|q\| = r\}$ of radius $r > 0$,
- the origin $\mho = \{0\}$.

This is summarized in Section 4.3, together with the corresponding short little groups. In either case, the short little group is $\mathrm{Spin}(j)$ for some $j \in \mathbb{N}^+$, so its irreps can be looked up.

Orbit $o \subset \mathbb{R}^{d-1}$	Reference $k_\circledast \in o$	Short little group ℓ
$r\mathbb{S}$, $r > 0$	$k_\circledast = (1, 0, \cdots, 0)$	$\mathrm{Spin}(d-2)$
$\mho = \{0\}$	$k_\circledast = 0$	$\mathrm{Spin}(d-1)$

TABLE 4.3. The orbit $o \subset \mathbb{R}^{d-1}$ and associated short little group ℓ.

The possibility $o = r\mathbb{S}$ leads to an irrep $\varsigma : \mathrm{E}^*(1, d) \to \mathrm{U}(\mathcal{V})$ of the little group $\mathrm{E}^*(d-1)$ which is *infinite*-dimensional, since it will consist of wavefunctions on $r\mathbb{S}$ valued in an irrep of the short little group. This exotic possibility leads to *continuous spin*. It does not appear to be realized in nature.

So, we focus on the case where the orbit o is $\mho = \{0\}$. This means that null rotations act trivially on the little group irrep. The whole little group irrep is then unitarily equivalent to the short little group irrep. So, we can conclude: *for each irrep of* $\mathrm{Spin}(d-1)$, *we get exactly one irrep of* $\mathrm{E}^*(d-1)$.

When $d = 3$, then the relevant short little group is $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. The irreps of $\mathrm{U}(1)$ have the form $\chi_j : e^{i\theta} \mapsto e^{ij\theta}$, where $j \in \mathbb{Z}$. Then, $\{\chi_j\}_{j\in\mathbb{Z}}$ is a complete set of irreps.

PROPOSITION 4.22. *Choosing* $p_{\circledast} = (1, 0, 0, 1)$, *so that* $L \subseteq \mathrm{SL}(2, \mathbb{C})$ *is given by eq.* (4.89), *the irrep* $\varsigma : L \to \mathbb{C}$ *induced by* χ_j *has the form* $S \mapsto a^j$. ■

To summarize, in ς, the null rotations all act trivially, but the rotations around k act nontrivially.

4.4. Standard boosts and Wigner rotations. For each $p \in V_+\backslash\{p_{\circledast}\}$, we want to find a restricted Lorentz matrix $D[p]$ such that $D[p] : p_{\circledast} \mapsto p$, where $p_{\circledast} = (1, 0, \cdots, 0, 1)$ is the future-directed null vector whose spatial component we call $\mathbf{z} = (0, 0, \cdots, 1)$. If $p = (p_0, \mathbf{p})$ is a null vector, with $p_0 > 0$ and $\mathbf{p} \notin \operatorname{span} \mathbf{z}$, then there is a unique spatial rotation $R(\mathbf{p})$ acting as the identity on $\{\mathbf{z}, \mathbf{p}\}^\perp$ and mapping $R(\mathbf{p}) : \mathbf{z} \mapsto \hat{\mathbf{p}}$. Then, we can take

$$D[p] = R(\mathbf{p})D[q] \tag{4.94}$$

for $q = p_0 p_{\circledast}$, if we have already defined $D[q]$. So, it suffices to define $D[q]$ for such q.

The natural choice is to define $D[q] = \Lambda(v\mathbf{z})$, the usual speed v pure boost in the z-direction (eq. (1.15) but with z in place of x), for some $v \in \mathbb{R}$. Specifically, $\Lambda(v)p_{\circledast} = q$ when

$$\sqrt{\frac{1-v}{1+v}} = p_0, \qquad \text{i.e.} \qquad v = \frac{1-p_0^2}{1+p_0^2}. \tag{4.95}$$

So,

$$D[p] = R(\mathbf{p})\Lambda\Big(\frac{1-p_0^2}{1+p_0^2}\mathbf{z}\Big) = \Lambda\Big(\frac{1-p_0^2}{1+p_0^2}\hat{\mathbf{p}}\Big)R(\mathbf{p}). \tag{4.96}$$

The next ingredient is the Wigner rotation $W(\Lambda, p) = D[p]^{-1}\Lambda D[\Lambda^{-1}p]$. The computations of these can be found in [Tal22, §9.6.2].

5. Parity and time-reversal ($\star$) [*]

A. Existence of the Radon–Nikodym derivative (Proof of Proposition 4.2)

Fix $p_{\circledast} \in O$. Let $D[p] \in \mathrm{Spin}(1, d)$ denote the standard boosts, with $D[p]p_{\circledast} = p$, discussed above. We are assuming that these have been chosen to depend continuously on p. We prove the following, which is a more precise version of Proposition 4.2.

PROPOSITION 4.23. *Let* $\mathcal{D}$ *denote the Gårding domain, eq.* (3.62). *Suppose that* $\phi, \psi \in \mathcal{D}$, *with* $\phi = \operatorname{avg}_\theta \phi_0$ *and* $\psi = \operatorname{avg}_\vartheta \psi_0$ *for* $\psi_0, \phi_0 \in \mathcal{H}$, $\theta, \vartheta \in C_c^\infty(G; \mathbb{R})$. *Consider, for each* $p, q \in O$, $h \in \mathrm{P}^*$.

$$f_p[h](q) \overset{\text{def}}{=} \int_{\mathbb{R}^{1,d} \rtimes L[p_{\circledast}]} \theta(D[p]\ell D[q]^{-1}h^{-1})\vartheta(D[p]\ell D[q]^{-1})\,\mathrm{d}\ell, \tag{4.97}$$

where the integral is done using the Haar measure on $\mathbb{R}^{1,d} \rtimes L[p_{\circledast}]$. *Then:*

(i) For each fixed p, the function $\mathrm{P}^* \times O \ni (h, q) \mapsto f_p[h](q)$ *lies in* $C_c^0(\mathrm{P}^* \times O)$.

(ii) Moreover, this depends continuously on p, in the sense that for each fixed $p_0 \in O$, *there exists some open neighborhood* $U \subset O$ *of* p_0 *and a compact subset* $K \Subset \mathrm{P}^* \times O$ *such that, for all* $p \in U$, $\operatorname{supp} f_p[-](-) \subseteq K$, *and the map* $(h, q, p) \mapsto f_p[h](q)$ *lies in* $C^0(\mathrm{P}^* \times O^2)$.

(iii) For any Borel $E \subseteq \mathbb{R}^{1,d}$, $\mu_{\phi,\psi}(E) = \int_{E\cap O}\int_{\mathrm{P}^*} \underbrace{\langle \phi_0, \rho(h)\Pi(f_p[h])\psi_0\rangle}_{\in C_c^0(\mathrm{P}^*)}\,\mathrm{d}h\,\mathrm{d}\mu(p)$.

In other words, the Radon–Nikodym derivative $\mathrm{d}\mu_{\phi,\psi}/\,\mathrm{d}\mu(p)$ *exists and is*

$$\frac{\mathrm{d}\mu_{\phi,\psi}}{\mathrm{d}\mu}(p) = \int_{\mathrm{P}^*} \langle \phi_0, \rho(h)\Pi(f_p[h])\psi_0\rangle\,\mathrm{d}h. \tag{4.98}$$

■

PROOF. The statements (i) and (ii) on the regularity, needed to make sense of the integrals in (iii), are straightforward consequences of the compact support of θ, ϑ, so we will focus on the more difficult proof of (iii). Let $E \in \mathrm{Borel}(\mathbb{R}^{1,d})$. We want to compute

$$\begin{aligned}&\int_O 1_E(p) \int_{\mathrm{P}^*} \langle \phi_0, \rho(h)\Pi(f_p[h])\psi_0\rangle \,\mathrm{d}h \,\mathrm{d}\mu(p)\\ &= \int_{\mathrm{P}^*} \Big\langle \phi_0, \rho(h)\Pi\Big(\int_O 1_E(p) \int_{\mathbb{R}^{1,d}\rtimes L[p_*]} \theta(D[p]\ell D[\bullet]^{-1}h^{-1})\vartheta(D[p]\ell D[\bullet]^{-1}) \,\mathrm{d}\ell \,\mathrm{d}\mu(p)\Big)\psi_0\Big\rangle \,\mathrm{d}h. \end{aligned} \tag{4.99}$$

The key thing is to recognize that, setting $g = D[p]\ell$, then $\mathrm{d}\ell \,\mathrm{d}\mu(p) = \mathrm{d}g$ (this is one way in which $\mathrm{d}\mu(p)$ is defined). So, the argument of $\Pi(\bullet)$ above is the function

$$q \mapsto \int_{\mathrm{P}^*} 1_E(\Lambda(g)p_*)\theta(gD[q]^{-1}h^{-1})\vartheta(gD[q]^{-1}) \,\mathrm{d}g \tag{4.100}$$

where $\Lambda(g)$ is the linear part of g. Using the invariance of the Haar measure, we can replace $gD[q]^{-1}$ with g for each individual q, giving

$$\int_{\mathrm{P}^*} 1_E(\Lambda(g)D[q]p_*)\theta(gh^{-1})\vartheta(g) \,\mathrm{d}g = \int_{\mathrm{P}^*} 1_E(\Lambda(g)q)\theta(gh^{-1})\vartheta(g) \,\mathrm{d}g. \tag{4.101}$$

So,

$$\Pi(\cdots) = \int_{\mathrm{P}^*} \theta(gh^{-1})\vartheta(g)\Pi(1_E \circ \Lambda(g)) \,\mathrm{d}g = \int_{\mathrm{P}^*} \theta(gh^{-1})\vartheta(g)\rho(g^{-1})\Pi(1_E)\rho(g) \,\mathrm{d}g. \tag{4.102}$$

Plugging this into the above, we have arrived at:

$$\int_O 1_E(p) \int_{\mathrm{P}^*} \langle \phi_0, \rho(h)\Pi(f_p[h])\psi_0\rangle \,\mathrm{d}h \,\mathrm{d}\mu(p) = \int_{\mathrm{P}^*}\int_{\mathrm{P}^*} \theta(gh^{-1})\vartheta(g)\langle \phi_0, \rho(hg^{-1})\Pi(1_E)\rho(g)\psi_0\rangle \,\mathrm{d}g \,\mathrm{d}h. \tag{4.103}$$

We can now use Fubini's theorem to swap the order of integration, making the integral over h the inner integral, and then defining $t = gh^{-1}$; then $\mathrm{d}t = \mathrm{d}h$ (using the bi-invariance of the Haar measure), and the right-hand side above becomes

$$\int_{\mathrm{P}^*}\int_{\mathrm{P}^*} \theta(t)\vartheta(g)\langle \phi_0, \rho(t^{-1})\Pi(1_E)\rho(g)\psi_0\rangle \,\mathrm{d}g \,\mathrm{d}t = \langle \phi, \Pi(1_E)\psi\rangle = \mu_{\phi,\psi}(E). \tag{4.104}$$

□

B. Casimir operators [*]

Problems and exercises

EXERCISE 4.1: Prove that, if p, q lie in the same Lorentz orbit O, then, for either $G = \mathrm{SO}(1,d), \mathrm{Spin}(1,d)$, the stabilizers of p, q in G are conjugate.

EXERCISE 4.2: Let $p, q \in \mathbb{R}^{1,d}$ be nonzero elements in distinct Lorentz orbits.

(a) Show that, if $p \in V_\pm$ and $q \in V_\mp$, then no Lorentz-closed subsets $U \ni p$, $V \ni q$ satisfy $U \cap V = \varnothing$.

(b) Prove that, excepting the case noted in the previous part, U, V can be chosen such that $U \cap V = \varnothing$.

EXERCISE 4.3: Check carefully that the induced representation π, eq. (4.48), has spectrum $\sigma = \bar{O}$ and little group representation ς.

EXERCISE 4.4: Show that the induced representation eq. (4.48) does not depend on the choice of reference momentum p_* in the following sense: if $p_{*,j} \in O$, $j = 1, 2$ are two choices of reference momenta with little groups $L[p_{*,j}]$, then, given a continuous unitary representation $\varsigma : L[p_{*,1}] \to \mathrm{U}(\mathcal{V})$, if we define

$$\begin{aligned} \varsigma' : L[p_{*,2}] &\to \mathrm{U}(\mathcal{V}) \\ A &\mapsto \varsigma(D[p_{*,2}]^{-1} A D[p_{*,2}]), \end{aligned}$$

then the induced representation constructed using the data $(p_{*,2}, \varsigma')$ is unitarily equivalent to that using the data $(p_{*,1}, \varsigma)$.

Exercise 4.5: Check that the matrix Λ in eq. (6.45) lies in $\mathrm{O}(2,1)$.

Exercise 4.6: Explain how the irreps of $\mathrm{P}^*(1,d)$ with spectrum $\sigma = X_{m,-}, V_-$ are related to those with spectrum $X_{m,+}, V_+$.

Problem 4.1: Show that the conclusion of Proposition 4.1 holds under the weaker assumption that ρ is primary.

Problem 4.2:
- (i) Show that the induced representation π (eq. (4.48)) is primary if and only if ς is.
- (ii) Conclude that every finite-spin primary representation of $\mathrm{P}^*(1,d)$ with spectrum $\sigma \neq \{0\}, Y_\Gamma$. can be decomposed into a countable direct sum of copies of the same irrep.
- (iii) (Optional.) Repeat the above for the remaining cases. You may use without proof any facts you want about the representation theory of $\mathrm{Spin}(1,d)$.

Problem 4.3 (Tachyons): Classify the irreps of $\mathrm{P}^*(1,3)$ whose spectrum is Y_Γ for some $\Gamma > 0$.

Hint: the little group is $\mathrm{Spin}(1,2)$*, so you need the (unitary) irreps of this group. Problem 2.2 says* $\mathrm{Spin}(1,2) \cong \mathrm{SL}(2,\mathbb{R})$*, and you can just look up the irreps of this group.*

Problem 4.4 (Exotic vacua): Classify the nontrivial irreps of $\mathrm{P}^*(1,3)$ whose spectrum is $\{0\}$.

Hint: look up the irreps of $\mathrm{SL}(2,\mathbb{C})$*.*

CHAPTER 5

Relativistic wave mechanics: massive case

Wigner's classification of particles in terms of irreps of the Poincaré group

$$\mathrm{P}^*(1,d) = \mathbb{R}^{1,d} \rtimes \widetilde{\mathrm{SO}(1,d)}, \qquad d \geq 1 \tag{5.1}$$

is highly abstract. For each $m \geq 0$ and little group representation $s : L \to \mathrm{U}(\mathcal{V})$, we have constructed an irrep $\pi_{m,s}$ describing particles of mass m and spin s. The Hilbert space consists of certain vector-valued functions

$$\Psi : X_{m,+} \to \mathcal{V} \tag{5.2}$$

on the "mass shell"

$$X_{m,+} = \{(E, \mathbf{p}) \in \mathbb{R}^{1,d} : E^2 = m^2 + \|\mathbf{p}\|^2, E \geq 0\}. \tag{5.3}$$

The effect of a Lorentz transformation combines an action on the argument with a complicated action on the values via the little group representation.

A more down-to-earth approach is **wave mechanics**. In wave mechanics, particles are described via wavefunctions *on spacetime*, satisfying some linear first- or second-order PDE. The wavefunctions are valued in some finite-dimensional representation $\mathcal{T}$ of the Lorentz group, all of which arise as suitably symmetrized/anti-symmetrized spaces of tensors. In physicists' parlance, wavefunctions may have Lorentz or spinor *indices*.

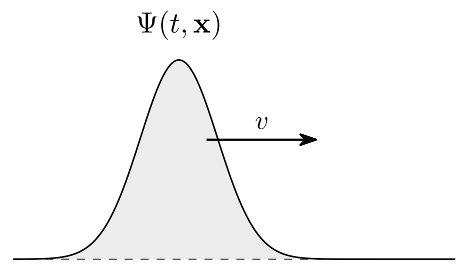


FIGURE 5.1. A schematic scalar wavefunction Ψ, shown on one time slice.

The (actual, not abstract) inverse Fourier transform converts functions on momentum space to functions on spacetime. This immediately yields a realization of $\pi_{m,s}$ as a space of functions on spacetime, *but valued in the space* $\mathcal{V}$ *on which the little group representation acts, not* $\mathcal{T}$, acting on those indices in an unnatural way. The missing mathematical ingredient needed for the wave-mechanical realization of $\pi_{m,s}$ is the "method of polarization tensors," which will involve a p-dependent embedding $\mathcal{V} \hookrightarrow \mathcal{T}$ to "untwist" the Lorentz action. The p-dependence is predetermined by the standard boosts $D[p]$ used to construct $\pi_{m,s}$, so the only representation-theoretic input required is the decomposition of $\mathcal{T}$ into irreps of the little group. Thus:

The wave-mechanical realizations of the $\pi_{m,s}$ are constructed using two things:

(I) the inverse Fourier transform,

(II) knowledge of the equivariant embeddings $\mathcal{V} \hookrightarrow \mathcal{T}$,

where equivariant means equivariant with respect to the little group L.

Let

$$S : \widetilde{\mathrm{SO}(1,d)} \to \mathrm{GL}(\mathcal{T}) \tag{5.4}$$

denote the actual representation acting on $\mathcal{T}$.

WARNING: Note that S is not required to be unitary and will typically not be, except for the trivial case. Because the Lorentz group is non-compact, its finite-dimensional representations need not be unitarizable, and in fact never are, unless trivial.

The set of $\mathcal{T}$-valued functions $\Psi : \mathbb{R}^{1,d} \to \mathcal{T}$ carries a natural P^*-action:

$$\begin{aligned} T\Psi(x) &= S(\Lambda)\Psi \circ T^{-1} \\ &= S(\Lambda)\Psi(\Lambda^{-1}(x-a)) \end{aligned} \tag{5.5}$$

for any Poincaré transformation $T = (a, \Lambda) \in \mathrm{P}^*$. A PDE is said to be *relativistic* (a "relativistic wave equation") if $T\Psi$ solves it whenever Ψ does, for all $T \in \mathrm{P}^*$. Synonyms include Poincaré-invariant[1].

The set $\mathcal{X}$ of weak solutions of a relativistic wave equation constitutes a representation of P^*. Since $\mathcal{X}$ does not come with an inner product, it is not a Hilbert space, and so it does not make sense to ask whether our representation is unitary. In fact, $\mathcal{X}$ is much too large, containing some very irregular distributions. A goal is to find a dense Hilbertizable Poincaré-closed subspace

$$\mathcal{H} \subseteq \mathcal{X} \tag{5.6}$$

on which the representation is unitarizable. That is, for some choice of inner product making $\mathcal{H}$ into a Hilbert space, the action of P^* is unitary. Then, analyzing *this* representation, we may decompose it into irreps:

$$\mathcal{H} \cong \bigoplus \pi_\bullet. \tag{5.7}$$

This is what it means to understand the "particle content" of a relativistic wave equation. In the massive examples that we consider here, a suitable $\mathcal{H} \subseteq \mathcal{X}$ can always be found. This is not always so when massless fields are involved. Thus, this lecture is devoted to the massive case, though we will handle the massless case when it is costless to do so.

One minor caveat is that we choose to restrict attention to *positive* energy representations of the Poincaré group, but relativistic wave equations admit both positive and negative energy solutions. "Positive-energy" just means that the Fourier transform $\mathcal{F}\Psi \in \mathcal{S}'$ is supported in the $E \geq 0$ half-space

$$\{E \geq 0\} = \{(E, \mathbf{p}) \in \mathbb{R}^{1,d} : E \geq 0\}, \tag{5.8}$$

and analogously for negative energy. The negative energy $\pi_\bullet$'s are in bijection with the positive energy $\pi_\bullet$'s, so one could easily analyze their contribution to the solution space of any relativistic wave equation. We find it a bit simpler just to restrict $\mathcal{X}$ to contain only positive-energy solutions. Then $\mathcal{H}$ only contains positive energy irreps.

This lecture consists of two threads.

(1) In the first, we begin with specific examples of relativistic wave equations, and then try to analyze their particle content, the $\pi_{m,s}$'s that appear. We only consider second-order equations (including first-order).

Second-order relativistic wave equations imply that each component solves the massive wave equation (with the same mass), so it will suffice to restrict attention to the $\mathcal{T}$-valued wave equation, for various $\mathcal{T}$.

(2) In the second thread, we begin with an irrep of P^* and try to engineer a relativistic wave equation containing that particle content.

The upshot is that, for $m > 0$, all of the irreps $\pi_{m,s}$ arise from wave mechanics.

The quantum mechanical analysis of relativistic wave equations is a rich topic. We will see below a diversity of behavior. The source of this diversity is the interplay between the finite-dimensional representation theory of the Lorentz group and the unitary representation theory of the Poincaré group, governed by the little group. We covered the latter in previous lectures. The former we discuss below, in §5.B.

[1]"Covariant" and "equivariant" may also be used.

Fields transform under finite-dimensional representations of the Lorentz group $\mathrm{Spin}(1,d)$, whereas particles transform according to infinite-dimensional unitary representations of the Poincaré group $\mathrm{P}^*(1,d)$.

To appreciate this distinction, keep in mind that the restriction of a unitary representation of the Poincaré group $\mathrm{P}^*(1,d)$ to the Lorentz subgroup $\mathrm{Spin}(1,d)$ is some highly reducible direct integral of unitary representations of the latter group, and these representations are either trivial or infinite-dimensional. We are still doing quantum mechanics, but the Lorentz representation S is playing an auxiliary role, so there is no reason for it to be unitary.

Remark: Actually, rather than *functions* on spacetime, it is technically simpler to work with *distributions* on spacetime. Wavefunctions are therefore $\mathcal{T}$-valued (tempered) distributions,

$$\Psi \in \mathcal{S}'(\mathbb{R}^{1,3};\mathcal{T}) = \mathcal{S}(\mathbb{R}^{1,3};\mathcal{T}^*)^*. \tag{5.9}$$

Given a (constant-coefficient) differential operator P, we use $\mathcal{X} \subseteq \mathcal{S}'(\mathbb{R}^{1,3};\mathcal{T})$ to denote the set of positive-energy distributional solutions Ψ of $P\Psi = 0$.

Thus, the PDE is relativistic if $\mathcal{X}$ is closed under the action of the Poincaré group.

Remark: When S descends to a representation of $\mathrm{SO}(1,d)$, then P^* can be replaced by P in the preceding paragraphs.

Remark 5.1. It turns out that any finite-dimensional representation of

$$\widetilde{\mathrm{SO}(1,2)} = \widetilde{\mathrm{SL}(2,\mathbb{R})} \tag{5.10}$$

automatically descends to a representation of the double cover

$$\mathrm{Spin}(1,2) \cong \mathrm{SL}(2,\mathbb{R}). \tag{5.11}$$

Since $\widetilde{\mathrm{SO}(1,d)} = \mathrm{Spin}(1,d)$ for $d \geq 3$, we can stop writing tildes and declare that the object of interest is a finite-dimensional representation of the double cover $\mathrm{Spin}(1,d)$, regardless of d. We actually already did this above. ■

1. Basic examples (Klein–Gordon, Dirac)

1.1. Scalar fields. The simplest example is that of the scalar field. The scalar wave equation with mass $m \geq 0$ reads

$$\Box\phi + m^2\phi = 0, \quad \phi \in \mathcal{D}'(\mathbb{R}^{1,d}) \tag{5.12}$$

where $\Box = \partial_t^2 - \triangle$ is the d'Alembertian. If $m > 0$, this is also known as the *Klein–Fock–Gordon equation*.[2] If $m = 0$, it is known as the *d'Alembert equation*. In either case, we will refer to the equation as the massive/massless wave equation.

The representation $S : \mathrm{SO}(1,3) \to \mathcal{T}$ here is just the trivial one. The transformation law is the simplest possible:

$$(T\phi)(x) = \phi(\Lambda^{-1}(x-a)) \tag{5.13}$$

for $T = (a,\Lambda) \in \mathrm{P}$. Scalar fields have no additional indices. Everyone agrees about the values a scalar field takes.

Proposition 5.2. *The scalar wave equation is Poincaré-covariant:*

$$\Box\phi + m^2\phi = 0 \Longrightarrow (\Box + m^2)(\phi \circ T^{-1}) = 0, \tag{5.14}$$

for any Poincaré transformation $T \in \mathrm{P}$. ■

[2] In the West, Fock is usually given short shrift and omitted from the name, despite publishing his work in the same journal as Klein and Gordon and doing so before Gordon. Guiltily, we will follow suit and refer to the equation as that of "Klein–Gordon."

We use this opportunity to practice physicists' index notation. Skip to §5.A if you have not seen this before, or need a brush-up.

PROOF. Let $\tilde{x} = T^{-1}(x) = \Lambda^{-1}(x-a)$. Thus,

$$\tilde{x}^\mu = (\Lambda^{-1})^\mu{}_\nu (x^\nu - a^\nu), \tag{5.15}$$

$$\partial \tilde{x}^\mu / \partial x^\nu = (\Lambda^{-1})^\mu{}_\nu. \tag{5.16}$$

Via the chain rule,

$$\begin{aligned}\Box(\phi \circ T^{-1}(x)) = -\eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \phi(\tilde{x}) = -\eta^{\mu\nu} \frac{\partial \tilde{x}^\sigma}{\partial x^\mu} \frac{\partial \tilde{x}^\tau}{\partial x^\nu} \frac{\partial^2 \phi(\tilde{x})}{\partial \tilde{x}^\sigma \partial \tilde{x}^\tau} &= -\eta^{\mu\nu} (\Lambda^{-1})^\sigma{}_\mu (\Lambda^{-1})^\tau{}_\nu \frac{\partial^2 \phi(\tilde{x})}{\partial \tilde{x}^\sigma \partial \tilde{x}^\tau} \\ &= -\eta^{\sigma\tau} \frac{\partial^2 \phi(\tilde{x})}{\partial \tilde{x}^\sigma \partial \tilde{x}^\tau} = (\Box\phi)(\tilde{x}),\end{aligned} \tag{5.17}$$

where in going from one line to the next we used eq. (5.95).

So, $(\Box + m^2)(\phi \circ T^{-1}) = ((\Box + m^2)\phi) \circ T^{-1}$. $\square$

The solution space

$$\mathcal{X} = \{\text{positive energy } \phi \in \mathcal{S}'(\mathbb{R}^{1,d}) : (\Box + m^2)\phi = 0\}, \tag{5.18}$$

which we now know to host a representation of P, can be described explicitly using the Fourier transform: it consists of those ϕ of the form

$$\begin{aligned}\phi(x) &= \int_{X_{m,+}} e^{-i\langle p,x\rangle} a(\mathbf{p}) \,\mathrm{d}\mu_{X_{m,+}}(p) \\ &= \int_{\mathbb{R}^d} e^{ip_0 t - i\mathbf{p}\cdot\mathbf{x}} a(\mathbf{p}) \frac{\mathrm{d}^d\mathbf{p}}{2\sqrt{m^2 + \|\mathbf{p}\|^2}}, \quad p^0 = \sqrt{m^2 + \|\mathbf{p}\|^2}\end{aligned} \tag{5.19}$$

for $a \in \mathcal{S}'(\mathbb{R}^d)$. Let $\mathcal{H} \subset \mathcal{X}$ consist of those ϕ of this form for

$$a \in (m^2 + \|\mathbf{p}\|^2)^{1/4} L^2(\mathbb{R}^d_{\mathbf{p}}). \tag{5.20}$$

PROPOSITION 5.3. *The subspace $\mathcal{H} \subseteq \mathcal{X}$ is Poincaré-closed.* ■

PROOF. The action of a spacetime translation T_a on ϕ is $(T_a\phi)(x) = \phi(x-a)$, and

$$\phi(x-a) = \int_{\mathbb{R}^d} \frac{e^{ip_0 t - i\mathbf{p}\cdot\mathbf{x}} (e^{i\langle p,a\rangle} a(\mathbf{p}))\,\mathrm{d}^d\mathbf{p}}{2\sqrt{m^2 + \|\mathbf{p}\|^2}}. \tag{5.21}$$

The product $e^{i\langle p,a\rangle} a(\mathbf{p})$ lies in $(m^2 + \|\mathbf{p}\|^2)^{1/4} L^2(\mathbb{R}^d_{\mathbf{p}})$ if and only if a does.

Similarly, the action of a Lorentz transformation T_Λ on ϕ is $(T_\Lambda \phi)(x) = \phi(\Lambda^{-1}x)$, and

$$\begin{aligned}\phi(\Lambda^{-1}x) = \int_{X_{m,+}} e^{-i\langle p,\Lambda^{-1}x\rangle} a(\mathbf{p}) \,\mathrm{d}\mu_{X_{m,+}}(p) &= \int_{X_{m,+}} e^{-i\langle \Lambda p,x\rangle} a(\mathbf{p}) \,\mathrm{d}\mu_{X_{m,+}}(p) \\ &= \int_{X_{m,+}} e^{-i\langle p,x\rangle} a(\mathbf{q}(\mathbf{p})) \,\mathrm{d}\mu_{X_{m,+}}(p)\end{aligned} \tag{5.22}$$

for $\mathbf{q}(\mathbf{p})$ the spatial part of $q = \Lambda^{-1}p$. So, we want to show that

$$a(\mathbf{q}(\mathbf{p})) \in (m^2 + \|\mathbf{p}\|^2)^{1/4} L^2(\mathbb{R}^d_{\mathbf{p}}), \tag{5.23}$$

assuming that the same holds for a. That is,

$$\int_{\mathbb{R}^d} \frac{|a(\mathbf{q}(\mathbf{p}))|^2 \,\mathrm{d}^d\mathbf{p}}{\sqrt{m^2 + \|\mathbf{p}\|^2}} < \infty. \tag{5.24}$$

The left-hand side is

$$\int_{\mathbb{R}^d} |a(\mathbf{q}(\mathbf{p}))|^2 \, \mathrm{d}\mu_{X_{m,+}}(p) = \int_{\mathbb{R}^d} |a(\mathbf{p})|^2 \, \mathrm{d}\mu_{X_{m,+}}(p), \tag{5.25}$$

which is finite by virtue of the assumption $\phi \in \mathcal{H}$. $\square$

We can make $\mathcal{H}$ into a Hilbert space by endowing it with the norm

$$\|\phi\|_{\mathcal{H}} = \|a\|_{L^2(X_{m,+})} \propto \left\| \frac{a(\mathbf{p})}{(m^2 + \|\mathbf{p}\|^2)^{1/4}} \right\|_{L^2(\mathbb{R}^d_{\mathbf{p}})}. \tag{5.26}$$

The computation above shows that this norm is Poincaré invariant, so $\mathcal{H}$ becomes a unitary representation of the Poincaré group.

Claim: the map $\phi \mapsto a$ defines a unitary equivalence between ρ and $\pi_{m,0}$. Indeed,

$$\begin{aligned} \pi_{m,0} = L^2(X_{m,+}, \mu_{X_{m,+}}) &= L^2\Big(\mathbb{R}^d, \frac{\mathrm{d}^d\mathbf{p}}{\sqrt{m^2 + \|\mathbf{p}\|^2}}\Big) \\ &= (m^2 + \|\mathbf{p}\|^2)^{1/4} L^2(\mathbb{R}^d), \end{aligned} \tag{5.27}$$

so

$$\|\phi\|_{\mathcal{H}} = \|a\|_{\pi_{m,0}}; \tag{5.28}$$

the map is unitary. Above, in eq. (5.21), eq. (5.22), we calculated the effect of spacetime translations and Lorentz transformations on a, and the result was precisely the same thing as the action in $\pi_{m,0}$. That things work out this way should not be surprising, because the actual Fourier transform used here is accomplishing the same thing as the abstract Fourier transform used in the construction of $\pi_{m,0}$. They must interact with Poincaré transformations in the same way.

1.2. The spinor-valued Klein–Gordon equation. Next, we turn to Weyl spinors. For simplicity, we only discuss the $d = 3$ case, though any odd d is similar.

Let $\mathcal{S} = \mathcal{S}^{0,1}, \mathcal{S}^{1,0}$ denote one of the two basic spinorial representations of $\mathrm{SL}(2, \mathbb{C})$, either the defining representation or the conjugate representation. The simplest spinorial relativistic wave equation is

$$(\Box + m^2)\psi = 0 \tag{5.29}$$

for $\psi \in \mathcal{S}'(\mathbb{R}^{1,3}; \mathcal{S})$. Here, ψ is a two-component object, with two components denoted ψ^α if $\mathcal{S}$ is the fundamental representation and $\psi^{\dot\alpha}$ if $\mathcal{S}$ is the anti-fundamental representation. Given that $\Box$ is Poincaré-covariant, the spinor-valued wave equation is manifestly Poincaré-covariant as well.

PROPOSITION 5.4. *For each $m > 0$, the particle content in $(\Box + m^2)\Psi = 0$ is $\pi_{m,1/2}$.* $\blacksquare$

PROOF. As before, we can use the Fourier transform. The solution space

$$\mathcal{X} = \{\text{positive energy } \psi \in \mathcal{S}'(\mathbb{R}^{1,3}; \mathcal{S}) : (\Box + m^2)\psi = 0\} \tag{5.30}$$

consists of those ψ of the form eq. (5.19), except now a is $\mathcal{S}$-valued (so two-component spinors). Now restrict to the subspace $\mathcal{H} \subset \mathcal{X}$ consisting of those ψ where

$$a \in (m^2 + \|\mathbf{p}\|^2)^{1/4} L^2(X_{m,+}, \mu_{X_{m,+}}; E). \tag{5.31}$$

Here, E is just the trivial bundle over $\mathbb{R}^d_{\mathbf{p}}$ with fiber $\mathcal{S}$, except we need to use a p-dependent inner product, namely that such that

$$S(D[p]) : E_{\mathbf{0}} \to E_{\mathbf{p}} \tag{5.32}$$

is unitary. The inner product on $E_{\mathbf{0}}$ is the one making the restriction of S to $\mathrm{Spin}(d) \subset \mathrm{Spin}(1, d)$ unitary.

This is endowed with the L^2 norm:

$$\|\psi\|_{L^2} = \|a\|_{L^2(X_{m,+}, \mu_{X_{m,+}}; E)} \propto \left\| \frac{S(D[p])^{-1} a(\mathbf{p})}{(m^2 + \|\mathbf{p}\|^2)^{1/4}} \right\|_{L^2(\mathbb{R}^3_{\mathbf{p}}; E_{\mathbf{0}})}. \tag{5.33}$$

Then, $\psi \mapsto a$ is a unitary map. To see that it intertwines the natural $\mathrm{P}^*(1,3)$-action on ψ with the $\pi_{m,1/2}$ action on $L^2(X_{m,+}, \mu_{X_{m,+}}; E)$ is left as an exercise. $\square$ [Exercise 5.2.]

Alternatively, use the method of polarization tensors with $\mathcal{T} = \mathcal{S}$, the representation of $\mathrm{SL}(2,\mathbb{C})$ on which, when restricted to the little group $\mathrm{SU}(2)$, is the spin-1/2 representation thereof.

1.3. Dirac equation [*].

2. Example: the massive vector field and Proca's equation

The scalar example above is almost misleadingly easy. A meatier example is the massive vector-valued wave equation

$$\Box A + m^2 A = 0, \tag{5.34}$$

$A \in \mathcal{S}'(\mathbb{R}^{1,d}; \mathcal{T}^1)$, $m > 0$, and the closely related Proca equation. Here, $\mathcal{T}^1$ stands for the defining representation of the Lorentz group $\mathrm{SO}(1,d)$. In physicists' parlance, a vector field A "carries a Lorentz index." Under a Poincaré transformation $T = (a, \Lambda)$, it transforms in the following way:

$$\begin{aligned} T :&A \mapsto \Lambda A \circ T^{-1}, \\ &(TA)^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}(x-a)). \end{aligned} \tag{5.35}$$

Given that $\Box$ is Poincaré-covariant, the vector-valued wave equation is manifestly Poincaré-covariant as well.

Note the distinction between a vector field and a quadruple of scalars. The latter describes an element of $\mathcal{S}'(\mathbb{R}^{1,d}; (\mathcal{T}^0)^{\oplus D}) = \mathcal{S}'(\mathbb{R}^{1,d}; \mathbb{C}^D)$, $D = 1 + d$. These transform differently under Lorentz transformations; if $A = (A^0, \cdots, A^d)$ were a tuple of scalars,

$$\begin{aligned} T :&A \mapsto A \circ T^{-1}, \\ &(TA)^\mu(x) = A^\mu(\Lambda^{-1}(x-a)) \end{aligned} \tag{5.36}$$

would be the transformation law. The individual scalar components would transform individually, rather than being mixed up as they are in eq. (5.35). So, even if we identify

$$\mathcal{S}'(\mathbb{R}^{1,d}; \mathcal{T}^1) = \mathcal{S}'(\mathbb{R}^{1,d}; (\mathcal{T}^0)^{\oplus D}) \tag{5.37}$$

at the level of underlying sets,

$$\mathcal{S}'(\mathbb{R}^{1,d}; \mathcal{T}^1) \neq \mathcal{S}'(\mathbb{R}^{1,d}; (\mathcal{T}^0)^{\oplus D}) \tag{5.38}$$

as representations of the Poincaré group.

Even for an equation as simple as the vector-valued wave equation, analyzing its particle content is non-trivial, and, unlike for the scalar wave equation, the $m > 0$ and $m = 0$ cases need to be distinguished. The natural expectation is for the massive spin-1 particle, $\pi_{m,1}$ to show up — one source of this expectation is that the spin degrees-of-freedom of a phonon are described by the (d-dimensional) vector representation of $\mathrm{SO}(d)$. It would be surprising if the particle content of a vector field did not involve a vector particle. But A has $D = 1 + d$ components, not d; an element of the defining representation of $\mathrm{SO}(d)$ is a spatial vector, whereas A is a spacetime vector. This will serve as an introduction to the important method of *polarization tensors*, described in detail in §3. It is key in understanding all examples with spin ≥ 1. When it comes time to construct free quantum fields, the method of polarization tensors is a main tool.

2.1. Scalar phonons, the Lorenz gauge condition, and Proca's equation. As mentioned above, the vector field A has one more degree-of-freedom than an element of $\pi_{m,1}$. Unless the PDE constrains A's components so that they are not all essentially independent (it does not), we should expect another particle in the content of the PDE. It should have one internal degree-of-freedom, which means it has to be a scalar, $\pi_{m,0}$. The total particle content should therefore be

$$\pi_{m,0} \oplus \pi_{m,1}. \tag{5.39}$$

Particles described by the $\pi_{m,0}$ here are called "scalar phonons."

Intuitively, the scalar phonon arises because there is a Poincaré-invariant way to construct solutions of the vector-valued massive wave equation from solutions of the scalar wave equation: if ϕ satisfies the scalar equation $\Box\phi + m^2\phi = 0$, then the gradient A, $A^\mu = \partial^\mu \phi$, satisfies the vector-valued equation. Indeed,

$$(\Box + m^2)\partial^\mu \phi = \partial^\mu(\Box + m^2)\phi = 0. \tag{5.40}$$

Such solutions are "secretly scalars." Not all of their components are independent.

Let $\mathcal{X} \subset \mathcal{S}'(\mathbb{R}^{1,d}; \mathcal{T}^1)$ denote the set of positive-energy solutions of eq. (5.34), and let $\mathcal{Y} \subset \mathcal{X}$ denote the set of scalar phonons:

$$\mathcal{Y} = \{\nabla\phi : \text{positive energy } \phi \in \mathcal{S}'(\mathbb{R}^{1,d}) \text{ s.t. } \Box\phi + m^2\phi = 0\} = \nabla\mathcal{X}_0, \tag{5.41}$$

where $\mathcal{X}_0 = \{\text{positive energy } \phi \in \mathcal{S}'(\mathbb{R}^{1,d}) : \Box\phi + m^2\phi = 0\}$. Then, ordinary phonons, described by $\pi_{m,1}$, should morally be the orthogonal complement to $\mathcal{Y}$ in $\mathcal{X}$ — but $\mathcal{X}$ is not a Hilbert space, so this does not make sense.

Instead, we should attempt to identify a Poincaré-closed linear-algebraic complement $\mathcal{Z} \subset \mathcal{X}$ to $\mathcal{Y}$. This can be done using the *Lorenz*[3] *gauge condition*

$$\partial_\mu A^\mu = 0, \tag{5.42}$$

which says that A is divergence-free. So, let

$$\mathcal{Z} = \{A \in \mathcal{X} : \partial_\mu A^\mu = 0\} \tag{5.43}$$

denote the set of solutions satisfying Lorenz gauge.

PROPOSITION 5.5.
- *Each of $\mathcal{Y}, \mathcal{Z}$ is closed under the action of the Poincaré group.*
- *$\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$. That is, $\mathcal{Y} \cap \mathcal{Z} = \{0\}$, and $\mathcal{Y} + \mathcal{Z} = \mathcal{X}$.*

■

PROOF.
- The definitions of $\mathcal{Y}, \mathcal{Z}$ are manifestly Poincaré invariant, owing to the invariance of the gradient and divergence operators.
- First, consider $\phi \in \mathcal{X}_0$, and let $A = \nabla\phi \in \mathcal{Y}$. Then, $\partial_\mu A^\mu = \partial_\mu \partial^\mu \phi = \Box\phi = -m^2\phi$. So, since $m > 0$, A satisfies the Lorenz gauge condition if and only if $\phi = 0$. This shows that $\mathcal{Y} \cap \mathcal{Z} = \{0\}$.[4]

  Next, consider arbitrary $A \in \mathcal{X}$, and let $\varphi = \partial_\mu A^\mu$ denote its divergence. This satisfies $\Box\varphi + m^2\varphi = 0$, so $\varphi \in \mathcal{X}_0$, and its gradient is $\partial^\mu\varphi = \partial^\mu\partial_\nu A^\nu \in \mathcal{Y}$. Now consider

  $$B^\mu = A^\mu + m^{-2}\partial^\mu\varphi = A^\mu + m^{-2}\partial^\mu\partial_\nu A^\nu \in \mathcal{X}. \tag{5.44}$$

  This satisfies

  $$\begin{aligned}\partial_\mu B^\mu = \partial_\mu A^\mu + m^{-2}\Box\partial_\nu A^\nu &= \partial_\mu A^\mu + m^{-2}\partial_\nu \Box A^\nu \\ &= \partial_\mu A^\mu - \partial_\nu A^\nu = 0.\end{aligned} \tag{5.45}$$

  That is, B is divergence free: $B \in \mathcal{Z}$. So, $A = B - m^{-2}\nabla\varphi \in \mathcal{Y} + \mathcal{Z}$.

[3] Hendrik Lorentz (with the transformations) and Ludvig Lorenz are two different physicists. Because Lorenz's gauge constraint is the Lorentz-invariant one, no confusion should arise from conflating the two physicists.

[4] This is our first hint that something goes wrong when $m = 0$.

□

The wave equation $\Box A + m^2 A = 0$ and the Lorenz gauge condition $\operatorname{div} A = 0$ can be combined into a single PDE

$$QA = 0, \tag{5.46}$$

for

$$Q = \begin{pmatrix} (\Box + m^2) I_D \\ \nabla^\intercal \end{pmatrix} \in \operatorname{Diff}^2(\mathbb{R}^{1,d}; \mathbb{C}^{(D+1)\times D}), \tag{5.47}$$

where $\nabla^\intercal = (-\partial_0, \partial_1, \cdots, \partial_d)$. Perhaps surprisingly, this is equivalent to a PDE of the form $PA = 0$ for $P \in \operatorname{Diff}^2(\mathbb{R}^{1,d}; \mathbb{C}^{D\times D})$. *Proca's equation* reads

$$\Box A^\mu - \partial^\mu \partial_\nu A^\nu + m^2 A^\mu = 0. \tag{5.48}$$

It is manifestly Poincaré-invariant.

Proposition 5.6. *Proca's equation is equivalent to the conjunction*

$$\begin{cases} \Box A + m^2 A = 0, \\ \operatorname{div} A = 0 \end{cases} \tag{5.49}$$

of $\Box A + m^2 A$ *and* $\partial^\mu A_\mu = 0$. ■

Proof. Equation (5.49) immediately implies Proca's equation. The converse follows from taking the divergence of both sides of Proca's equation:

$$\partial_\mu (\Box A^\mu - \partial^\mu \partial_\nu A^\nu + m^2 A^\mu) = 0, \tag{5.50}$$

which yields the Lorenz gauge condition $\operatorname{div} A$. Once the Lorenz gauge condition is known, Proca's equation simplifies to $\Box A + m^2 A$. □

So, the set $\mathcal{Z}$ defined by eq. (5.43) is

$$\mathcal{Z} = \{\text{positive energy } A \in \mathcal{S}'(\mathbb{R}^{1,d}; \mathcal{T}^1) : \Box A^\mu - \partial^\mu \partial_\nu A^\nu + m^2 A^\mu = 0\}, \tag{5.51}$$

the set of tempered distributional solutions of Proca's equation. Notice that the constraint $\partial_\mu A^\mu = 0$ removes exactly one degree of freedom from the D available in A_μ. Thus, the solution space $\mathcal{Z}$ has $D - 1 = d$ independent components. This matches the dimension of the spin-1 representation of the massive little group $\mathrm{SO}(d)$. This suggests very strongly that the particle content of Proca's equation is $\pi_{m,1}$.

2.2. Finding $\pi_{m,1}$. As a reminder: our goal is to find a Poincaré-closed subspace $\mathcal{H} \subseteq \mathcal{Z}$ on which the representation of the Poincaré group is unitarizable, and we are expecting one unitarily equivalent to the spin-1 representation $\pi_{m,1}$. We already know how to find in $\mathcal{X}_0$ and thus $\mathcal{Y}$ a copy of $\pi_{m,0}$ (by our discussion of the Klein–Fock–Gordon equation), so once we have found $\pi_{m,1}$, our analysis of the particle content of the massive vector-valued wave equation will be complete.

As in the treatment of the scalar field, the PDE will be solved via the Fourier transform, $\mathcal{F}$, which maps $\mathcal{Z}$ bijectively onto the set of $f \in \mathcal{S}'(\mathbb{R}^{1,d}; \mathbb{C}^D)$ such that

- $(p^2 + m^2) f(p) = 0$ (wave equation),
- $p \cdot f(p) = 0$ (Lorenz gauge condition).

These can be combined to

$$(p^2 + m^2) f(p) - p(p \cdot f(p)) = 0. \tag{Proca}$$

As usual, we restrict attention to positive energy solutions: let

$$\mathcal{S}'(X_{m,+}; E) = \{f \in \mathcal{S}'(\mathbb{R}^{1,d}; \mathbb{C}^D) : (p^2 + m^2) f = p \cdot f = 0, \operatorname{supp} f \subset \mathbb{R}^+ \times \mathbb{R}^d\}. \tag{5.52}$$

Thus, $\mathcal{F}$ maps $\mathcal{Z}$ onto $\mathcal{S}'(X_{m,+};E)$. As the notation suggests, we can think of $\mathcal{S}'(X_{m,+};E)$ as consisting of distributional sections of a bundle $E \to X_{m,+}$ whose total space,

$$E \subset X_{m,+} \times \mathbb{C}^D, \tag{5.53}$$

is defined by

$$E = \{(p, A) \in X_{m,+} \times \mathbb{C}^D : p \cdot A = 0\}, \tag{5.54}$$

This is a $\mathbb{C}^d$-bundle over $X_{m,+}$.

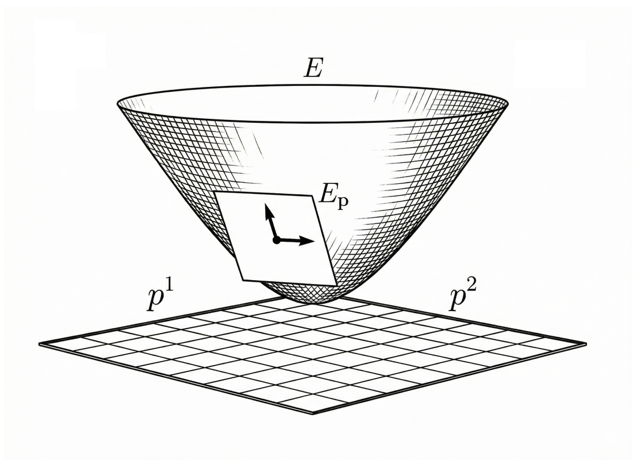


FIGURE 5.2. The fiber $E_{\mathbf{p}}$ over $\mathbf{p}$, depicted in the $d = 2$ case.

Consider $p = ((m^2 + \|\mathbf{p}\|^2)^{1/2}, \mathbf{p})$ for $\mathbf{p} \neq 0$. Consider the element

$$\epsilon_{\|} = (\|\mathbf{p}\|^2 (m^2 + \|\mathbf{p}\|^2)^{-1/2}, \mathbf{p}) \tag{5.55}$$

of the fiber E_p of E over p. Since its spatial component is parallel to $\mathbf{p}$, this is known as a *longitudinal polarization vector*. A direct complement to $\epsilon_{\|}$ within E_p is the set of $\epsilon_{\perp} \in \mathbb{R}^4$ of the form $(0, \mathbf{q})$ for $\mathbf{q} \perp \mathbf{p}$. These are known as *transverse polarization vectors*.

A basis of the fiber of $E \to X_{m,+}$ over the rest vector $\rho = (m, \mathbf{0}) \in X_{m,+}$ is given by the three *polarization vectors*

$$\begin{aligned} \epsilon_1(\rho) &= (0, 1, 0, \cdots, 0), \\ \epsilon_2(\rho) &= (0, 0, 1, 0, \cdots, 0), \\ &\vdots \end{aligned} \tag{5.56}$$

and so on. More generally, for any $p \in X_{m,+}$, let $\epsilon_j(p) = D[p]\epsilon_j(\rho)$ for each $j \in \{1, \dots, d\}$, where $D[p]$ is the pure boost taking ρ to p. Each polarization vector $\epsilon_1, \cdots, \epsilon_d$ is a smooth section of $X_{m,+} \times \mathbb{C}^D$, and in fact of E, since

$$\begin{aligned} p \cdot \epsilon_j(p) &= (D[p]\rho) \cdot (D[p]\epsilon_j(\rho)) \\ &= \rho \cdot \epsilon_j(\rho) = 0 \end{aligned} \tag{5.57}$$

for each $p \in X_{m,+}$. Since $\epsilon_1(p), \cdots, \epsilon_d(p)$ are linearly independent for each $p \in X_{m,+}$, $\{\epsilon_j\}_{j=1}^d$ is a section of the frame bundle of E, providing a trivialization

$$E \cong X_{m,+} \times \mathbb{C}^d \tag{5.58}$$

of E.

Proposition 5.7. *For* $\mathbf{f} \in \mathcal{S}'(X_{m,+})^d$*, let* $f^j \in \mathcal{S}'(X_{m,+})$ *denote the* j*th component of* $\mathbf{f}$*. Then,*

$$\mathcal{S}'(X_{m,+})^d \ni \mathbf{f} \mapsto \int_{X_{m,+}} e^{-ip\cdot x} f^j(p)\epsilon_j(p)\,\mathrm{d}\mu_{X_{m,+}}(p) \in \mathcal{Z} \tag{5.59}$$

defines a bijection $\mathcal{S}'(X_{m,+})^d \to \mathcal{Z}$*.* ■

Proof. We have a bijection $\mathcal{S}'(X_{m,+})^d \to \mathcal{S}'(X_{m,+}; E)$ which sends a tuple$\mathbf{f}$ of tempered distributions f^j to

$$\delta_{X_{m,+}}(p) f^j(p)\epsilon_j(p) \in \mathcal{S}'(X_{m,+}; E). \tag{5.60}$$

Composing this with $\mathcal{F}^{-1} : \mathcal{S}'(X_{m,+}; E) \to \mathcal{Z}$, we get a bijection $\mathcal{S}'(X_{m,+}; \mathbb{C})^d \to \mathcal{Z}$. This is the map described above. □

Let us now transport the action of the Poincaré group on $\mathcal{Z}$ back to $\mathcal{S}'(X_{m,+}; \mathbb{R}^d)$ via this bijection.

Proposition 5.8. *For each Poincaré transformation* $T = (a, \Lambda)$*, we have a commutative diagram*

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "A \mapsto TA"] & \mathcal{Z} \\ \mathcal{S}'(X_{m,+})^d \arrow[u] \arrow[r] & \mathcal{S}'(X_{m,+})^d \arrow[u] \end{tikzcd} \tag{5.61}$$

in which the vertical maps are eq. (5.59) *and the bottom horizontal map sends* $\mathbf{f}$ *to*

$$e^{ip\cdot a} W(\Lambda, p)^{\ell}{}_{j} \epsilon_\ell(p) f^j(\Lambda^{-1}p)\delta_{X_{m,+}}, \tag{5.62}$$

where $W(\Lambda, p) \in \mathrm{SO}(d)$ *is the Wigner rotation, defined by*

$$D[p]^{-1}\Lambda D[\Lambda^{-1}p] = \begin{pmatrix} 1 & 0 \\ 0 & W(\Lambda, p) \end{pmatrix}. \tag{5.63}$$

■

Proof. The image of $\mathbf{f}$ under the composition of the left vertical map and the top horizontal map is

$$\begin{aligned} \int_{X_{m,+}} e^{-ip\cdot(\Lambda^{-1}(x-a))} f^j(p)\Lambda\epsilon_j(p)\,\mathrm{d}\mu(p) &= \int_{X_{m,+}} e^{-i(\Lambda p)\cdot(x-a)} f^j(p)\Lambda D[p]\epsilon_j(\rho)\,\mathrm{d}\mu(p) \\ &= \int_{X_{m,+}} e^{-ip\cdot(x-a)} f^j(\Lambda^{-1}p)\Lambda D[\Lambda^{-1}p]\epsilon_j(\rho)\,\mathrm{d}\mu(p). \end{aligned} \tag{5.64}$$

Now applying the Fourier transform, we get

$$e^{ip\cdot a} f^j(\Lambda^{-1}p)\Lambda D[\Lambda^{-1}p]\epsilon_j(\rho)\delta_{X_{m,+}}. \tag{5.65}$$

We need to figure out how to write $\Lambda D[\Lambda^{-1}p]\epsilon_j(\rho)$ as a linear combination of $D[p]\epsilon_\ell(\rho)$:

$$\Lambda D[\Lambda^{-1}p]\epsilon_j(\rho) = c_j^{\ell} D[p]\epsilon_\ell(\rho) \tag{5.66}$$

for some c_j^{ℓ}. That is,

$$D[p]^{-1}\Lambda D[\Lambda^{-1}p]\epsilon_j(\rho) = c_j^{\ell}\epsilon_\ell(\rho). \tag{5.67}$$

So, c_j^{ℓ} is the ℓth component of the D-vector $D[p]^{-1}\Lambda D[\Lambda^{-1}p]\epsilon_j(\rho)$, which is the ℓth entry in the jth column of the matrix $D[p]^{-1}\Lambda D[\Lambda^{-1}p]$. (Note that we are counting starting from 0, but $j, \ell \in \{1, \cdots, d\}$.)

The conclusion is

$$e^{ip\cdot a} f^j(\Lambda^{-1}p)\Lambda D[\Lambda^{-1}p]\epsilon_j(\rho)\delta_{X_{m,+}} = e^{ip\cdot a} f^j(\Lambda^{-1}p)(D[p]^{-1}\Lambda D[\Lambda^{-1}p])^{\ell}{}_{j} D[p]\epsilon_\ell(\rho)\delta_{X_{m,+}}. \tag{5.68}$$

□

PROPOSITION 5.9. *Let $\mathcal{H} \subset \mathcal{Z}$ denote the image under eq. (5.59) of $L^2(X_{m,+}, \mu_{X_{m,+}})^d$. This set is closed under the action of the Poincaré group. Endowing $\mathcal{H}$ with the pushforward of the inner product on $L^2(X_{m,+}, \mu_{X_{m,+}})^d$, the resultant representation of the Poincaré group is unitarily equivalent to $\pi_{m,1}$.* ∎

PROOF. The map eq. (5.59) is unitary, and it defines a unitary equivalence between the representation of interest and one on $L^2(X_{m,+}, \mu_{X_{m,+}})^d = L^2(X_{m,+}, \mu_{X_{m,+}}; \mathbb{C}^d)$ used to define $\pi_{m,1}$. □

The Proca equation is not just a toy model; it describes the massive W and Z bosons which carry the electroweak force in the Standard Model.

3. The method of polarization tensors

The central step in the treatment of Proca's equation was the definition of the polarization vectors $\epsilon_j(\rho)$, $j = 1, \cdots d$. These are Lorentz vectors, with D components, but we only have d of them. Their span $\mathcal{V} = \mathbb{C}^d$ can be thought of as the vector representation of the little group $\mathrm{SO}(d)$ *embedded inside* the vector representation $\mathcal{T}^1 = \mathbb{R}^{1,d}$ of the Lorentz group $\mathrm{SO}(1,d)$. Nice solutions of Proca's equation were characterized as the (inverse) Fourier transforms of $\mathcal{T}^1$-valued functions ψ on $X_{m,+}$ such that

$$\psi(p) \in D[p]\mathcal{V} \tag{5.69}$$

for all $p \in X_{m,+}$, i.e. of sections of the vector bundle $E \subset X_{m,+} \times \mathcal{T}^1$ whose fiber E_p over p is $\{p\} \times D[p]\mathcal{V}$.

This suggests a general method to construct relativistic wave equations with specified particle content — beginning with a mass $m \geq 0$ and a unitary representation $\varsigma : L \to \mathrm{U}(\mathcal{V})$ of the little group

$$L \cong \begin{cases} \mathrm{SU}(2) & (m > 0), \\ \mathrm{E}^*(2) & (m = 0) \end{cases} \tag{5.70}$$

on some finite-dimensional Hilbert space $\mathcal{V}$, find an embedding $\mathcal{V} \subseteq \mathcal{T}$ of $\mathcal{V}$ into some other finite-dimensional vector space $\mathcal{T}$ endowed with a (typically non-unitary) representation $S : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathcal{T})$, which is required to *extend* ς in the sense that

$$S(\Lambda)|_{\mathcal{V}} = \varsigma(\Lambda) \tag{5.71}$$

for all $\Lambda \in L$. In other words, if $\mathcal{W}$ is a linear-algebraic complement to $\mathcal{V}$ in $\mathcal{T}$, then $S(\Lambda)$ has the block form

$$S(\Lambda) = \begin{pmatrix} \varsigma(\Lambda) & A : \mathcal{W} \to \mathcal{V} \\ 0 & B : \mathcal{W} \to \mathcal{W} \end{pmatrix} \tag{5.72}$$

with respect to the decomposition $\mathcal{T} = \mathcal{V} \oplus \mathcal{W}$.

We can now consider the vector bundle $E \subset X_{m,+} \times \mathcal{T}$ over $X_{m,+}$ defined by

$$E = \{(p, v) \in X_{m,+} \times \mathcal{T} : v \in S(D[p])\mathcal{V}\}, \tag{5.73}$$

where $D[p]$ is the standard boost taking the reference momentum $p_{\circledast}$ to p. Each fiber

$$E_p = \{p\} \times S(D[p])\mathcal{V} \tag{5.74}$$

of E is isomorphic, as a vector space, to $\mathcal{V}$ via $\mathcal{V} \ni v \mapsto (p, S(D[p])v) \in E_p$; in this way, the inner product on $\mathcal{V}$ can be transported to E_p. This lets us speak of the Hilbert space

$$L^2(X_{m,+}, \mu_{X_{m,+}}; E) = \{\psi \in L^2(X_{m,+}, \mu_{X_{m,+}}; \mathcal{T}) : \psi(p) \in S(D[p])\mathcal{V} \text{ a.e.}\} \tag{5.75}$$

of L^2-sections of E. The norm here is

$$\begin{aligned} \|\psi\|^2_{L^2(X_{m,+},\mu_{X_{m,+}};E)} &= \int_{X_{m,+}} \|\psi(p)\|^2_{E_p} \, \mathrm{d}\mu_{X_{m,+}}(p), \\ \|\psi(p)\|_{E_p} &= \|S(D[p])^{-1}\psi(p)\|_{\mathcal{V}}. \end{aligned} \tag{5.76}$$

Let

$$\mathcal{H} = \mathcal{F}^{-1} L^2(X_{m,+}, \mu_{X_{m,+}}; E), \tag{5.77}$$

which we endow with the inner product for which $\mathcal{F}^{-1} : L^2(X_{m,+}, \mu_{X_{m,+}}; E) \to \mathcal{H}$ is unitary. Note that $u \in \mathcal{H} \implies (\Box + m^2)u = 0$.

Theorem. *The Hilbert space $\mathcal{H}$ is a Poincaré-closed subspace of $\mathcal{S}'(\mathbb{R}^{1,3}; \mathcal{T})$. The representation of $\mathrm{P}^*(1,3)$ on $\mathcal{H}$ is unitary. If ς is irreducible, the representation just defined is unitarily equivalent to the irrep $\pi_{m,s}$ constructed using the little group representation ς.* ■

Proof. See [Tal22, §9.5]. □

Above, we required for clarity's sake that $\mathcal{V}$ literally be a subset of $\mathcal{T}$, but this is unnecessary. All we needed was for $\mathcal{V}$ to be equivalent to a subspace of $\mathcal{T}$. More precisely, this means that we possess an L-equivariant embedding

$$\iota : \mathcal{V} \hookrightarrow \mathcal{T}. \tag{5.78}$$

This means that the diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\iota"] \arrow[d, "\varsigma(g)"'] & \mathcal{T} \arrow[d, "S(g)"] \\ \mathcal{V} \arrow[r, "\iota"'] & \mathcal{T} \end{tikzcd} \tag{5.79}$$

commutes for all little group elements $g \in L$.

The construction of relativistic wave equations having $\pi_{m,s}$ as particle content is thereby reduced to finding embeddings $\mathcal{V} \hookrightarrow \mathcal{T}$, realizing ς as $S|_L$. The following proposition may be of use in determining for which pairs $(\mathcal{V}, \mathcal{T})$ such an embedding exists. We will mostly make use of it in the massless case.

Proposition 5.10. *Fix groups $L \subseteq G$. Let $\mathcal{V}$ be a finite-dimensional Hilbert space and $\varsigma : L \to \mathrm{U}(\mathcal{V})$ an irreducible unitary representation of L. Additionally, let $\mathcal{T}$ be a finite-dimensional vector space and $S : G \to \mathrm{GL}(\mathcal{T})$ denote a representation of G. Then:*

(a) $\exists$ an embedding $\iota : \mathcal{V} \hookrightarrow \mathcal{T}$ of L-representations if and only if the representation $\mathcal{T} \otimes \mathcal{V}^*$ of L has the trivial representation as a subrepresentation.

(b) If there are $N \in \mathbb{N}^+$ copies of the trivial representation, then we have a $\mathrm{U}(N)$'s worth of possible embeddings.

■

Proof.

(a) Note that the vector space $\mathcal{T} \otimes \mathcal{V}^*$ can be identified with the space of linear maps $\mathcal{V} \to \mathcal{T}$. For example, a pure tensor has the form $\psi \otimes \lambda$ for $\psi \in \mathcal{T}$ and $\lambda \in \mathcal{V}^*$, so

$$\mathcal{V} \ni v \mapsto \psi\lambda(v) \tag{5.80}$$

is a linear map $\mathcal{V} \to \mathcal{T}$. Identifying a general $\Psi \in \mathcal{T} \otimes \mathcal{V}^*$ with a linear map in this way, the action of $\Lambda \in L$ on Ψ is $(L\Psi)(v) = S(L)\Psi(\varsigma(L)^{-1}v)$.

- 'If:' Suppose that $\mathcal{T} \otimes \mathcal{V}^*$ has a nonzero vector Ψ invariant under L. Now consider the map $\mathcal{V} \to \mathcal{T}$ given by $v \mapsto \Psi(v)$; L-invariance says $\Psi(v) = S(L)\Psi(\varsigma(L)^{-1}v)$, i.e.

$$\Psi(\varsigma(L)v) = S(L)\Psi(v). \tag{5.81}$$

So, Ψ is a morphism of L-representations.

Equation (5.81) implies that $\ker\Psi \subset \mathcal{V}$ is a subrepresentation of ς. Because ς is assumed to be irreducible, it must be $\{0\}$ or all of $\mathcal{V}$. It cannot be all of $\mathcal{V}$, since $\Psi \neq 0$. So, $\ker\Psi = \{0\}$.
- 'Only if:' Suppose that we are given an embedding $\iota : \mathcal{V} \hookrightarrow \mathcal{T}$ of L-representations. We can interpret this as a $\Psi \in \mathcal{T} \otimes \mathcal{V}^*$, and via the same calculation above, L-invariance follows from eq. (5.81).

(b) Exercise. [Link]

□

Now let us carry this out for $m > 0$. (The $m = 0$ case will appear later.) Then, the relevant little group is $L = \mathrm{SU}(2)$.

PROPOSITION 5.11. *Let $j, k, \ell \in 2^{-1}\mathbb{N}$. An embedding $\mathcal{S}^\ell \to \mathcal{S}^{j,k}$ exists if and only if*
- *the parity of $2j + 2k$ matches that of 2ℓ*
- *$|j - k| \leq \ell \leq j + k$*

both hold. ■

PROOF. Via Clebsch–Gordan,

$$\mathcal{S}^{j,k} = \bigoplus_{\substack{\kappa=|j-k| \\ \kappa-|j-k|\in\mathbb{N}}}^{j+k} \mathcal{S}^\kappa. \tag{5.82}$$

So, we only have a copy of $\mathcal{S}^\ell$ under the stated conditions. □

PROOF TWO. The question is whether $\mathcal{S}^{j,k} \otimes \mathcal{S}^{\ell*} = \mathcal{S}^{j,k} \otimes \mathcal{S}^\ell$, when considered as a representation of the little group $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$, has a copy of the trivial representation $\mathcal{S}^0 = \mathbb{C}$. By Clebsch–Gordan,

$$\mathcal{S}^{j,k} \otimes \mathcal{S}^\ell = \bigoplus_{\substack{\kappa=|j-k| \\ \kappa-|j-k|\in\mathbb{N}}}^{j+k} (\mathcal{S}^\kappa \otimes \mathcal{S}^\ell). \tag{5.83}$$

Again by Clebsch–Gordan, $\mathcal{S}^\kappa \otimes \mathcal{S}^\ell$ has the trivial representation in it if and only if $\kappa = \ell$. So, the question is whether one of the κ's in the direct sum above is equal to ℓ, which holds if and only if the parity of $2(j + k)$ matches that of 2ℓ and $|j - k| \leq \ell \leq j + k$ holds. □

THEOREM. *Fix $j, k \in 2^{-1}\mathbb{N}$ and $m > 0$. The particle content of the $\mathcal{S}^{j,k}$-valued massive wave equation $\Box\phi + m^2\phi = 0$ is given by*

$$\bigoplus_{\substack{s=|j-k|, \\ s-|j-k|\in\mathbb{N}}}^{j+k} \pi_{m,s}. \tag{5.84}$$

■

PROOF. Via the method of polarization tensors, the particle content is the same as the decomposition of $\mathcal{S}^{j,k}$ into irreps of the little group $\mathrm{SU}(2)$. This is described by Proposition 5.17. □

EXAMPLE 5.12. For any $j \in 2^{-1}\mathbb{N}$, the particle content of the $\mathcal{S}^{j,0}$- or $\mathcal{S}^{0,j}$-valued massive wave equation is $\pi_{m,j}$. We already saw this when $j \in \{0, 1/2\}$ (using the notation $\mathcal{T}^0$ in place of $\mathcal{S}^{0,0}$). The first novel case is $j = 1$, when $\mathcal{S}^{1,0}$ and $\mathcal{S}^{0,1}$ are the representations of the Lorentz group describing left/right-handed anti-symmetric four-by-four matrices, respectively. The particle content of either theory is a single $\pi_{m,1}$. ■

EXAMPLE 5.13. The particle content of the $\mathcal{S}^{1/2,1/2}$-valued massive wave equation is $\pi_{m,0} \oplus \pi_{m,1}$. This confirms what we already knew from our study of Proca's equation, since $\mathcal{S}^{1/2,1/2} \cong \mathcal{T}^1$. ■

EXAMPLE 5.14. Consider the representation $\mathcal{T} = \mathrm{Sym}^2_0 \mathcal{T}^1$ of the Lorentz group consisting of symmetric traceless four-by-four matrices. This is $\mathcal{S}^{1,1}$. The particle content of the $\mathcal{T}$-valued Klein–Gordon equation $(\Box + m^2)g = 0$ with mass $m > 0$ is

$$\pi_{m,0} \oplus \pi_{m,1} \oplus \pi_{m,2}; \tag{5.85}$$

we have a scalar graviton, a vector graviton, and an ordinary (albeit massive) graviton, these having spins $0, 1, 2$ respectively. ■

4. Derivative fields (⋆) [*]

5.A. Primer on index notation

Physicists' index notation consists of the convention that certain symbols ("indices") are reserved to denote components of vectors, covectors, matrices, tensors, etc. of fixed size, with repeated indices summed in accordance with the Einstein summation convention. This means that, in any expression, each index appears at most twice on each side of the '=' sign. When an index appears twice, it must appear once as a superscript and once as a subscript and it is to be summed over its allowed values. This is the *Einstein summation convention.* Its provenance is Einstein's seminal paper on general relativity, in which the $\sum$'s became unwieldy. When an index appears once in an expression (that is, when it is unpaired), it must appear on both sides of the equation, and it must appear at the same height on both sides. Then, the equation is read as stating that, for all possible values of the unpaired indices, equality holds.

Mathematically, the distinction between raised indices and lowered indices reflects the distinction between vectors $\in \mathcal{T}$ and covectors $\in \mathcal{T}^*$. An isomorphism $\mathcal{T} \cong \mathcal{T}^*$ is the same thing as a non-degenerate bilinear pairing $\mathcal{T}^2 \to \mathbb{C}$. When a *canonical* isomorphism exists, the distinction between the two sorts of indices can be dropped, and they can freely be written as superscripts or subscripts. In special relativity, where $\mathcal{T} = \mathbb{R}^{1,d}$, there exist two different isomorphisms:

- The "usual" isomorphism identifies $\mathcal{T} = \mathcal{T}^*$ via the pairing
$$\mathcal{T}^2 \ni (x, y) \mapsto x^\intercal y. \tag{5.86}$$
That is, a vector $x \in \mathbb{R}^{1,d}$ is identified as covector $y \mapsto x^\intercal y$.
- The "natural" isomorphism uses instead the pairing
$$\mathcal{T}^2 \ni (x, y) \mapsto x^\intercal \eta y, \qquad \eta = \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix}. \tag{5.87}$$
That is, a vector $x \in \mathbb{R}^{1,d}$ is identified as a covector $y \mapsto x^\intercal \eta y$.

The latter identification is the natural one in Lorentzian geometry, but the former is the one implicit in matrix arithmetic — just turn the vector on its side. When we speak of the "components" of a covector, it is that one we are using. It is the existence of these two distinct duality pairings that merits distinguishing raised from lower indices in relativity. Otherwise one loses track of the signs introduced by η.

Usually, the meaning of any index can be inferred from the object on which it appears.

REMARK: A rule of thumb is that mid-alphabet Greek letters

$$\mu, \nu, \sigma, \lambda, \tau \in \{0, \ldots, d\} \tag{5.88}$$

refer to Lorentz indices and mid-alphabet Roman letters

$$i, j, k, \ell, m, n \in \{1, \ldots, d\} \tag{5.89}$$

refer to spatial indices. When working with spinors, $\alpha, \beta, \dot\alpha, \dot\beta$ are common choices; the dotted indices are usually used for distinguishing the indices of a right-handed Weyl spinor from a left-handed Weyl spinor. When labeling the generators of a Lie algebra, early-alphabet Roman letters $a, b, c \cdots$ are standard.

5.A.1. Notation for matrices. Let $D \in \mathbb{N}$. Given a D-by-D matrix M, the notation used to denote its entries depends on whether we want to think of M as a linear map $\mathbb{C}^D \ni x \mapsto Mx$ or a bilinear form $(\mathbb{C}^D)^2 \ni (x, y) \mapsto x^\intercal My$.

- In the former case, its entries are labeled "${M^\mu}_\nu$." Specifically, ${M^\mu}_\nu$ denotes the entry in the μth row and νth column of M. Assuming that μ, ν are Lorentz indices, so that we start counting at $\mu, \nu = 0$, then
$$M = \begin{bmatrix} {M^0}_0 & {M^0}_1 & \cdots \\ {M^1}_0 & {M^1}_1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.90}$$
Thus, $(Mx)^\mu = {M^\mu}_\nu x^\nu$, for all $x \in \mathbb{C}^D$.
- When the matrix is being used to represent a bilinear form, the entry in the μth row and νth column is denoted $M_{\mu\nu}$, so that
$$M = \begin{bmatrix} M_{00} & M_{01} & \cdots \\ M_{10} & M_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.91}$$
Thus, $x^\intercal My = x^\mu M_{\mu\nu} y^\nu$.

Despite being closely related, bilinear forms and linear transformations are not the same thing. Mathematically, this tracks the distinction between $\mathcal{T} \otimes \mathcal{T}^*$ (the space of linear transformations) and $\mathcal{T}^* \otimes \mathcal{T}^*$ (the space of bilinear forms). Of course, the distinction loses its bite if we have a canonical isomorphism $\mathcal{T} \cong \mathcal{T}^*$, which we usually do not.

Warning: Remember the space in front of 'ν' in "${M^\mu}_\nu$;" indices on matrices are conventionally placed northwest to southeast:
$$M\searrow.$$
This is to accommodate conventions physicists have about raising and lowering indices using η. We need to leave space to lower μ into. Great caution is required if you abbreviate 'M^μ_ν,' though you can sometimes get away with it. Incaution can give rise to ambiguous notation, or even a contradiction.

A Lorentz matrix Λ is (of course) thought of as a linear transformation, and the Minkowski metric η is thought of as a quadratic form. So, the matrix identity $\Lambda^\intercal \eta \Lambda = \eta$ reads
$${(\Lambda^\intercal)^\mu}_\sigma \eta_{\sigma\tau} {\Lambda^\tau}_\nu = \eta_{\mu\nu}, \tag{5.92}$$
where, per the Einstein summation convention, repeated indices are summed over their possible values. For a physicist, eq. (5.92) throws a syntax error. There are two breaches of proper index notation:

(i) repeated indices are supposed to appear in pairs in which one member is in the superscript and one is in the subscript, but here σ appears twice as a subscript,
(ii) μ appears as a superscript on the left-hand side and a subscript on the right-hand side, violating the requirement that unpaired indices appear at the same height on both sides of the equation.

Nevertheless, the equation is true. Both oddities are fixed upon rewriting the expression without using $\intercal$:
$$\boxed{{\Lambda^\sigma}_\mu \eta_{\sigma\tau} {\Lambda^\tau}_\nu = \eta_{\mu\nu}.} \tag{5.93}$$
This is the condition for Λ to be a Lorentz matrix, written in standard index notation.

We will need a slight variant of the previous identity. Because $\Lambda^\intercal$ is Lorentz whenever Λ is Lorentz, we can apply eq. (5.93) to $\Lambda^\intercal$ to get ${(\Lambda^\intercal)^\sigma}_\mu \eta_{\sigma\tau} {(\Lambda^\intercal)^\tau}_\nu = \eta_{\mu\nu}$, i.e.
$${\Lambda^\mu}_\sigma \eta_{\sigma\tau} {\Lambda^\nu}_\tau = \eta_{\mu\nu}. \tag{5.94}$$

Again, we have gotten something that looks syntactically weird (though it is correct), with mismatched index structure. To fix this, physicists define $\eta^{\mu\nu} = \eta_{\mu\nu}$, and then the identity becomes

$$\boxed{\Lambda^\mu{}_\sigma \eta^{\sigma\tau} \Lambda^\nu{}_\tau = \eta^{\mu\nu}.} \tag{5.95}$$

Let us emphasize that this is equivalent to eq. (5.93), but via non-trivial algebra (that any one-sided inverse of a finite matrix is a two-sided inverse).

5.A.2. Raising and lowering indices. Given some expression $M_{\cdots}{}^{\cdots}{}_{\mu}{}^{\cdots}{}_{\cdots} \in \mathbb{C}$ with various indices, physicists use the shorthand

$$M_{\cdots}{}^{\cdots\,\mu\,\cdots}{}_{\cdots} \stackrel{\text{def}}{=} \eta^{\mu\nu} M_{\cdots}{}^{\cdots}{}_{\nu}{}^{\cdots}{}_{\cdots}, \tag{5.96}$$

"raising" the index μ, and similarly for lowering an index (using $\eta_{\mu\nu}$). This notation is self-consistent, and consistent with the dual notation used for η, owing to

$$\eta^{\mu\nu}\eta_{\nu\sigma} = \delta^\mu_\sigma. \tag{5.97}$$

For instance, $\eta_{\mu\nu}$ should be related to $\eta^{\mu\nu}$ by a lowering of two indices:

$$\eta_{\mu\nu} \stackrel{?}{=} \eta_{\mu\tau}\eta^{\tau\sigma}\eta_{\sigma\nu}. \tag{5.98}$$

Indeed, the right-hand side is $\delta^\sigma_\mu \eta_{\sigma\nu} = \eta_{\mu\nu}$.

For example, if $x \in \mathbb{R}^{1,d}$ is a vector, and x^μ denotes its components, then

$$x_\mu = \begin{cases} -x^0 & (\mu = 0), \\ x_j & (\mu = j), \end{cases} \tag{5.99}$$

where $j \in \{1, \ldots, d\}$ is any spatial index. Similarly, if M is the matrix representing a bilinear form, then

$$M^{\mu\nu} = \begin{cases} -M_{\mu\nu} & (\mu = 0 \text{ XOR } \nu = 0), \\ M_{\mu\nu} & (\text{otherwise}). \end{cases} \tag{5.100}$$

(The same formula applies whenever we raise/lower both indices on a two-index object.) The reason why $\eta^{\mu\nu} = \eta_{\mu\nu}$ is that η has no entries in the zeroth row or column except the -1 on the diagonal, so the 'XOR' case of eq. (5.100) never applies.

If Λ is a Lorentz transformation, then the entries of Λ^{-1} can be expressed by raising and lowering indices of Λ:

$$(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu. \tag{5.101}$$

[Exercise 5.1] See the exercises at the end of this lecture.

WARNING: This illustrates the danger of ignoring whitespace; if you write "Λ^μ_ν," do you mean the entries of Λ or the entries of Λ^{-1}?

REMARK: Here, we discussed raising and lowering Lorentz indices, but similar remarks apply in other settings, e.g. when working with a non-chiral representation of the Lorentz group, or when working with the adjoint representation of a semisimple Lie algebra. All that is needed is a distinguished bilinear form (filling the role of η) to do the raising and lowering.

The raising/lowering conventions reach peak usefulness when the distinguished bilinear form is equivariant under some relevant symmetry group. The reason why raising and lowering indices is useful in Lorentzian geometry is that η is Lorentz-equivariant. The reason why raising and lowering indices is useful in the theory of semisimple Lie algebras is that the Killing form is equivariant with respect to the adjoint action of the Lie group. The existence of an equivariant bilinear form means that the representation under investigation is self-dual. It is therefore fortunate that all non-chiral representations of the Lorentz group are self-dual.

5.B. Finite-dimensional representations of the Lorentz group

Now we recall some facts about the (continuous) finite-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$. Because this group is simply connected, it suffices to consider representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{o}(1,3)$, which can be considered either as a six-dimensional real Lie algebra or a three-dimensional complex Lie algebra. The complexification $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ is a six-dimensional complex Lie algebra. Continuous finite-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$ are the same thing as finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$.

Proposition 5.15. $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ ■

Note that this is not true without the complexification!

Proof. [*] □

So, an irrep of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ is a pair of irreps of $\mathfrak{su}(2)_{\mathbb{C}}$. For each $j,k \in 2^{-1}\mathbb{N}$, we have an irrep $\mathcal{S}^{j,k}$, also denoted (j,k), in which the left $\mathfrak{su}(2)_{\mathbb{C}}$ is represented using the spin-j irrep and the right copy is represented using the spin-k irrep.

Modulo sign conventions:

Proposition 5.16. *(a) $\mathcal{S}^{1/2,0}$ is equivalent to the defining representation of* $\mathrm{SL}(2,\mathbb{C})$*,*
(b) $\mathcal{S}^{0,1/2}$ is equivalent to the anti-defining representation $S \mapsto S^*$,
(c) $\mathcal{S}^{j,k}$ is equivalent to $\mathcal{S}^{(j,0)} \otimes \mathcal{S}^{(0,k)}$.
(d) $\mathcal{S}^{(j,0)} \subset (\mathcal{S}^{(1/2,0)})^{\otimes 2j}$, and $\mathcal{S}^{(0,k)} \subset (\mathcal{S}^{(0,1/2)})^{\otimes 2k}$

■

An object transforming according to $\mathcal{S}^{j,k}$ is a tensor

$$T = \left\{T^{\alpha_1\cdots\alpha_j,\dot\alpha_1\cdots\dot\alpha_k}\right\}_{\alpha_\bullet,\dot\alpha_\bullet=0,1} \tag{5.102}$$

with j "undotted indices" and k "dotted indices," obeying certain symmetrization conditions on its indices. The transformation law is

$$(ST)^{\beta_1\cdots\beta_j,\dot\beta_1\cdots\dot\beta_k} = S^{\alpha_1}{}_{\beta_1}\cdots S^{\alpha_j}{}_{\beta_j}(S^*)^{\dot\alpha_1}{}_{\dot\beta_1}\cdots(S^*)^{\dot\alpha_k}{}_{\dot\beta_k}T^{\beta_1\cdots\beta_j,\dot\beta_1\cdots\dot\beta_k}. \tag{5.103}$$

Proposition 5.17. *Let $j,k \in 2^{-1}\mathbb{N}$. The irrep $\mathcal{S}^{j,k}$ of* $\mathrm{SL}(2,\mathbb{C})$*, when considered as a representation of the subgroup* $\mathrm{SU}(2)$*, decomposes in the following way:*

$$\mathcal{S}^{j,k} \cong \bigoplus_{\substack{\ell=|j-k|\\ \ell-|j-k|\in\mathbb{N}}}^{j+k} \mathcal{S}^{\ell} \tag{5.104}$$

where, for $\ell \in 2^{-1}\mathbb{N}$, $\mathcal{S}^{\ell}$ is the $2\ell+1$-dimensional irrep of $\mathrm{SU}(2)$*.* ■

Proof. As an $\mathrm{SU}(2)$ irrep, $\mathcal{S}^{(j,k)} \cong \mathcal{S}^j \otimes \mathcal{S}^k$. Now we use the Clebsch–Gordan decomposition, which tells us that a tensor product of $\mathcal{S}^j, k$ □

Example 5.18. As a representation of $\mathrm{SU}(2)$:

- $\mathcal{S}^{1/2,0} = \mathcal{S}^{1/2}$,
- $\mathcal{T}^1 = \mathcal{S}^{1/2,1/2} \cong \mathcal{S}^0 \oplus \mathcal{S}^1$
- $\mathcal{S}^{1,0} \cong \mathcal{S}^0 \oplus \mathcal{S}^1$.
- $\mathcal{S}^{1,1} \cong \mathcal{S}^1 \otimes \mathcal{S}^1 \cong \mathcal{S}^0 \oplus \mathcal{S}^1 \otimes \mathcal{S}^2$.

■

Exercises and problems

EXERCISE 5.1: To get some practice with raising and lowering indices, prove that, for any Lorentz matrix $\Lambda \in \mathrm{O}(1,3)$, the identity $(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$ holds.

EXERCISE 5.2: Check the final claim in §1.2.

PROBLEM 5.1: Consider the bundle $E \subset X_{m,+} \times \mathbb{C}^D$ used in our discussion of Proca's equation. We defined the longitudinal polarization vector $\epsilon_{\parallel}$, which is a smooth section of E, except at $p_{*} = (m, 0, \cdots, 0)$. We also defined the spaces of transverse polarizations $(0, \mathbf{q})$, $\mathbf{q} \perp \mathbf{p}$ over $\mathbf{p}$. This forms a sub-bundle $E_{\perp}$ (punctured at p_{*}). Does $E_{\perp}$ admit a smooth non-vanishing section?

Hint. The answer depends on the number d of spatial dimensions.

CHAPTER 6

Relativistic wave mechanics: massless case

The previous lecture analyzed the particle content of relativistic wave equations with mass. One might expect that the massless case is similar. It is not, except for examples with few indices (scalar and spinor fields), owing to the great differences in the representation theory of SU(2), $\mathrm{E}^*(2)$, the massive and massless little groups, respectively. The simplest example illustrating the discrepancy is the vector-valued wave equation

$$(\Box + m^2)A = 0, \qquad A \in \mathcal{D}'(\mathbb{R}^{1,3}; \mathcal{T}^1) \tag{6.1}$$

with mass $m \geq 0$. The $m > 0$ case, describing phonons and the $W^{\pm}, Z$-bosons, has already been discussed. The $m = 0$ case

$$\Box A = 0 \tag{6.2}$$

is of central importance in electrodynamics, where it describes the photon, and in chromodynamics, where it describes the gluon (sans self-interactions). The difficulties involved in treating it are not merely technical. Rather, they are closely related to the fundamental notion of *gauge invariance*. For both photons and gluons, the relevant Poincaré representation is $\pi_{0,-1} \oplus \pi_{0,+1}$; the photon/gluon has helicity ± 1. But identifying this Poincaré representation in the solution space of the massless wave equation requires understanding gauge invariance. This lecture is devoted to this problem. In the final section, we present the application of the method of polarization tensors to general massless wave equations.

We only discuss $d = 3$ here.

1. Example: the massless vector field, first attempt

We saw in the previous lecture that, when $m > 0$, the particle content of the vector-valued wave equation is $\pi_{m,0} \oplus \pi_{m,1}$, describing a scalar and a phonon. The phonon accounts for three internal degrees-of-freedom, one longitudinal polarization and two transverse polarizations. The scalar phonon accounts for a single internal degree-of-freedom. Since $3 + 1 = 4$, all four degrees-of-freedom present in the four-vector A are accounted for.

Now consider the $m = 0$ case. A natural guess would be that the particle content is related to the particle content in the $m > 0$ case, by taking some sort of $m \to 0^+$ limit. Indeed, the $\pi_{m,0}$ limits to a $\pi_{0,0}$ (scalar photon); the former consisted of the gradients

$$A^\mu = \partial^\mu \phi \tag{6.3}$$

of (nice) solutions of the scalar equation $(\Box + m^2)\phi = 0$. This works equally well if $m = 0$, so we have found a $\pi_{0,0}$ in the particle content of the massless equation. This accounts for 1 out of the 4 components of A.

The $m \to 0^+$ limit of the $\pi_{m,1}$ in the particle content of the massive equation is more complicated. Keeping in mind that the massless irreps $\pi_{0,h}$, $h \in 2^{-1}\mathbb{Z}$, are all one-dimensional, the natural expectation would be that the representation "splits"

$$\pi_{m,1} \rightsquigarrow \pi_{0,-1} \oplus \pi_{0,0} \oplus \pi_{0,1} \tag{6.4}$$

in the $m \to 0^+$ limit. The photon is supposed to be described by $\pi_{0,-1} \oplus \pi_{0,+1}$, so this looks promising. Unfortunately, *the proposed splitting is completely wrong.* Instead, the $\pi_{m,1}$ will limit to a non-unitarizable representation of the Poincaré group, which, owing to its non-unitarizability, has no quantum-mechanical interpretation at all.

The problem is not difficult to understand. By the method of polarization tensors, we know that the particle content of $\Box A = 0$ can be found by finding the copies of the various unitarizable little group representations in the Lorentz representation $\mathcal{T}^1$. Since this representation is finite-dimensional, the only possibilities are the finite-dimensional representations of the little group, $L \cong \mathrm{E}^*(2)$. Finite-dimensionality required that the null rotations $N \in L$ act trivially. But on $\mathcal{T}^1$, null rotations do not act trivially. Indeed, this is the defining representation, and the null rotations are matrices $N \neq I_4$, so they do not act like I_4. This means that $\mathcal{T}^1$ does not decompose into unitarizable representations of $\mathrm{E}^*(2)$.

Why did we not encounter this pathology in the massive case? The reason ultimately has to do with the fact that the massive little group $\cong \mathrm{SU}(2)$ is compact, whereas the massless little group $\cong \mathrm{E}^*(2)$ is non-compact. Finite-dimensional representations of compact Lie groups are automatically unitarizable, and consequently irreducibility is equivalent to indecomposability, but this is not true for non-compact groups. E.g. the Lorentz group has no nontrivial finite-dimensional unitary representations.

Returning to $\mathcal{T}^1$, the particle content has to be found within the subspace $\mathcal{N} \subseteq \mathcal{T}^1$ on which null rotations act trivially. This obviously includes the fiducial momentum $p_{\circledast}$, since the little group $L \ni N$ is the stabilizer of this vector. In fact,

$$\mathcal{N} = \operatorname{span}_{\mathbb{R}} p_{\circledast}. \tag{6.5}$$

That is, the only vectors fixed by all null rotations are multiples of $p_{\circledast}$. To prove this, it suffices to prove the analogous statement when there are two spatial dimensions, since given any vector $p \in \mathbb{R}^{1,3}$ the spatial parts $\mathbf{p}_{\circledast}, \mathbf{p} \in \mathbb{R}^3$ are contained together in a two-dimensional subspace. The formula for a null rotation of $\mathbb{R}^{1,2}$ fixing $p_{\circledast} = (1, 0, 1)$ was

$$\Lambda = \begin{bmatrix} 1 + e^2/2 & e & -e^2/2 \\ e & 1 & -e \\ e^2/2 & e & 1 - e^2/2 \end{bmatrix} \tag{6.6}$$

for $e > 0$. So,

$$\Lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + be + (a-c)e^2/2 \\ b + (a-c)e \\ c + be + (a-c)e^2/2 \end{bmatrix}. \tag{6.7}$$

This is equal to (a, b, c) if and only if $a = c$ and $b = 0$. Therefore $(a, b, c) = (a, 0, a) = ap_{\circledast}$.

Returning to the particle content of the massless vector-valued wave equation, this means that the only particle content is the copy of $\pi_{0,0}$ we have already found, the scalar photon. Indeed, recall that the correspondence between Lorentz subrepresentations of $\mathcal{T}^1$ and Poincaré subrepresentations of the solution space of $\Box A = 0$ is specifying the allowed polarization vectors $\varepsilon(p_{\circledast}) \in \mathcal{T}^1$. The polarization vector of a gradient $A = \mathrm{d}\phi$ at $p_{\circledast} \in V_+$ is exactly $p_{\circledast}$.

2. Example: Maxwell's equations

We have failed to find the expected photon in the solution space of $\Box A = 0$. This disaster forces us to reconsider A as the object hosting the photon. Recall that, in classical electrodynamics, the four-potential plays an auxiliary role. The dynamical equations, Maxwell's equations, are formulated in terms of the field-strength tensor $F = \mathrm{d}A$. The description in terms of A has some built-in redundancy. If we change $A \rightsquigarrow A + \mathrm{d}\phi$, by adding to it a gradient, then

$$F = \mathrm{d}(A + \mathrm{d}\phi) = \mathrm{d}A \tag{6.8}$$

remains unchanged, owing to $\mathrm{d}^2 = 0$. This suggests we directly analyze Maxwell's equations.

The field strength tensor F can be regarded as a (distributional) differential form $F \in \mathcal{D}'(\mathbb{R}^{1,3}; \wedge^2 \mathcal{T}^{1*})$, i.e. an anti-symmetric matrix

$$F = \begin{bmatrix} 0 & F_{01} & F_{02} & F_{03} \\ F_{10} & 0 & F_{12} & F_{13} \\ F_{20} & F_{21} & 0 & F_{23} \\ F_{30} & F_{31} & F_{32} & 0 \end{bmatrix} \tag{6.9}$$

whose entries $F_{\mu\nu} = -F_{\nu\mu}$ are numerical distributions. *Maxwell's equations* read

$$\begin{cases} \partial^\mu F_{\mu\nu} = 0, \\ \partial_{[\mu} F_{\nu\tau]} = 0, \end{cases} \tag{6.10}$$

where

$$\partial_{[\mu} F_{\nu\tau]} = \frac{1}{3}(\partial_\mu F_{\nu\tau} + \partial_\nu F_{\tau\mu} + \partial_\tau F_{\mu\nu}) \tag{6.11}$$

is the complete anti-symmetrization of $\partial_\mu F_{\nu\tau}$ in its three Lorentz indices $\mu, \nu.\tau$. Maxwell's equations are manifestly relativistically covariant when presented in this way. (See also **??**) We will discuss two reformulations of Maxwell's equations below, one in terms of the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ (in §2.1) and one in terms of differential forms (in §2.2). Each reformulation has its advantages and its disadvantages. The reformulation in terms of electric and magnetic fields is the most elementary, but at the cost of obscuring relativistic covariance. The reformulation in terms of differential forms is particularly elegant, but it takes some work to unpack.

Our main goal in this section (carried out in §2.2) is to prove that the particle content of Maxwell's equations is exactly $\pi_{0,-1} \oplus \pi_{0,+1}$. We will see that the two possible helicities $h = \pm 1$ correspond to the two circular polarizations of light.

2.1. The electric and magnetic fields: $\mathbf{E}, \mathbf{B}$ ($\star$). The electric and magnetic fields $\mathbf{E} = \{E_j\}_{j=1}^3$, $\mathbf{B} = \{B_j\}_{j=1}^3$ are defined by

$$F = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix} = \begin{pmatrix} 0 & -\mathbf{E}^\intercal \\ \mathbf{E} & \mathbf{F} \end{pmatrix}, \tag{6.12}$$

where $\mathbf{F}$ is the matrix whose (j,k)th entry is $\varepsilon_{jk\ell} B_\ell$. In terms of the electric and magnetic fields, the Maxwell equation $\partial^\mu F_{\mu\nu} = 0$ reads:

$$\nabla \cdot \mathbf{E} = 0, \tag{6.13}$$

$$\partial_t E_1 = \partial_2 B_3 - \partial_3 B_2, \quad \partial_t E_2 = -\partial_1 B_3 + \partial_3 B_1, \quad \partial_t E_3 = \partial_1 B_2 - \partial_2 B_1. \tag{6.14}$$

The first of these, $\nabla \cdot \mathbf{E}$, is *Gauss's law* (in vacuo). The remaining three equations can be summarized as

$$\nabla \times \mathbf{B} = \partial_t \mathbf{E}. \tag{6.15}$$

This is known as the *Ampère–Maxwell law*. Ampère's original form was $\nabla \times \mathbf{B} = 0$ (in vacuo, or more generally in the absence of electric currents). The $\partial_t \mathbf{E}$ on the right-hand side was a correction due to Maxwell.

The remaining Maxwell equation $\partial_{[\mu} F_{\nu\tau]} = 0$ consists of $\binom{4}{3} = 4$ independent equations. The four are:

$$\partial_{[1} F_{23]} = 0 \iff \nabla \cdot \mathbf{B} = 0, \tag{6.16}$$

$$\begin{aligned} \partial_{[0} F_{12]} = 0 &\iff \partial_t B_3 + \partial_1 E_2 - \partial_2 E_1 = 0, \\ \partial_{[0} F_{31]} = 0 &\iff \partial_t B_2 + \partial_3 E_1 - \partial_1 E_3 = 0, \\ \partial_{[0} F_{23]} = 0 &\iff \partial_t B_1 + \partial_2 E_3 - \partial_3 E_2 = 0. \end{aligned} \tag{6.17}$$

The first of these, $\nabla \cdot \mathbf{B} = 0$, is the magnetic analogue of Gauss's law. It is usually stated to hold everywhere, not just in vacuo, which has the significance of excluding the existence of magnetic monopoles. (Even if magnetic monopoles were to exist, the equation would hold in vacuo.) The remaining three equations can be summarized as

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}. \tag{6.18}$$

This is *Faraday's law of induction.*

To summarize, Maxwell's equations, eq. (6.10), when written in terms of the electric and magnetic fields (defined by eq. (6.12)), are equivalent to

$$\begin{cases} \nabla \cdot \mathbf{E} = 0, \\ \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \\ \nabla \times \mathbf{B} = \partial_t \mathbf{E}. \end{cases} \tag{6.19}$$

This is the form of Maxwell's equations presented to young students.

PROPOSITION 6.1 (Electromagnetic duality). *If the pair* $(\mathbf{E}, \mathbf{B})$ *solves Maxwell's equations, in the form eq.* (6.19)*, then so does the pair* $(\mathbf{B}, -\mathbf{E})$. ■

PROOF. Let $\mathbf{E}' = \mathbf{B}$ and $\mathbf{B}' = -\mathbf{E}$. Then, eq. (6.19) says that

$$\begin{cases} \nabla \cdot \mathbf{B}' = 0, \\ \nabla \cdot \mathbf{E}' = 0, \\ \nabla \times \mathbf{B}' = \partial_t \mathbf{E}', \\ \nabla \times \mathbf{E}' = -\partial_t \mathbf{B}'. \end{cases} \tag{6.20}$$

These are just Maxwell's equations for the pair $(\mathbf{E}', \mathbf{B}')$, except written out of order relative to the presentation in eq. (6.19). □

REMARK: Electromagnetic duality may surprise the reader, since electric effects are much more prominent than magnetic effects in everyday life. Apparently, this is due not to an asymmetry in the laws of electromagnetism but rather the fact that electric monopoles are abundant, whereas magnetic monopoles do not exist at all, or are rare enough to have escaped unambiguous detection so far.

The sign in the duality transformation $\bigstar : (\mathbf{E}, \mathbf{B}) \mapsto (\mathbf{B}, -\mathbf{E})$ has a consequence: $\bigstar^2 : (\mathbf{E}, \mathbf{B}) \mapsto (-\mathbf{E}, -\mathbf{B})$, i.e.

$$\bigstar^2 = -\,\mathrm{id}\,. \tag{6.21}$$

Its eigenvalues are purely imaginary, specifically $\pm i$, and its eigenvectors are necessarily complex-valued. Say that a pair $\mathbf{F} = (\mathbf{E}, \mathbf{B})$ is *left-handed* if $\bigstar\mathbf{F} = i\mathbf{F}$ and *right-handed* if $\bigstar\mathbf{F} = -i\mathbf{F}$. Left-handedness means that $\mathbf{B} = i\mathbf{E}$ and right-handedness means $\mathbf{B} = -i\mathbf{E}$. Any $\mathbf{F}$ can be decomposed

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- \tag{6.22}$$

into left- and right-handed parts, $\mathbf{F}_\pm = 2^{-1}(\mathbf{F} \pm i\bigstar\mathbf{F})$.

In terms of F,

$$\bigstar : \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{bmatrix}. \tag{6.23}$$

2.2. Reformulation in terms of differential forms ($\star$). The Faraday tensor F can be associated with a (distributional) two-form

$$\bar{F} = 2^{-1} F_{\mu\nu}\,\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}. \tag{6.24}$$

The action of a Poincaré transformation T on F is equivalent to pulling-back $\bar{F}$ by T:

PROPOSITION 6.2. *$\overline{TF} = T^{-1*}\bar{F}$, where $(\bullet)^*$ denotes the operation that pulls-back differential forms via the transformation $\bullet$.* ■

PROOF. If $T = (a, \Lambda)$ is a Poincaré transformation, $T : x \mapsto \Lambda x + a$, then its inverse is $T^{-1} : x \mapsto \Lambda^{-1}(x - a)$, so

$$\begin{aligned} T^{-1*}\bar{F} &= 2^{-1} F_{\mu\nu}(\Lambda^{-1}(x-a))\,\mathrm{d}(\Lambda^{-1}(x+a))^{\mu} \wedge \mathrm{d}(\Lambda^{-1}(x-a))^{\nu} \\ &= 2^{-1} F_{\mu\nu}(\Lambda^{-1}(x-a))\,\mathrm{d}(\Lambda^{-1}x)^{\mu} \wedge \mathrm{d}(\Lambda^{-1}x)^{\nu} \\ &= 2^{-1} F_{\mu\nu}(\Lambda^{-1}(x-a))(\Lambda^{-1})^{\mu}{}_{\mu'}(\Lambda^{-1})^{\nu}{}_{\nu'}\,\mathrm{d}x^{\mu'} \wedge \mathrm{d}x^{\nu'} \\ &= 2^{-1} F_{\mu\nu}(\Lambda^{-1}(x-a))\Lambda_{\mu'}{}^{\mu}\Lambda_{\nu'}{}^{\nu}\,\mathrm{d}x^{\mu'} \wedge \mathrm{d}x^{\nu'} = \overline{TF}. \end{aligned} \tag{6.25}$$

□

Going forwards, no confusion will arise from conflating F with $\bar{F}$, so we just drop the $\bar{}$ over $\bar{F}$.

The economy provided by the language of differential forms can be seen from the second of Maxwell's equations $\partial_{[\mu}F_{\nu]\tau}$, which just says $\mathrm{d}F = 0$, where d is the exterior derivative. In order to recast the remaining of Maxwell's equations, we need to introduce the Minkowski–Hodge star $\bigstar$. This is the unique linear operator $\bigstar$ on distributional forms, sending p-forms to $(4-p)$-forms for each $p \in \{0, 1, 2, 3\}$, such that

- $\bigstar c\omega = c\bigstar\omega$
- $\omega \wedge \bigstar\zeta = -(\omega, \zeta)_{\wedge\mathcal{T}^{1*}}\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$

for any constant-coefficient forms $\omega, \zeta \in \wedge\mathcal{T}^{1*}$ and numerical distribution $c \in \mathcal{D}'(\mathbb{R}^{1,3})$. Here, $(-,-)_{\wedge\mathcal{T}^{1*}}$ denotes the bilinear form on $\wedge\mathcal{T}^{1*}$ induced by the Minkowski pairing on $\mathcal{T}^1$.

REMARK: The fact that there exists an operator $\bigstar$ with the claimed properties, and the fact that it is unique, follows from the fact that the map $f : \wedge^p\mathcal{T}^{1*} \times \wedge^{4-p}\mathcal{T}^{1*} \to \mathbb{C}$ defined by

$$f(\omega, \zeta)\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 = \omega \wedge \zeta \tag{6.26}$$

for $\omega \in \wedge^p\mathcal{T}^{1*}$, $\zeta \in \wedge^{4-p}\mathcal{T}^{1*}$ is a duality pairing.

Explicitly, the action of the Minkowski–Hodge star on 2-forms is given by

$$\bigstar\,\mathrm{d}x_{\mu} \wedge \mathrm{d}x_{\nu} = \frac{1}{2}\varepsilon_{\mu\nu\tau\sigma}\eta^{\tau\lambda}\eta^{\sigma\varsigma}\,\mathrm{d}x_{\lambda} \wedge \mathrm{d}x_{\varsigma}, \tag{6.27}$$

where

$$\varepsilon_{\mu\nu\tau\sigma} = \begin{cases} 0 & (\mu,\nu,\tau,\sigma \text{ not all distinct}) \\ \operatorname{sgn}(\mu,\nu,\tau,\sigma) & (\text{otherwise}) \end{cases} \tag{6.28}$$

is the four-dimensional Levi–Civita symbol. (Here, $\operatorname{sgn}(\mu,\nu,\tau,\sigma)$ is the sign of the permutation $(\mu+1, \nu+1, \tau+1, \sigma+1) \in \mathfrak{S}_4$. So, $\operatorname{sgn}(\mu,\nu,\tau,\sigma)$ is $+1$ if an even number of its arguments are out of order and -1 otherwise.)

LEMMA 6.3. For any restricted Poincaré transformation T, the identity $T^*\bigstar = \bigstar T^*$ holds. ■

That is, $\bigstar$ is Poincaré-invariant.

PROOF. This is almost obvious from the characterization of $\bigstar$ given above. Indeed, the wedge product is invariant under all coordinate transformations whatsoever, the vector space of constant-coefficient forms is invariant under all affine transformations, the volume form $\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$ is invariant under all volume-preserving transformations (including all affine transformations whose linear part has determinant one, such as restricted Poincaré transformations), and the form $\langle -, - \rangle_{\wedge \mathcal{T}^{1*}}$ is Lorentz-invariant, essentially by definition. □

Let us verify directly, using the formula eq. (6.27), that $\bigstar$ commutes with Λ^* when applied to 2-forms, for any Lorentz transformation Λ. Then:

$$\begin{aligned}
\bigstar \Lambda^*(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu) &= \bigstar(\mathrm{d}({\Lambda_\mu}^\lambda x_\lambda) \wedge \mathrm{d}({\Lambda_\nu}^\rho x_\rho)) = {\Lambda_\mu}^\lambda {\Lambda_\nu}^\rho \bigstar(\mathrm{d}x_\lambda \wedge \mathrm{d}x_\rho) \\
&= \frac{1}{2} {\Lambda_\mu}^\lambda {\Lambda_\nu}^\rho \varepsilon_{\lambda\rho\tau\sigma} \eta^{\tau\varrho} \eta^{\sigma\varsigma} \, \mathrm{d}x_\varrho \wedge \mathrm{d}x_\varsigma \\
&= \frac{1}{2} {\Lambda_\mu}^\lambda {\Lambda_\nu}^\rho \varepsilon_{\lambda\rho\tau\sigma} {\Lambda_{\tau'}}^\tau \eta^{\tau'\varrho'} {\Lambda_{\varrho'}}^\varrho {\Lambda_{\sigma'}}^\sigma \eta^{\sigma'\varsigma'} {\Lambda_{\varsigma'}}^\varsigma \, \mathrm{d}x_\varrho \wedge \mathrm{d}x_\varsigma \\
&= \frac{\det(\Lambda^{-1})}{2} \varepsilon_{\mu\nu\tau'\sigma'} \eta^{\tau'\varrho'} {\Lambda_{\varrho'}}^\varrho \eta^{\sigma'\varsigma'} {\Lambda_{\varsigma'}}^\varsigma \, \mathrm{d}x_\varrho \wedge \mathrm{d}x_\varsigma \\
&= \frac{1}{2} \varepsilon_{\mu\nu\tau'\sigma'} \eta^{\tau'\varrho'} {\Lambda_{\varrho'}}^\varrho \eta^{\sigma'\varsigma'} {\Lambda_{\varsigma'}}^\varsigma \, \mathrm{d}x_\varrho \wedge \mathrm{d}x_\varsigma \\
&= \frac{1}{2} \Lambda^*(\varepsilon_{\mu\nu\tau'\sigma'} \eta^{\tau'\varrho'} \eta^{\sigma'\varsigma'} \, \mathrm{d}x_{\varrho'} \wedge \mathrm{d}x_{\varsigma'}) = \Lambda^* \bigstar(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu).
\end{aligned} \tag{6.29}$$

Let us calculate the effect of the Hodge star on the electric and magnetic fields. (This should agree with the notation used in the previous subsection, where $\bigstar : (\mathbf{E}, \mathbf{B}) \mapsto (\mathbf{B}, -\mathbf{E})$.) Since $\bigstar \, \mathrm{d}x_k \wedge \mathrm{d}t = -2^{-1} \varepsilon_{k\ell m} \, \mathrm{d}x_\ell \wedge \mathrm{d}x_m$ and $\bigstar \, \mathrm{d}x_k \wedge \mathrm{d}x_\ell = \varepsilon_{k\ell m} \, \mathrm{d}x_m \wedge \mathrm{d}t$ (note that $t = x^0$, not x_0), and since

$$F = 2^{-1} F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = E_j \, \mathrm{d}x_j \wedge \mathrm{d}t + 2^{-1} \varepsilon_{jk\ell} B_j \, \mathrm{d}x_k \wedge \mathrm{d}x_\ell \tag{6.30}$$

we have

$$\begin{aligned}
\bigstar F &= -2^{-1} E_k \varepsilon_{k\ell m} \, \mathrm{d}x_\ell \wedge \mathrm{d}x_m + 2^{-1} \varepsilon_{jk\ell} \varepsilon_{k\ell m} B_j \, \mathrm{d}x_m \wedge \mathrm{d}t \\
&= B_j \, \mathrm{d}x_j \wedge \mathrm{d}t - 2^{-1} E_j \varepsilon_{jk\ell} \, \mathrm{d}x_k \wedge \mathrm{d}x_\ell = F(\mathbf{B}, -\mathbf{E}).
\end{aligned} \tag{6.31}$$

So, the effect of the Hodge star is to reverse the electric field and then swap the electric and magnetic fields, as desired.

We can now reformulate Maxwell's equations in terms of differential forms.

Proposition 6.4. *Maxwell's equations are equivalent to*

$$\begin{cases} \mathrm{d}F = 0 \\ \mathrm{d}\bigstar F = 0. \end{cases} \tag{6.32}$$

■

Proof. By definition, $\mathrm{d}F = 2^{-1} \partial_\mu F_{\nu\tau} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\tau$, so

$$\mathrm{d}F = 0 \iff \partial_{[\mu} F_{\nu\tau]} = 0. \tag{6.33}$$

This is the second of Maxwell's equations in Equation (6.10). We saw in our discussion of $\mathbf{E}, \mathbf{B}$ that the other equation in eq. (6.10), $\partial^\mu F_{\mu\nu} = 0$, is equivalent to the other with $(\mathbf{E}, \mathbf{B})$ replaced with $(\mathbf{B}, -\mathbf{E})$. Thus,

$$\mathrm{d}\bigstar F = 0 \iff \partial^\mu F_{\mu\nu} = 0. \tag{6.34}$$

□

This formulation of Maxwell's equations makes electromagnetic duality obvious: F satisfies eq. (6.32) if and only if $\bigstar F$ does (since $\bigstar^2 F = -F$).

2.3. Particle content of Maxwell's equations. Let

$$\mathcal{X} = \{\text{positive energy } F \in \mathcal{S}'(\mathbb{R}^{1,3}; \wedge^2 \mathcal{T}^{1*}) : \partial^\mu F_{\mu\nu}, \partial_{[\mu} F_{\nu\tau]} = 0\} \tag{6.35}$$

denote the space of positive-energy tempered solutions of Maxwell's equations. Via the Fourier transform, $\mathcal{X}$ consists of the inverse Fourier transforms of $\wedge^2\mathcal{T}^{1*}$-valued functions $\hat{F}(p)$ on V_+ satisfying

$$p^\mu \hat{F}_{\mu\nu}(p) = p_{[\mu}\hat{F}_{\nu\tau]}(p) = 0 \tag{6.36}$$

for each $p \in V_\pm$, or more generally distributional sections of the bundle $E \subset V_+ \times \wedge^2 \mathcal{T}^{1*}$ whose fiber over p is specified by these equations. The fiber over p consists of generalized plane waves with four-momentum exactly p. As discussed earlier, this is two-dimensional (as a complex vector space).

Because F solving Maxwell implies $\bigstar F$ solving Maxwell, the space of distributional solutions of Maxwell's equations decomposes into left-handed and right-handed subspaces, characterized by $\bigstar F = \mp iF$. Thus, each of the fibers, defined by eq. (6.36), must split into two one-dimensional subspaces of left/right-handed tensors:

$$E = C_- \oplus C_+. \tag{6.37}$$

The points in $C_\pm(p) \subset \wedge^2\mathcal{T}^{1*}$ are the field-strength tensors describing circularly polarized plane waves. The L^2-sections of $C_\pm$ can be shown to be $\pi_{0,\pm 1}$. Indeed, circularly polarized plane waves are multiplied by a factor of $e^{\pm i\theta}$ upon rotating them by θ-degrees about their direction of propagation.

3. Example: the massless vector field, revisited

When it comes to building interacting theories, physicists want to work with A, not F. We saw that we cannot find the photon Hilbert space $\pi_{0,-1} \oplus \pi_{0,+1}$ as a *subspace* $\mathcal{H} \subseteq \mathcal{X}$ of the space $\mathcal{X} = \{\text{positive energy } A \in \mathcal{S}'(\mathbb{R}^{1,3}; \mathcal{T}^1) : \Box A = 0\}$ of solutions of $\Box A = 0$. But an alternative construction, exploiting gauge invariance, works. The idea is to find Poincaré-closed subspaces $\mathcal{B} \subseteq \mathcal{Z} \subseteq \mathcal{X}$ and form the quotient

$$\mathcal{Q} = \mathcal{Z}/\mathcal{B}. \tag{6.38}$$

Then, $\mathcal{H} \subseteq \mathcal{Q}$ will be a dense Hilbertizable subspace of $\mathcal{Q}$. Roughly, $\mathcal{Z}$ is the set of vector fields satisfying a certain "gauge condition," the Lorenz[1] gauge condition, and $\mathcal{B}$ is the set of "pure gauge" vector fields, corresponding to the subspace $\pi_{0,0}$ of scalar photons. The quotient $\mathcal{H}$ inherits an action of the Poincaré group. *This* representation will be unitarizable and unitarily equivalent to $\pi_{0,-1} \oplus \pi_{0,1}$. No additional $\pi_{0,0}$ appears.

Concretely, let

$$\begin{aligned} \mathcal{Z} &= \{A \in \mathcal{X} : \partial_\mu A^\mu = 0\} \\ \mathcal{B} &= \{A \in \mathcal{X} : \exists \phi \in \mathcal{X}_0 \text{ s.t. } A^\mu = \partial^\mu \phi\}. \end{aligned} \tag{6.39}$$

That each of these is closed under the action of the Poincaré group is obvious from their manifestly covariant definition. The important inclusion $\mathcal{B} \subseteq \mathcal{Z}$ follows immediately from the definition of $\mathcal{X}_0 = \{\text{positive energy } \phi \in \mathcal{S}'(\mathbb{R}^{1,3}) : \Box\phi = 0\}$. So the definition of $\mathcal{Q} = \mathcal{Z}/\mathcal{B}$ makes sense.

Consider the map $\mathcal{Q} \ni A \bmod \mathcal{B} \mapsto F = \mathrm{d}A^\flat$. This is well-defined, because changing A by an element $\partial^\mu\phi$ changes $A^\flat$ by $\mathrm{d}\phi$, and $\mathrm{d}^2\phi = 0$. For $F = \mathrm{d}A^\flat$, the Maxwell equation $\mathrm{d}F = 0$ holds automatically. The other Maxwell equation $\mathrm{d}\bigstar F = 0$ reads $\mathrm{d}\bigstar\mathrm{d}A^\flat = 0$, and this is equivalent to $\Box A = 0$. So, the map above lands in the solution space $\mathcal{X}_{\mathrm{Maxwell}}$ of Maxwell's equations. In fact, the Poincaré lemma (that closed forms on $\mathbb{R}^N$ are exact) tells us that this map is surjective. It is also evidently Poincaré-equivariant. So, we can conclude that the particle content of $\mathcal{Q}$ is $\pi_{0,-1} \oplus \pi_{0,+1}$ from the fact that this is the particle content of Maxwell's equations. [Exercise 6.1]

[1] Lorenz and Lorentz are different physicists.

4. The method of polarization tensors (massless case)

Let us use $\mathcal{R}^j : \mathrm{E}^*(2) \to \mathbb{C}$ to refer to the unitary representation of $\mathrm{E}^*(2) = \mathbb{R}^2 \rtimes \mathrm{Spin}(2)$, $\mathrm{Spin}(2) = \mathbb{R}/4\pi\mathbb{Z}$, given by $(a, [\theta \bmod 4\pi\mathbb{Z}]) \mapsto e^{ij\theta}$. This kills the subgroup $\{(a, [0 \bmod 4\pi\mathbb{Z}]) : a \in \mathbb{R}^2\} \subset \mathrm{E}^*(2)$ consisting of null rotations Note that $\mathcal{R}^{j*} = \mathcal{R}^{-j}$.

We want to find the subspaces of $\mathcal{S}^{j,k}$ equivalent to the irreps $\mathcal{R}^\bullet$. Let $\mathcal{N}^{j,k} \subseteq \mathcal{S}^{j,k}$ denote the subspace of $\mathcal{S}^{j,k}$ on which null rotations (in the little group) act trivially. This subspace is important, because all of the copies of $\mathcal{R}^\bullet$ have to sit inside it.

In order to compute $\mathcal{N}^{j,k}$, it is useful to realize $\mathcal{S}^{j,k}$, as a vector space, as a tensor product $\mathcal{S}^{j,k} = \mathcal{S}^j \otimes \mathcal{S}^k$, where $\mathcal{S}^\ell$ is the spin-ℓ representation of SU(2). Then, letting $|\ell, m\rangle$ denote the element of $\mathcal{S}^\ell$ with eigenvalue m under $\mathsf{J}^3 \in \mathfrak{su}(2)_\mathbb{C}$, the vectors

$$|j, m, k, m'\rangle = |j, m\rangle \otimes |k, m'\rangle \tag{6.40}$$

give a basis for $\mathcal{S}^{j,k}$. The action of the Lorentz generators $\mathsf{J}^\bullet, \mathsf{K}^\bullet$ on $\mathcal{S}^{j,k}$ are given by

$$\begin{aligned} \mathsf{J}_i &= J_i^{(j)} \otimes I_k + I_j \otimes J_i^{(k)} \\ \mathsf{K}_i &= -iJ_i^{(j)} \otimes I_k + iI_j \otimes J_i^{(k)}, \end{aligned} \tag{6.41}$$

where $J_i^{(\ell)}$ is the matrix representing J_i on $\mathcal{S}^\ell$.

Proposition 6.5. *Fix $j, k \in 2^{-1}\mathbb{N}$. The subspace $\mathcal{N}^{j,k}$ is one-dimensional, being spanned by the element $|j, -j, k, k\rangle$ of $\mathcal{S}^{j,k}$.* ■

Proof. The null rotations are generated by $\mathsf{K}_1 + \mathsf{J}_{13} = \mathsf{K}_1 - \mathsf{J}_2$ and $\mathsf{K}_2 + \mathsf{J}_{23} = \mathsf{K}_2 + \mathsf{J}_1$. Suppose $v \otimes w \in \mathcal{S}^{j,k}$ is killed by $\mathsf{K}_1 - \mathsf{J}_2$. Compute

$$\begin{aligned} (\mathsf{K}_1 - \mathsf{J}_2) v \otimes w &= -i((\mathsf{J}_1^{(j)} - i\mathsf{J}_2^{(j)})v) \otimes w + iv \otimes ((\mathsf{J}_1^{(k)} + i\mathsf{J}_2^{(k)})w) \\ (\mathsf{K}_2 + \mathsf{J}_1) v \otimes w &= ((\mathsf{J}_1^{(j)} - i\mathsf{J}_2^{(j)})v) \otimes w + v \otimes ((\mathsf{J}_1^{(k)} + i\mathsf{J}_2^{(k)})w). \end{aligned} \tag{6.42}$$

These vanish together if and only if

$$\begin{aligned} (\mathsf{J}_1^{(j)} - i\mathsf{J}_2^{(j)})v &= 0 \\ (\mathsf{J}_1^{(k)} + i\mathsf{J}_2^{(k)})w &= 0 \end{aligned} \tag{6.43}$$

Note that $i\mathsf{J}_1 \pm \mathsf{J}_2$ are *ladder operators*. For the above vanishings, v must be lowest weight, and w must be highest weight. □

Example 6.6. Consider the vector representation $\mathcal{T}^1 \cong \mathcal{S}^{1/2,1/2}$. Then, $\mathcal{N}$ consists of the vectors $p \in \mathbb{R}^{1,3}$ that are fixed by null rotations fixing $p_* = (1, 0, 0, 1)$. Almost by construction,

$$\mathcal{N} = \operatorname{span}_\mathbb{R} p_*. \tag{6.44}$$

The inclusion $\supseteq$ really is by construction. To prove $\subseteq$, it suffices to prove the analogous inclusion on $\mathbb{R}^{1,2}$, that the only eigenspace fixed by

$$\Lambda = \begin{bmatrix} 1 + e^2/2 & e & -e^2/2 \\ e & 1 & -e \\ e^2/2 & e & 1 - e^2/2 \end{bmatrix} \tag{6.45}$$

for $e > 0$ is $(1, 0, 1)$. You are asked to prove this as an exercise. ■

Consequently, a $\mathcal{S}^{j,k}$-valued massless equation can have at most *one* $\pi_{0,\bullet}$ in it. This mismatch between the number of degrees-of-freedom in a $\mathcal{S}^{j,k}$ valued object and the one degree-of-freedom described by $\pi_{0,\bullet}$ is due to the presence of null rotations. Most of the degrees-of-freedom in a $\mathcal{S}^{j,k}$-valued equation give a non-unitarizable representation of the Poincaré group and so admit no quantum mechanical interpretation.

The only remaining thing is to figure out *which* $\pi_{0,\bullet}$ can be present.

PROPOSITION 6.7. *The subspace $\mathcal{N}^{j,k}$, when regarded as a representation of $L[p_{\divideontimes}]$, is a copy of $\mathcal{R}^{k-j}$.* ■

PROOF. Recalling that $p_{\divideontimes} = (1, 0, 0, 1)$, the remaining generator of the little group is J_3. On the tensor representation $\mathcal{S}^{j,k} = \mathcal{S}^j \otimes \mathcal{S}^k$,

$$\mathsf{J}^3 = \mathsf{J}_3^{(j)} \otimes I_k + I_j \otimes \mathsf{J}_3^{(k)}. \tag{6.46}$$

The z-component of angular momentum of $|j, m, k, m'\rangle$ is therefore $m+m'$. For the vector $|j, -j, k, k\rangle$ which spans $\mathcal{N}_{j,k}$, this is $k - j$. □

6.A. Example: the Weyl equations

The simplest wave equation whose particle content includes spinors is the *Weyl equation*. This is a first-order PDE whose solutions are two-component spinors ψ. Actually there are two Weyl equations:

$$\begin{aligned} \sigma^\mu \partial_\mu \psi_{\mathrm{L}} &= 0, \\ \bar{\sigma}^\mu \partial_\mu \psi_{\mathrm{R}} &= 0, \end{aligned} \tag{6.47}$$

where $\sigma = (I_2, \sigma_1, \sigma_2, \sigma_3)$ and $\bar{\sigma} = (-I_2, \sigma_1, \sigma_2, \sigma_3)$. The $\psi_{\mathrm{R}}, \psi_{\mathrm{L}}$ are spinor-valued:

$$\psi_{\mathrm{L}} \in \mathcal{D}'(\mathbb{R}^{1,3}; \mathcal{S}^{1/2,0}) \quad \text{and} \quad \psi_{\mathrm{R}} \in \mathcal{D}'(\mathbb{R}^{1,3}; \mathcal{S}^{0,1/2}), \tag{6.48}$$

where $\mathcal{S}^{1/2,0}$ is the fundamental representation of $\mathrm{SL}(2, \mathbb{C})$, while $\mathcal{S}^{0,1/2}$ is the conjugate-fundamental representation. Consequently, under a spinorial Lorentz transformation $S \in \mathrm{SL}(2, \mathbb{C})$,

$$\begin{aligned} \psi_{\mathrm{L}} &\rightsquigarrow S\psi_{\mathrm{L}}, \\ \psi_{\mathrm{R}} &\rightsquigarrow S^{-1\dagger}\psi_{\mathrm{R}}. \end{aligned} \tag{6.49}$$

Given these two transformation laws, the Weyl equations are Poincaré invariant. In demonstrating this, the key observation is that $\sigma, \bar{\sigma}$ "transform like a four-vector." The meaning of this slightly mysterious phrase is:

LEMMA 6.8. For any $S \in \mathrm{SL}(2, \mathbb{C})$, let $\Lambda = \pi(S)$ be the corresponding Lorentz matrix. Then,

$$\begin{aligned} S^{-1}\bar{\sigma}^\mu S^{-1\dagger} &= \Lambda^\mu{}_\nu \bar{\sigma}^\nu \\ S^\dagger \sigma^\mu S &= \Lambda^\mu{}_\nu \sigma^\nu. \end{aligned} \tag{6.50}$$

■

PROOF. The Bloch map Σ has the form $\Sigma(x) = \bar{\sigma}_\mu x^\mu$ (note the lowered index). So, the definition of $\Lambda = \pi(S)$ can be written $\bar{\sigma}_\nu (\Lambda x)^\nu = S\bar{\sigma}_\mu S^\dagger x^\mu$. For this to hold for all x means

$$\Lambda^\nu{}_\mu \bar{\sigma}_\nu = S\bar{\sigma}_\mu S^\dagger. \tag{6.51}$$

Noting that $\bar{\sigma}_\mu = \sigma^\mu$, this last identity can be written $\Lambda^\nu{}_\mu \sigma^\nu = S\sigma^\nu S^\dagger$. Writing $\Lambda^\nu{}_\mu = (\Lambda^\intercal)^\mu{}_\nu$ and using $\pi(S^\dagger) = \pi(S)^\intercal$ finishes the proof of the second identity.

Raising the indices in eq. (6.51) gives $S\bar{\sigma}^\mu S^\dagger = \Lambda_\nu{}^\mu \bar{\sigma}^\nu$. Using $\Lambda_\nu{}^\mu = (\Lambda^{-1})^\mu{}_\nu$ finishes the proof. □

We can now prove the Poincaré-invariance of the Weyl equations:

PROPOSITION 6.9. *Let $T = (a, S) \in \mathrm{P}^*(1, 3)$ denote a spinorial Poincaré transformation. Then,*

$$\begin{aligned} \sigma^\mu \partial_\mu \psi_{\mathrm{L}} = 0 &\iff \sigma^\mu \partial_\mu (S\psi_{\mathrm{L}} \circ \Lambda^{-1}) = 0, \\ \bar{\sigma}^\mu \partial_\mu \psi_{\mathrm{R}} = 0 &\iff \bar{\sigma}^\mu \partial_\mu (S^{-1\dagger}\psi_{\mathrm{R}} \circ \Lambda^{-1}) = 0 \end{aligned} \tag{6.52}$$

for $\Lambda = \pi(S)$. ■

PROOF. The two computations are similar, so we just do the left-handed version. We have

$$\begin{aligned}\sigma^\mu\partial_\mu(S\psi_{\mathrm{L}}\circ\Lambda^{-1}) = \sigma^\mu S\partial_\mu(\psi_{\mathrm{L}}\circ\Lambda^{-1}) &= (\Lambda^{-1})^\nu{}_\mu\sigma^\mu S(\partial_\nu\psi_{\mathrm{L}})\circ\Lambda^{-1} \\ &= (\Lambda^{-1})^\nu{}_\mu S^{-1\dagger}(S^\dagger\sigma^\mu S)(\partial_\nu\psi_{\mathrm{L}})\circ\Lambda^{-1} \\ &= (\Lambda^{-1})^\nu{}_\mu S^{-1\dagger}(\Lambda^\mu{}_\tau\sigma^\tau)(\partial_\nu\psi_{\mathrm{L}})\circ\Lambda^{-1} \\ &= S^{-1\dagger}(\sigma^\nu\partial_\nu\psi_{\mathrm{L}})\circ\Lambda^{-1}.\end{aligned} \tag{6.53}$$

So ψ_{L} solves the Weyl equation if and only if $S\psi_{\mathrm{L}}\circ\Lambda^{-1}$ does. □

What is the particle content of the Weyl equation? The natural guess is $\pi_{0,\pm1/2}$, with one sign for the left-handed equation and one for the right-handed equation.

At this point, the reader may be wondering if it possible to add a "mass term" to the Weyl equation to get $\pi_{m,1/2,\pm}$ – that is, to model a massive spin-1/2 particle. Something should prevent this, because massive spin-1/2 particles have two internal spin degrees-of-freedom, whereas massless spin-1/2 particles only have one, so it would seem that Weyl spinors do not possess enough components to host a copy of $\pi_{m,1/2}$. But why not just consider

$$(\sigma^\mu\partial_\mu + m)\psi_{\mathrm{L}} = 0, \tag{6.54}$$

or the left-handed analogue? The problem is that this does not transform properly under Lorentz transformations: if $S\in\mathrm{SL}(2,\mathbb{C})$, then

$$(\sigma^\mu\partial_\mu + m)(S\psi_{\mathrm{L}}\circ\Lambda^{-1}) = S^{-1\dagger}(\sigma^\nu\partial_\nu)\psi_{\mathrm{L}}\circ\Lambda^{-1} + Sm\psi_{\mathrm{L}}\circ\Lambda^{-1}. \tag{6.55}$$

Consequently, we only get relativistic invariance if $m=0$.

WARNING: The left vs. right conventions in this subsection are dependent on the sign conventions used for the metric. Notes that use $(+,-,-,-)$ signature (Peskin & Schroeder, Weinberg, etc.) will disagree with those that use $(-,+,+,+)$ signature (us).

Exercises and problems

EXERCISE 6.1: Prove that, for $A\in\mathcal{D}'(\mathbb{R}^{1,3};\mathcal{T}^1)$ satisfying the Lorenz gauge condition, $\mathrm{d}{\bigstar}\mathrm{d}(A^\flat)$ is equivalent to $\Box A=0$.

CHAPTER 7

The Schrödinger picture

The formalism of defining a Hilbert space structure directly on a space of wavefunctions on spacetime is known as the **Heisenberg picture**. In the competing **Schrödinger picture,** one instead considers a Hilbert space of wavefunctions depending on $\mathbf{x} \in \mathbb{R}^d$ alone. These are then taken to represent the state of a particle at some moment in time. The wavefunction is then taken to evolve in time. Usually, students are more comfortable with the Schrödinger picture than the Heisenberg picture, but the Heisenberg picture is often preferable in relativistic quantum mechanics. This is because it comes closer to treating time and space on equal footing, as it makes no reference to a preferred foliation of spacetime via spacelike hypersurfaces.

Let π denote a (positive-energy, finite spin) irrep of the Poincaré group $\mathrm{P}^*(1,d)$, $d \in \mathbb{N}^{\geq 2}$, describing a particle of some mass $m \geq 0$, realized wave-mechanically. So,

$$\pi \subseteq \{\text{positive energy } \Psi \in \mathcal{S}'(\mathbb{R}^{1,d};\mathcal{T}) : (\Box + m^2)\Psi = 0\} \tag{7.1}$$

for some Lorentz representation $S : \widetilde{\mathrm{SO}(1,d)} \to \mathrm{GL}(\mathcal{T})$. As we discussed previously, this involves finding a subspace $\mathcal{V} \subseteq \mathcal{T}$, closed under the action of the little group L, on which $S|_L$ is equivalent to π's little group representation.

Recall that any element $\Psi \in \pi$ has the form

$$\Psi(x) = \int_{\mathbb{R}^d} \psi(\mathbf{p}) \frac{e^{-it\sqrt{m^2+\|\mathbf{p}\|^2}+i\mathbf{x}\cdot\mathbf{p}}\,\mathrm{d}^d\mathbf{p}}{(m^2+\|\mathbf{p}\|^2)^{1/4}} \tag{7.2}$$

for some $\psi \in \mathcal{S}'(\mathbb{R}^d;\mathcal{T})$. Consequently,

$$\pi \subseteq C^0(\mathbb{R}_t;\mathcal{S}'(\mathbb{R}^d;\mathcal{T})) \tag{$\star$}$$

Consider, for each $t \in \mathbb{R}$, the evaluation map $\mathrm{eval}_t : \pi \to \mathcal{S}'(\mathbb{R}^d;\mathcal{T})$, sending $\Psi \in \pi$ to $\mathrm{eval}_t \Psi = \Psi(t,-)$. This is a linear injection (as follows from the injectivity of the Fourier transform), so we can port all of the structure on π, the inner product and the Poincaré representation, to the image $\mathrm{eval}_t(\pi) \subset \mathcal{S}'(\mathbb{R}^d;\mathcal{T})$. Because the norm $\|\Psi\|_\pi$ does not depend on the phase of $\psi(\mathbf{p})$, *the subspace* $\mathrm{eval}_t(\pi)$ *does not depend on* t, and so we write

$$\mathrm{slice}(\pi) = \mathrm{eval}_0(\pi). \tag{7.3}$$

The map eval_t lets us identify elements of π with their constant time slices $\in \mathrm{slice}(\pi)$, which transform in a certain way among themselves under Poincaré transformations.

Spatial translations and rotations work in the expected way — $\mathcal{S}'(\mathbb{R}^d;\mathcal{T})$ carries a manifest action of the group $\mathrm{E}^*(d)$ of spatial isometries. However, $\mathcal{S}'(\mathbb{R}^d;\mathcal{T})$ admits no "obvious" action under the generator $H \in \mathfrak{p}$ of time translations nor under boosts, so the action ported from π is doing something nontrivial. The effect of a boost is to mix the solution at different times. A moving observer will not agree with the laboratory frame about what the "constant time" slices $\mathrm{eval}_*(\Psi)$ are, because they will not agree about which spacetime loci are simultaneous. This is the tradeoff between the Heisenberg picture, in which the Hilbert space consists of solutions of a relativistic PDE on spacetime, and the Schrödinger picture, in which the Hilbert space consists of time-slices of those solutions. The latter consists of simpler objects, but the cost of choosing a distinguished spacelike hyperplane $\Sigma_t = \{(t,\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$ is breaking manifest Lorentz covariance.

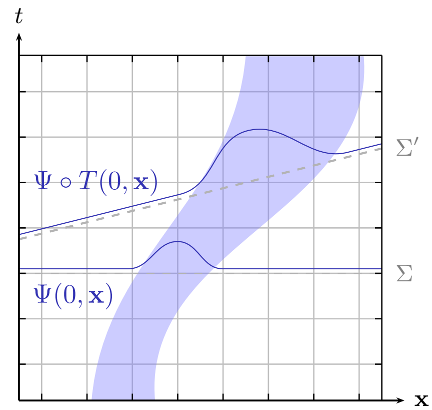


FIGURE 7.1. Restricting a scalar wavefunction $\Psi : \mathbb{R}^{1,d} \to \mathbb{C}$ to different spacelike hyperplanes Σ, Σ' (which can differ by a time-translation and/or a boost) gives different ways of extracting from Ψ a function on *space* $\mathbb{R}^d$. In special relativity, nothing is special about the laboratory time slices $\Sigma_t = \{(t, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$, and so the manifestly covariant Heisenberg picture is usually preferred to the Schrödinger picture.

The effect of time translation, which induces a flow on slice(π), involves solving a first-order partial/pseudodifferential equation with specified initial data. Indeed, a positive-energy solution $\phi \in \mathcal{S}'(\mathbb{R}^{1,d}; \mathcal{T})$ is a solution of the *half-Klein–Gordon equation*

$$\big(- i\partial_t + \sqrt{m^2 - \Delta}\,\big)\Psi = 0. \tag{7.4}$$

The object $\sqrt{m^2 - \Delta}$ is not a partial differential operator but rather a *pseudodifferential* operator (and more specifically a Fourier multiplier):

$$\sqrt{m^2 - \Delta}\Phi(\mathbf{x}) = \mathcal{F}^{-1}_{\mathbf{p}\to\mathbf{x}}\Big(\sqrt{m^2 + \|\mathbf{p}\|^2}\mathcal{F}_{\mathbf{y}\to\mathbf{p}}\Phi(\mathbf{y})\Big), \tag{7.5}$$

where $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ denotes the *spatial* Fourier transform.

Now let's consider the function space $\mathrm{eval}_t(\pi)$ in more detail.

1. Scalar case

When $\mathcal{T} = \mathbb{C}$, the distribution ψ in eq. (7.2) lies in $L^2(\mathbb{R}^d)$, and the norm on π is $\|\Psi\|_\pi = \|\psi\|_{L^2}$. Consequently, we can refine $(\star)$ to

$$\boxed{\pi \subseteq \begin{cases} C^0(\mathbb{R}_t; H^{1/2}(\mathbb{R}^d_{\mathbf{x}})) & (m > 0), \\ C^0(\mathbb{R}_t; \dot{H}^{1/2}(\mathbb{R}^d_{\mathbf{x}})) & (m = 0), \end{cases}} \tag{$\star\star$}$$

where H^s denotes the usual order $s \in \mathbb{R}$ L^2-based *Sobolev space*, and $\dot{H}^s$ is its homogeneous counterpart. So

$$H^s(\mathbb{R}^d) = (-\Delta + 1)^{-s/2} L^2(\mathbb{R}^d). \tag{7.6}$$

Thus, $\Psi(t, -) \in H^{1/2}$ means that it is half an order smoother than a generic element of L^2. The homogeneous space is similar but without the requirement that f itself lies in L^2. For simplicity, we restrict attention to $s < d/2$ (recall that $d \geq 2$).[1] Then:

$$\dot{H}^s(\mathbb{R}^d) = \{\check{g} \in \mathcal{S}'(\mathbb{R}^d) : g \in \|\mathbf{p}\|^{-s} L^2(\mathbb{R}^d_{\mathbf{p}}) \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)\}. \tag{7.7}$$

[1] Generally, the spaces $\dot{H}^s(\mathbb{R}^d)$ are defined as equivalence classes of elements of $\mathcal{S}'(\mathbb{R}^d)$ modulo polynomials. The problem is that constants want to lie in the space whenever $s > 0$, because $\|\mathbf{p}\|^s\delta = 0$. The problem just gets worse as s increases, because any fixed derivative of δ is killed by $\|\mathbf{p}\|^s$ if s is large enough. This is why the quotienting is done. When $s < d/2$, the same effect as quotienting is achieved by requiring that Fourier transforms lie in L^1_{loc}. This is what we did above.

We can endow H^s and $\dot{H}^s$ with norms in many ways. For $\dot{H}^s$, one choice of norm (modulo arbitrary constants of proportionality) is

$$\|f\|_{\dot{H}^{1/2}} = \left\| \sqrt{\|\mathbf{p}\|}\hat{f}(\mathbf{p}) \right\|_{L^2(\mathbb{R}^d)} = \left[\int_{\mathbb{R}^d} \|\mathbf{p}\| |\hat{f}(\mathbf{p})|^2 \,\mathrm{d}^d\mathbf{p} \right]^{1/2}. \tag{7.8}$$

This makes $\dot{H}^{1/2}$ a Hilbert space. For $H^{1/2}$, we have one natural choice for each $m > 0$:

$$\|f\|_{H^{1/2};m} = \|(m^2 + \|\mathbf{p}\|^2)^{1/4}\hat{f}(\mathbf{p})\|_{L^2} = \left[\int_{\mathbb{R}^d} (m^2 + \|\mathbf{p}\|^2)^{1/2} |\hat{f}(\mathbf{p})|^2 \,\mathrm{d}^d\mathbf{p} \right]^{1/2}. \tag{7.9}$$

Each of these makes $H^{1/2}$ into a Hilbert space. Let $\|-\|_{H^{1/2};0} = \|-\|_{\dot{H}^{1/2}}$.

These norms correspond to the norm on π:

$$\begin{aligned} \|\Psi\|_\pi = \|\psi\|_{L^2(\mathbb{R}^d)} &= \|(m^2 + \|\mathbf{p}\|^2)^{1/4}\hat{\Psi}(t,-)\|_{L^2(\mathbb{R}^d)} \\ &= \|\Psi(t,-)\|_{H^{1/2};m}. \end{aligned} \tag{7.10}$$

Thus, it is with respect to the $H^{1/2}$ or $\dot{H}^{1/2}$ norm on $\mathrm{eval}_t(\pi)$ that the Poincaré representation, ported from π, is unitary.

Born's rule, in its abstract form, tells us that the expectation value of any observable O when measured, while the system is in state Ψ, is $\langle O \rangle = \langle \Psi, O\Psi \rangle_{\mathcal{H}}$, where $\mathcal{H}$ is the system's Hilbert space. A natural sort of observable is the location of the particle. The probability that the particle is found in some open region $U \subset \mathbb{R}^d$ when its position is measured should be $\langle 1_U \rangle$, but this does not quite make sense, because multiplication by 1_U is not a bounded operator on $H^{1/2}$. If $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$ then $\langle \chi \rangle$ is well-defined; so, we can take

$$\chi = \rho_\epsilon * 1_U \approx 1_U, \quad 0 < \epsilon \ll 1, \tag{7.11}$$

where $\rho_\epsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$ is a standard mollifier (a smooth approximation of δ), and then use $\langle \chi \rangle$ as a proxy for the ill-defined $\langle 1_U \rangle$. Unfortunately, because of the fractional 1/2 derivative in the definition of the $H^{1/2}$ inner product, the quantity $\langle \chi \rangle$ depends on the values of $\Psi(t,-)$ outside $\operatorname{supp}\chi$.

This "nonlocality" is an unpalatable conclusion to some. It has even led to the insistence that the Klein–Gordon equation does not constitute a physically acceptable one-particle theory, necessitating an embedding into a quantum field theory. Nevertheless, scalar particles *do* exist. The Higgs boson is an elementary example, albeit unstable. Composite examples, like the Helium-4 nucleus in its ground state (α-particle), or pions, are modeled by the one-particle theory. This conclusion is not vitiated by the existence of internal degrees-of-freedom, which are relevant to the spectrum of excited states. Moreover, the spectrum of any quantum field theory decomposes as a direct integral of Poincaré irreps, and some of these will likely be spin-0. The conclusions above therefore apply, unpalatable or not.

Remark 7.1 (Comparison with non-relativistic QM). In Schrödinger's non-relativistic wave mechanics, the wavefunction of a particle at time $t \in \mathbb{R}$ is an element

$$\Psi(t,-) \in L^2(\mathbb{R}^d), \tag{7.12}$$

and *Born's rule* tells us that the probability of finding the particle in an open set $U \subset \mathbb{R}^d$ at that time is

$$\int_U |\Psi(t,\mathbf{x})|^2 \,\mathrm{d}^d\mathbf{x} = \langle \Psi(t,-), 1_U \Psi(t,-) \rangle_{L^2(\mathbb{R}^d)}. \tag{7.13}$$

Thus, the probability of finding a particle at $\mathbf{x} \in \mathbb{R}^d$ only has to do with the value of $\Psi(t,-)$ at $\mathbf{x}$, a very intuitive result.

The reason why non-relativistic quantum mechanics uses the L^2 inner product and relativistic quantum mechanics uses the $H^{1/2}$ inner product has to do with the difference between their symmetry groups (Galilean vs. Poincaré) and corresponding "dispersion relations." The origin of the '1/2' is the denominator in the Lorentz-invariant measure

$$\mathrm{d}\mu_{X_{m,+}} = \delta(p^2+m^2)\,\mathrm{d}^{1+d}p = \frac{\mathrm{d}^d\mathbf{p}}{2\sqrt{m^2+\|\mathbf{p}\|^2}} \tag{7.14}$$

on the Lorentz orbit $X_{m,+}$. In non-relativistic quantum mechanics, the place of the hyperboloid $X_{m,+}$ is taken by

$$W_m = \{p\in\mathbb{R}^{1,d} : p^0 = \|\mathbf{p}\|^2/2m\}. \tag{7.15}$$

The Galilean-invariant measure on this paraboloid is

$$\mathrm{d}\mu_{W_m} = \delta(2mp^0 - \|\mathbf{p}\|^2)\,\mathrm{d}^{1+d}p = \mathrm{d}^d\mathbf{p}/2m. \tag{7.16}$$

Notice the absence of a power of $\|\mathbf{p}\|$ in the denominator. So, $L^2(\mathbb{R}^d)$ is the right Hilbert space to describe the time-slices of wavefunctions in non-relativistic quantum mechanics. This is how Born's rule, in its original form, follows from Galilean symmetry. ■

The problem with the L^2 norm vis-à-vis Lorentzian relativity is not difficult to understand. Consider a solution $\phi\in\pi$ to the Klein–Gordon equation trapped in a box $[-1,1]^d$, which is stationary in the laboratory frame, and whose worldtube is $\mathbb{R}_t\times[-1,1]^d\subset\mathbb{R}^{1,d}$. For simplicity, we assume that ϕ is in a standing wave state, so $|\phi(t,\mathbf{x})|^2$ is independent of t. From the perspective of a moving observer, the volume of the box is contracted by a factor of $\gamma = 1/\sqrt{1-v^2}$, where v is the speed of the observer. But all observers agree about the values of scalar fields at fixed spacetime loci, so, from the perspective of the moving observer, the L^2-norm of ϕ has been contracted by a factor of $\gamma^{1/2}$.

2. General representations

The discussion above requires modification when $\mathcal{T}$ is nontrivial. Then, since the Lorentz group has no nontrivial unitary finite-dimensional representations, $\mathcal{T}$ is *not* unitary. It comes with no canonical norm, so "$\|-\|_{L^2(\mathbb{R}^d;\mathcal{T})}$" is not automatically defined. Of course, $\mathcal{T}$ is finite-dimensional, so there exists some norm on it. This is good enough as far as defining the topology is concerned, as all norms on a finite-dimensional vector space give the same topology. But for Born's rule $\mathbb{E}[O] = \langle\Psi, O\Psi\rangle_\pi$, we need to find the exact norm that matches π.

Returning to the definition of π, it appears we need a different norm for each momentum $\mathbf{p}$. The function $\psi:\dot{\mathbb{R}}^d\to\mathcal{T}$ in eq. (7.2) is a section of some vector bundle

$$E\subset\mathbb{R}^d\times\mathcal{T} \tag{7.17}$$

over $\mathbb{R}^d$ if $m>0$ or $\dot{\mathbb{R}}^d$ if $m=0$. The fiber of this bundle over $\mathbf{p}$ is $E_{\mathbf{p}} = D[p]\mathcal{V}$, where $\mathcal{V}\subset\mathcal{T}$ is the (unitary!) little group representation used in the wave-mechanical realization of π and $D[p] : p_{\circledast}\mapsto p$ is the standard boost taking the reference momentum $p_{\circledast}\in X_{m,+}$ to p. The norm on $E_{\mathbf{p}}$ is transported from $\mathcal{V}$, meaning

$$\|\psi\|_{E_{\mathbf{p}}} = \|S(D[p])^{-1}\psi\|_{\mathcal{V}}. \tag{7.18}$$

Here we are using the norm on $\mathcal{V}$ that makes the little group representation $S|_{L[p_{\circledast}]}$ unitary. The norm $\|-\|_\pi$ can now be written

$$\|\Psi\|_\pi = \|\psi\|_{L^2(\mathbb{R}^d;E)} = \Big[\int_{\mathbb{R}^d}\Big\|S\Big(D\Big[\sqrt{m^2+\|\mathbf{p}\|^2},\mathbf{p}\Big]\Big)^{-1}\psi\Big\|_{\mathcal{V}}^2\,\mathrm{d}^d\mathbf{p}\Big]^{1/2} \tag{7.19}$$

(up to constants of proportionality).

Annoyingly, $\mathcal{V}$ is not generally closed under the Lorentz group, so $\|-\|_{\mathcal{V}}$ cannot be used to define a norm on $\mathcal{T}\supset\mathcal{V}$. Suppose we have a (not necessarily definite) Hermitian form $Q:\mathcal{T}\to\mathbb{R}$ on $\mathcal{T}$.

We can define a quasinorm $\|-\|_{H^{1/2}(\mathbb{R}^d;\mathcal{T});m}$ by

$$\|f\|_{H^s(\mathbb{R}^d;\mathcal{T});m} = \left[\int_{\mathbb{R}^d} (m^2 + \|\mathbf{p}\|^2)^s Q(\hat{f}(\mathbf{p}))\,\mathrm{d}^d\mathbf{p}\right]^{1/2}. \tag{7.20}$$

If Q is not definite, then this is not really a norm, because it is not positive definite on all of $H^s(\mathbb{R}^d;\mathcal{T})$.

Suppose that Q restricts to $v \mapsto \|v\|^2_{\mathcal{V}}$ on $\mathcal{V}$. Additionally, let us suppose that Q transforms under Lorentz boosts in the following sensible way: $\exists k \in \mathbb{N}$ such that

$$\begin{aligned} Q(S(D[p])^{-1}v) &= \Big(\frac{m^2 + \|\mathbf{p}\|^2}{m^2 + \|\mathbf{p}_{\divideontimes}\|^2}\Big)^{-k/2} Q(v) \\ &= \begin{cases} \gamma^{-k} Q(v) & (m>0), \\ \|\mathbf{p}\|^{-k} Q(v) & (m=0), \end{cases} \end{aligned} \tag{7.21}$$

equivalently

$$Q(S(D[p])v) = \begin{cases} \gamma^{k} Q(v) & (m>0), \\ \|\mathbf{p}\|^{k} Q(v) & (m=0) \end{cases} \tag{7.22}$$

for all $p \in X_{m,+}$ and $v \in \mathcal{T}$. Here, γ is the Lorentz factor associated with the boosts $D[p]$, $D[p]^{-1}$. The term k has physical significance:

($k=0$) If $k=0$, then Q "transforms like a scalar," i.e. it is Lorentz invariant.[2] Examples in the $d=3$ case include

$$|\phi|^2, \quad \bar{\psi}\psi, \quad \bar{\psi}\gamma^5\psi, \quad A^{\mu*}A_\mu, \quad F^*_{\mu\nu}F^{\mu\nu} \tag{7.23}$$

where $\phi \in \mathbb{C}$ is a scalar, $\psi \in \mathbb{C}^4$ is a Dirac spinor, $A \in \mathbb{C}^4$ is a four-vector, and $F \in \wedge\mathbb{C}^4$ is an anti-symmetric matrix, respectively. These are known as (invariant) *mass terms.*

($k=1$) If $k=1$, then Q transforms like the 0th component ρ of a vector $j=(\rho,\mathbf{j})$, that is a *charge density.* Indeed, the ${\Lambda^0}_0$ matrix element of a boost is exactly γ. Examples include

$$\psi^\dagger\psi = \bar{\psi}\gamma^0\psi, \tag{7.24}$$

where ψ is a Dirac spinor as before. (Examples can also be constructed using Rarita–Schwinger fields.)

($k=2$) By the same reasoning as for $k=1$, if $k=2$ then Q transforms like the 00th component T^{00} of a symmetric traceless tensor $T^{\mu\nu}$, that is like an *energy density.* An example is the electromagnetic energy density

$$T^{00} = F^{0\alpha}F^0_{\ \alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}, \tag{7.25}$$

where $F \in \wedge\mathbb{C}^4$.

($k=3$) Et cetera.

Now let's evaluate the norm eq. (7.19). Explicitly,

$$\begin{aligned} \|\Psi\|^2_\pi &= \int_{\mathbb{R}^d} Q\Big(S\Big(D\Big[\sqrt{m^2+\|\mathbf{p}\|^2},\mathbf{p}\Big]\Big)^{-1}\psi\Big)\,\mathrm{d}^d\mathbf{p} \\ &= \int_{\mathbb{R}^d} \Big(1 + \frac{\|\mathbf{p}\|^2}{m^2}\Big)^{-k/2} Q(\psi)\,\mathrm{d}^d\mathbf{p} = \int_{\mathbb{R}^d} Q\Big(\Big(1+\frac{\|\mathbf{p}\|^2}{m^2}\Big)^{-k/4}\psi\Big)\,\mathrm{d}^d\mathbf{p} \\ &\propto \int_{\mathbb{R}^d} Q\Big((m^2 - \triangle)^{\frac{1-k}{4}}\Psi(t,\mathbf{x})\Big)\,\mathrm{d}^d\mathbf{x} = \|\Psi(t,-)\|^2_{H^{\frac{1-k}{2}}(\mathbb{R}^d;\mathcal{T});m}, \end{aligned} \tag{7.26}$$

[2] Because the Lorentz group has no nontrivial unitary finite-dimensional representations, the only way (besides trivial cases) this is possible is if Q is not definite.

where the last line used Parseval–Plancherel. In summary, the time slices of an element of π have $(1-k)/2$ orders of L^2-based Sobolev regularity:

$$\boxed{\pi \subseteq \begin{cases} C^0(\mathbb{R}_t; H^{\frac{1-k}{2}}(\mathbb{R}^d;\mathcal{T})) & (m>0), \\ C^0(\mathbb{R}_t; \dot{H}^{\frac{1-k}{2}}(\mathbb{R}^d;\mathcal{T})) & (m=0), \end{cases}} \tag{7.27}$$

and

$$\boxed{\|\Psi\|_\pi = \|\Psi(t,-)\|^2_{H^{\frac{1-k}{2}}(\mathbb{R}^d;\mathcal{T});m}} \tag{7.28}$$

for all $t \in \mathbb{R}$.

Something special happens when $k=1$:

$$\|\Psi\|_\pi = \|\Psi(t,-)\|_{L^2(\mathbb{R}^d;\mathcal{T})}. \tag{7.29}$$

Then, the π norm is just the L^2 norm on the time slices. Born's rule, in its original local form, is recovered, and $|\Psi(t,\mathbf{x})|^2$ can be interpreted as the probability density of finding the particle at $\mathbf{x}$. The fact that the Dirac representation admits a Q with $k=1$ is one reason why the Dirac equation is more palatable than the Klein–Gordon equation as a one-particle theory — we have a local probability density.

Remark: Often, one $\mathcal{T}$ admits multiple Q with different k. We saw this above for Dirac bispinors, two quadratic forms being $\psi \mapsto \bar{\psi}\psi, \psi^\dagger\psi$. But how can the right norm be $H^{1/2}(\mathbb{R}^d;\mathcal{T})$ with respect to one choice of Q and $L^2(\mathbb{R}^d;\mathcal{T})$ with respect to another? The key thing to remember is that $\psi \mapsto \bar{\psi}\psi$ is not positive definite. The actual components of $\psi(t,-)$ are generically no better than L^2, but when we form the *difference* $\bar{\psi}\psi$ between the norm-squared of various components, the most singular parts cancel, leaving a remainder with an extra half-order of regularity. This generalizes beyond the Dirac representation.

CHAPTER 8

The second-quantization functor and the CCR/CAR

First quantization is a mystery, but second quantization is a functor! [Attributed to Edward Nelson]

In ordinary quantum mechanics, the ket space of a multipartite system is the tensor product of the Hilbert spaces describing the individual systems. When all of the systems are described by the same "one-particle" Hilbert space $\mathcal{H}$, the Hilbert space describing the multi-particle system is the tensor space

$$\mathcal{T}\mathcal{H} = \overline{\bigoplus}_{j=0}^{\infty} \mathcal{H}^{\otimes j} = \overline{\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \mathcal{H}^{\otimes 3} \oplus \cdots}, \tag{8.1}$$

in which the subspace

$$\mathcal{H}^{\otimes j} = \mathcal{H}_{j\text{-particle}}, \tag{8.2}$$

consisting of all j-tensors, is interpreted as the space of all j-particle configurations.[1] The zero-particle Hilbert space $\mathbb{C} = \mathcal{H}^{\otimes 0}$ is spanned by a vector $\Omega \in \mathcal{T}\mathcal{H}$ representing the vacuum.

Nature does not seem to make much use of this possibility. Physicists attribute this to the *indistinguishability* of different instances of the same particle species. Multi-particle states like $\phi \otimes \psi$, $\psi \otimes \phi$, which differ only in which particles are in which states, are not considered distinct. It makes sense to say that "one electron is here and another over there," but not that "electron one is over here and electron two over there." So, physicists prescribe restricting attention to one of two subspaces,

$$\operatorname{Sym} \mathcal{H}, \wedge\mathcal{H} \subseteq \mathcal{T}\mathcal{H}, \tag{8.3}$$

the subspace of *totally symmetric* tensors and the subspace of *totally anti-symmetric tensors*, respectively:

$$\begin{aligned} \operatorname{Sym} \mathcal{H} &= \overline{\bigoplus}_{j=0}^{\infty} \operatorname{Sym}^j \mathcal{H}, \\ \wedge\mathcal{H} &= \overline{\bigoplus}_{j=0}^{\infty} \wedge^j \mathcal{H}, \end{aligned} \tag{8.4}$$

where $\operatorname{Sym}^j \mathcal{H} = \mathfrak{S}\mathcal{H}^{\otimes j}$, $\wedge^j\mathcal{H} = \mathfrak{A}\mathcal{H}^{\otimes j}$ respectively. Here,

$$\mathfrak{S} : \phi_1 \otimes \cdots \otimes \phi_j \mapsto \frac{1}{j!} \sum_{\sigma \in \mathfrak{S}_j} \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(j)} \tag{8.5}$$

is the symmetrization operator (initially defined on pure tensors and then extended to all of $\mathcal{T}\mathcal{H}$ linearly) and

$$\mathfrak{A} : \phi_1 \otimes \cdots \otimes \phi_j \mapsto \frac{1}{j!} \sum_{\sigma \in \mathfrak{S}_j} (-1)^\sigma \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(j)} \tag{8.6}$$

is the anti-symmetrization operator (also initially defined on pure tensors and then extended to all of $\mathcal{T}\mathcal{H}$ linearly); $\mathfrak{S}_j$ is the symmetric group on j objects. The maps $\mathfrak{S}, \mathfrak{A}$ are orthogonal projections onto $\operatorname{Sym} \mathcal{H}$, $\wedge\mathcal{H}$, respectively. The two Hilbert spaces $\operatorname{Sym} \mathcal{H}$, $\wedge\mathcal{H}$ are called the *bosonic* Fock space and the *fermionic* Fock space, respectively. A "boson" is a particle species whose multi-particle [Exercise 8.1]

[1] The bar over the orthogonal sum in eq. (8.1) denotes a completion, so $\mathcal{T}\mathcal{H}$ consists of sequences $\{\psi_j\}_{j=0}^\infty$ of $\psi_j \in \mathcal{H}^{\otimes j}$ such that $\sum_{j=0}^\infty \|\psi_j\|^2_{\mathcal{H}^{\otimes j}} < \infty$.

configurations are described by elements of $\operatorname{Sym}\mathcal{H}$. Such particles are said to obey Bose–Einstein statistics. Likewise, a "fermion" is a particle species whose multi-particle configurations are described by elements of $\wedge\mathcal{H}$. Such particles are said to obey Fermi–Dirac statistics.

This lecture is about the functors $\mathcal{H} \mapsto \operatorname{Sym}\mathcal{H}, \wedge\mathcal{H}$, the bosonic and fermionic *Fock functors*.

The Fock spaces will be shown to host "creation/annihilation" operators (a.k.a. "ladder" operators)

$$a(f), a(f)^\dagger : \mathcal{D} \to \mathcal{D}, \tag{8.7}$$

defined on some dense domain $\mathcal{D}$, which satisfy the canonical commutation relations (CCR) or canonical anti-commutation relations (CAR):

$$\begin{aligned} [a(f), a(g)^\dagger]_\pm &= \langle f, g\rangle_{\mathcal{H}} I, \\ [a(f), a(g)]_\pm &= [a(f)^\dagger, a(g)^\dagger]_\pm = 0, \end{aligned} \tag{8.8}$$

where $[A, B]_\pm$ is either the commutator $[A, B] = AB - BA$ or anti-commutator $\{A, B\} = AB + BA$ of A, B, depending on whether the Fock space is bosonic (commutator) or fermionic (anti-commutator).

Remark 8.1. The reader may not find the "metaphysical" justification above, the argument from indistinguishability, entirely convincing. For bosonic particles, an alternative justification can be given later: free bosonic particles are the quantized perturbations of a free quantized field. From this perspective, the ontologically basic entity is the quantum field — it just so happens that the bosonic Fock space is naturally isomorphic to the Hilbert space consisting of wavefunctions on the space of field configurations.

Parallel remarks apply to fermionic particles, except one has to talk about *Grassmann-valued* fields. If we are willing to countenance that, then we might as well bite the bullet and accept the argument from indistinguishability. For this reason, I personally prefer to take the Fock spaces as basic and fields as derived, at least in the discussion of free fields. ∎

0.1. Low-dimensional examples.

Example 8.2 ($j = 2$). $\operatorname{Sym}^2\mathcal{H} \oplus \wedge^2\mathcal{H} = \mathcal{H}^{\otimes 2}$, and

$$\begin{aligned} \operatorname{Sym}^2\mathcal{H} &= \{\phi\otimes\psi + \psi\otimes\phi : \phi, \psi \in \mathcal{H}\} \\ \wedge^2\mathcal{H} &= \{\phi\otimes\psi - \psi\otimes\phi : \phi, \psi \in \mathcal{H}\}. \end{aligned} \tag{8.9}$$

So, $\operatorname{Sym}^2\mathcal{H} \perp \wedge^2\mathcal{H}$, and every 2-particle state can be decomposed into a symmetric part and an anti-symmetric part. ∎

Example 8.3 ($j = 3$). We still have $\operatorname{Sym}^3\mathcal{H} \perp \wedge^3\mathcal{H}$, with

$$\operatorname{Sym}^3\mathcal{H} = \{\varphi\otimes\phi\otimes\psi + \text{permutations} : \varphi, \phi, \psi \in \mathcal{H}\} \tag{8.10}$$

$$\wedge^3\mathcal{H} = \{\varphi\otimes\phi\otimes\psi + \phi\otimes\psi\otimes\varphi + \psi\otimes\varphi\otimes\phi - (\varphi\leftrightarrow\phi) : \varphi, \phi, \psi \in \mathcal{H}\}. \tag{8.11}$$

Note that $\operatorname{Sym}^3\mathcal{H} \oplus \wedge^3\mathcal{H} \subsetneq \mathcal{H}^{\otimes 3}$; three-tensors cannot be decomposed into a purely symmetric part and a purely anti-symmetric part. ∎

0.2. Some notation. We assume throughout this section that $\mathcal{H}$ is infinite-dimensional, but everything applies in the finite-dimensional case, with minor notational adjustments.

Let $\mathcal{D} = \bigoplus_{j=0}^\infty \mathcal{H}^{\otimes j}$ denote the subspace of $\mathcal{T}\mathcal{H}$ consisting of multi-particle configurations with boundedly many particles. Here, the direct sum is the one in the category of inner product spaces which are not necessarily complete. So if $\psi \in \mathcal{D}$, then ψ is a linear combination of elements of $\mathcal{H}^{\otimes 0}, \cdots, \mathcal{H}^{\otimes N}$ for some finite N (depending on ψ).

On $\mathcal{D}$, we can define an operator $N : \mathcal{D} \to \mathcal{D}$ by setting $N\psi = j\psi$ whenever $\psi \in \mathcal{H}^{\otimes j}$ is a pure j-tensor and then extending this definition to all of $\mathcal{D}$ via linearity. This is the "number operator".

It extends continuously to a map $\mathcal{D}(N) \to \mathcal{TH}$, where $\mathcal{D}(N)$ consists of those $\psi \in \mathcal{H}$ such that

$$\sum_{j=0}^{\infty} j^2 \|\psi_j\|^2_{\mathcal{H}^{\otimes j}} < \infty, \tag{8.12}$$

where ψ_j is the component of ψ in $\mathcal{H}^{\otimes j}$, and where $\mathcal{D}(N)$ is endowed with the norm

$$\|\psi\|_{\mathcal{D}(N)} = \sqrt{\sum_{j=0}^{\infty} (j+1)^2 \|\psi_j\|^2_{\mathcal{H}^{\otimes j}}}. \tag{8.13}$$

Considered as an unbounded operator on $\mathcal{TH}$, the number operator N is self-adjoint, with domain $\mathcal{D}(N)$. [Problem 8.1]

If $\phi, \psi \in \mathcal{D}$, let

$$\begin{aligned} \phi \odot \psi &= \mathfrak{S}(\phi \otimes \psi) \\ \phi \wedge \psi &= \mathfrak{A}(\phi \otimes \psi). \end{aligned} \tag{8.14}$$

Then, $\odot$ defines an associative product on $\operatorname{Sym}\mathcal{H}$, and $\wedge$ defines an associative (but non-commutative) product on $\wedge\mathcal{H}$.

Let $\phi_1, \phi_2, \dots \in \mathcal{H}$ denote any orthonormal basis for $\mathcal{H}$. If $n_1, n_2, n_3, \dots \in \mathbb{N}$ and only a finite number of $n_1, n_2, n_3, \dots$ are nonzero, then let

$$|n_1, n_2, n_3, \dots\rangle_{\operatorname{Sym}\mathcal{H}} = \sqrt{\frac{N!}{n_1! n_2! \cdots}} \phi_1^{\otimes n_1} \odot \phi_2^{\otimes n_2} \odot \cdots \in \operatorname{Sym}\mathcal{H} \tag{8.15}$$

for $n_1, n_2, n_3, \dots \in \mathbb{N}$. Similarly, if $n_1, n_2, n_3, \dots \in \{0, 1\}$ and only a finite number of these are nonzero, then let

$$|n_1, n_2, n_3, \dots\rangle_{\wedge\mathcal{H}} = \sqrt{N!} \phi_1^{\otimes n_1} \wedge \phi_2^{\otimes n_2} \wedge \cdots \in \wedge\mathcal{H} \tag{8.16}$$

denote the wedge product of those finitely many ϕ_j for which $n_j \neq 0$.

PROPOSITION 8.4. *These are orthonormal bases of* $\operatorname{Sym}\mathcal{H}, \wedge\mathcal{H}$, *respectively.* ■

PROOF. The orthogonality follows immediately from the form of the inner product on $\mathcal{TH}$. The more nontrivial thing is that each of the vectors above has unit norm.

The symmetric product $\phi_1^{\otimes n_1} \odot \phi_2^{\otimes n_2} \odot \cdots$ can be written as the average over pure N-tensors $\psi_1 \otimes \psi_2 \otimes \cdots$ with the constraint that each $\psi_\bullet$ is some $\phi_\bullet$, with ϕ_j appearing among the $\psi_\bullet$'s exactly n_j times:

$$\phi_1^{\otimes n_1} \odot \phi_2^{\otimes n_2} \odot \cdots = \frac{1}{\#} \sum \psi_1 \otimes \psi_2 \otimes \cdots \tag{8.17}$$

where $\# = \binom{N}{n_1, n_2, \cdots} = N!/n_1! n_2! \cdots$ is the number of terms in the sum. Because the distinct $\psi_1 \otimes \psi_2 \otimes \cdots$ are orthonormal (owing to the orthonormality of the ϕ_j's),

$$\|\phi_1^{\otimes n_1} \odot \phi_2^{\otimes n_2} \odot \cdots\| = \sqrt{\frac{1}{\#^2} \sum 1} = \sqrt{\frac{1}{\#}}. \tag{8.18}$$

This shows that $|n_1, n_2, n_3, \dots\rangle_{\operatorname{Sym}\mathcal{H}}$, with the normalization constant appearing in eq. (8.15), is a unit vector.

The argument for the anti-symmetric product is completely analogous, except for some signs (in front of orthonormal vectors), and the relevant $\psi_1 \otimes \psi_2 \otimes \cdots$ are those in which the ϕ_j's for which $n_j = 1$ appear in some order. The number $\#$ of distinct possible $\psi_1 \otimes \psi_2 \otimes \cdots$ is therefore $\# = N!$. □

1. Creation/annihilation operators

First, for $f \in \mathcal{H}$, define $a(f)$ on all of $\mathcal{D}$ by specifying its action on pure tensors:

$$\begin{aligned} a(f)\Omega &= 0 \\ a(f)\psi \otimes \phi &= \sqrt{N+1}\langle f, \psi\rangle_{\mathcal{H}}\phi \end{aligned} \tag{8.19}$$

for any finite tensor $\phi \in \mathcal{T}\mathcal{H}$, $\psi \in \mathcal{H}$, and then extending linearly and continuously to each $\mathcal{H}^{\otimes j}$. Our convention on $\langle -, -\rangle$ is that it is anti-linear in the first slot and linear in the second. That is why ψ appears second — it has to appear in the linear slot, for $a(f)$ to be a well-defined linear operator.

So, $a(f)$ annihilates the vacuum and maps

$$a(f) : \mathcal{H}^{\otimes j} \to \mathcal{H}^{\otimes (j-1)} \tag{8.20}$$

for each $j \in \mathbb{N}^+$, with $a(f)|_{\mathcal{H}^{\otimes j}} = \langle f, -\rangle_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{H}}^{\otimes(j-1)}$. Thus, $a(f)$ "annihilates" an instance of f from the state to which it's applied, which is why it is called an annihilation operator. Modulo the particular choice of normalization $\sqrt{N+1}$ (which is chosen to make the algebra nicer later), its definition is natural.

1.1. Bosonic case.

Note that if ψ is a totally symmetric tensor, then so is $a(f)\psi$. For example,

$$a(f)(\psi \odot \phi \odot \varphi) = \frac{1}{\sqrt{3}}(\langle f, \psi\rangle_{\mathcal{H}}\phi \odot \varphi + \langle f, \phi\rangle_{\mathcal{H}}\varphi \odot \psi + \langle f, \varphi\rangle_{\mathcal{H}}\psi \odot \phi). \tag{8.21}$$

So, $a(f)$ restricts to an operator on the finite-particle subspace $\mathcal{D} \cap \operatorname{Sym} \mathcal{H}$ of the bosonic Fock space.

On the other hand, the *creation* operator $a(f)^\dagger : \mathcal{D} \cap \operatorname{Sym} \mathcal{H} \to \mathcal{D} \cap \operatorname{Sym} \mathcal{H}$ is defined by

$$a(f)^\dagger : \mathcal{D} \ni \psi \mapsto f \odot (\sqrt{N+1}\psi) = \sqrt{N}(f \odot \psi). \tag{8.22}$$

So, $a(f)^\dagger$ "creates" an instance of f. Modulo the choice of normalization, its definition is natural. Do not yet interpret the '$\dagger$' as an adjoint operation.

Note that $a(f)$ depends *anti*-linearly on f, whereas $a(f)^\dagger$ depends linearly on f:

$$\begin{aligned} a(\lambda f + g) &= \lambda^* a(f) + a(g), \\ a(\lambda f + g)^\dagger &= \lambda a(f)^\dagger + a(g)^\dagger. \end{aligned} \tag{8.23}$$

They are linear operators on Fock space, however.

Lemma 8.5.

$$a(\phi_1)|n_1, n_2, n_3, \dots\rangle_{\operatorname{Sym}\mathcal{H}} = \begin{cases} 0 & (n_1 = 0), \\ \sqrt{n_1}|n_1 - 1, n_2, n_3, \dots\rangle_{\operatorname{Sym}\mathcal{H}} & (\text{otherwise}). \end{cases} \tag{8.24}$$

■

Proof. The $n_1 = 0$ case is tautological, so assume $n_1 \geq 1$. Recall that we are using $\phi_1, \phi_2, \dots$ to denote an orthonormal basis for $\mathcal{H}$. Let $\# = n_1 + n_2 + \dots$ (i.e. the value of N on $|n_1, n_2, \cdots\rangle_{\operatorname{Sym}\mathcal{H}}$), and let $\psi_1, \dots, \psi_\# \in \{\phi_1, \phi_2, \dots\}$ be n_1 copies of ϕ_1, followed by n_2 copies of ϕ_2, and so on (in that order!). Then,

$$\begin{aligned} a(\phi_1)|n_1, n_2, \dots\rangle_{\mathrm{Sym}\,\mathcal{H}} &= \sqrt{\frac{\#!}{n_1!n_2!\dots}}\frac{1}{\#!}\sum_{\sigma\in\mathfrak{S}_\#} a(\phi_1)\psi_{\sigma(1)}\otimes\dots\otimes\psi_{\sigma(\#)} \\ &= \sqrt{\frac{(\#-1)!}{n_1!n_2!\dots}}\frac{1}{(\#-1)!}\sum_{\sigma\in\mathfrak{S}_\#}\langle\phi_1, \psi_{\sigma(1)}\rangle_{\mathcal{H}}\cdot\psi_{\sigma(2)}\otimes\dots\otimes\psi_{\sigma(\#)} \\ &= \sqrt{\frac{(\#-1)!}{n_1!n_2!\dots}}\frac{n_1}{(\#-1)!}\sum_{\sigma\in\mathfrak{S}_{\#-1}}\tilde\psi_{\sigma(1)}\otimes\dots\otimes\tilde\psi_{\sigma(\#)}, \end{aligned} \tag{8.25}$$

where $\tilde\psi_j = \psi_{1+j}$, using the fact that $\phi_1 \perp \psi_{\sigma(1)}$ unless $\psi_{\sigma(1)} = \phi_1$, which holds if and only if $\sigma(1)$ is one of $1, \dots, n_1$. Recognizing the right-hand side of eq. (8.25) as proportional to the symmetric product of $\psi_2, \dots, \psi_\#$:

$$\begin{aligned} a(\phi_1)|n_1, n_2, \dots\rangle_{\mathrm{Sym}\,\mathcal{H}} &= n_1\sqrt{\frac{(\#-1)!}{n_1!n_2!\dots}}\psi_2\odot\dots\odot\psi_\# \\ &= \sqrt{n_1}\cdot\sqrt{\frac{(\#-1)!}{(n_1-1)!n_2!\dots}}\psi_2\odot\dots\odot\psi_\# = \sqrt{n_1}|n_1-1, n_2, \dots\rangle_{\mathrm{Sym}\,\mathcal{H}}, \end{aligned} \tag{8.26}$$

as claimed. □

LEMMA 8.6. $a(\phi_1)^\dagger|m_1, m_2, \dots\rangle_{\mathrm{Sym}\,\mathcal{H}} = \sqrt{m_1+1}|m_1+1, m_2, \dots\rangle_{\mathrm{Sym}\,\mathcal{H}}$. ■

PROOF. Let $\# = m_1 + m_2 + \dots$.

$$\begin{aligned} a(\phi_1)^\dagger|m_1, m_2, \dots\rangle_{\mathrm{Sym}\,\mathcal{H}} &= \sqrt{\frac{\#!}{m_1!m_2!\dots}}a(\phi_1)(\phi_1^{\otimes m_1}\odot\phi_2^{\otimes m_2}\odot\cdots) \\ &= \sqrt{\frac{(\#+1)!}{m_1!m_2!\dots}}\phi_1^{\otimes(m_1+1)}\odot\phi_2^{\otimes m_2}\odot\dots = \sqrt{m_1+1}|m_1+1, m_2, \dots\rangle_{\mathrm{Sym}\,\mathcal{H}}. \end{aligned} \tag{8.27}$$

□

$$\begin{tikzcd} |0\rangle \arrow[r, bend left, "a^\dagger"] & |1\rangle \arrow[l, bend left, "a"] \arrow[loop above, "N"] \arrow[r, bend left, "a^\dagger/\sqrt{2}"] & |2\rangle \arrow[l, bend left, "a/\sqrt{2}"] \arrow[r, bend left, "a^\dagger/\sqrt{3}"] & \cdots \arrow[l, bend left, "a/\sqrt{3}"] \end{tikzcd}$$

FIGURE 8.1. The action of creation/annihilation operators.

PROPOSITION 8.7. *For any $f \in \mathcal{H}$, the (unbounded) operators $a(f), a(f)^\dagger$ are adjoints (as the notation suggests), in the sense that*

$$\langle a(f)\psi, \phi\rangle_{\mathrm{Sym}\,\mathcal{H}} = \langle\psi, a(f)^\dagger\phi\rangle_{\mathrm{Sym}\,\mathcal{H}}, \tag{8.28}$$

for all $\phi, \psi \in \mathcal{D} \cap \mathrm{Sym}\,\mathcal{H}$ and $f \in \mathcal{H}$. ■

PROOF. It suffices to prove the identity for $\phi = |m_1, m_2, \dots\rangle_{\mathrm{Sym}\,\mathcal{H}}$ and $\psi = |n_1, n_2, \dots\rangle_{\mathrm{Sym}\,\mathcal{H}}$, for some $m_1, n_1, m_2, n_2, \dots \in \mathbb{N}$, only finitely many of which are nonzero. Moreover, using the linearity/anti-linearity of the maps $f \mapsto a(f), a(f)^\dagger$, it suffices to consider the case when $f \in$

$\{\phi_1, \phi_2, \dots\}$, i.e. when f is an element of our preferred orthonormal basis for $\mathcal{H}$. Without loss of generality, we can order the ϕ_j's such that $f = \phi_1$.

Lemma 8.5 says

$$\begin{aligned}\langle a(f)\psi, \phi\rangle_{\operatorname{Sym}\mathcal{H}} &= \begin{cases} 0 & (n_1 = 0) \\ \sqrt{n_1}\langle n_1 - 1, n_2, \dots | m_1, m_2, \dots \rangle & (\text{otherwise}) \end{cases} \\ &= \begin{cases} 0 & (m_1 \neq n_1 - 1 \text{ or } m_k \neq n_k \text{ for some } k \in \mathbb{N}^{\geq 2}) \\ \sqrt{n_1} & (\text{otherwise}). \end{cases}\end{aligned} \tag{8.29}$$

Lemma 8.6 says

$$\begin{aligned}\langle \psi, a(f)^\dagger \phi\rangle_{\operatorname{Sym}\mathcal{H}} &= \sqrt{m_1 + 1}\langle n_1, n_2, \dots | m_1 + 1, m_2, \dots \rangle \\ &= \begin{cases} 0 & (m_1 + 1 \neq n_1 \text{ or } m_k \neq n_k \text{ for some } k \in \mathbb{N}^{\geq 2}) \\ \sqrt{m_1 + 1} & (\text{otherwise}). \end{cases}\end{aligned} \tag{8.30}$$

The conditions for $\langle a(f)\psi, \phi\rangle_{\operatorname{Sym}\mathcal{H}}, \langle \psi, a(f)^\dagger \phi\rangle_{\operatorname{Sym}\mathcal{H}}$ to be 0 coincide, and otherwise

$$\sqrt{n_1} = \sqrt{m_1 + 1}, \tag{8.31}$$

so $\langle a(f)\psi, \phi\rangle_{\operatorname{Sym}\mathcal{H}}, \langle \psi, a(f)^\dagger \phi\rangle_{\operatorname{Sym}\mathcal{H}}$ coincide then as well. □

Proposition 8.8 (CCR). *For all $f, g \in \mathcal{H}$,*

$$[a(f), a(g)^\dagger] = \langle f, g\rangle_{\mathcal{H}} \operatorname{id}_{\operatorname{Sym}\mathcal{H}}, \tag{8.32}$$

$$[a(f), a(g)] = [a(f)^\dagger, a(g)^\dagger] = 0, \tag{8.33}$$

as operators on $\mathcal{D}$. ■

Proof. The fact that $a(f)^\dagger, a(g)^\dagger$ commute follows from the definition eq. (8.22):

$$\begin{aligned} a(f)^\dagger a(g)^\dagger \psi &= f \odot (g \odot \psi) = f \odot g \odot \psi \\ a(g)^\dagger a(f)^\dagger \psi &= g \odot (f \odot \psi) = g \odot f \odot \psi, \end{aligned} \tag{8.34}$$

and the right-hand sides are equal because $\odot$ is a commutative multiplication operator.

Once it is known that $a(f)^\dagger, a(g)^\dagger$ commute, it follows (using Proposition 8.7) that $a(f), a(g)$ commute. So, eq. (8.33) holds.

It only remains to check eq. (8.32). Using the linearity/anti-linearity of the maps $f \mapsto a(f), a(f)^\dagger$, it suffices to consider the case when $f, g \in \{\phi_1, \phi_2, \dots\}$, and then it suffices to consider $f = \phi_1$, $g \in \{\phi_1, \phi_2\}$. We check that

$$[a(f), a(g)^\dagger]\psi = \begin{cases} 0 & (f \neq g) \\ \psi & (f = g) \end{cases} \tag{8.35}$$

for $\psi = |n_1, n_2, \dots\rangle_{\operatorname{Sym}\mathcal{H}}$.

Indeed, if $g, f = \phi_1$, then Lemma 8.5, Lemma 8.6 give

$$\begin{aligned} a(f)a(g)^\dagger \psi &= \sqrt{n_1 + 1} a(f) |n_1 + 1, n_2, \dots\rangle_{\operatorname{Sym}\mathcal{H}} = (n_1 + 1)|n_1, n_2, \dots\rangle_{\operatorname{Sym}\mathcal{H}}, \\ a(g)^\dagger a(f)\psi &= \sqrt{n_1} a(f) |n_1 - 1, n_2, \dots\rangle_{\operatorname{Sym}\mathcal{H}} = n_1 |n_1, n_2, \dots\rangle_{\operatorname{Sym}\mathcal{H}}, \end{aligned} \tag{8.36}$$

so eq. (8.32) holds, in this case. On the other hand, if $f = \phi_1$ and $g = \phi_2$, then Lemma 8.5, Lemma 8.6 instead give

$$\begin{aligned} a(f)a(g)^\dagger \psi &= \sqrt{n_2 + 1} a(f) |n_1, n_2 + 1, \dots\rangle_{\operatorname{Sym}\mathcal{H}} = \sqrt{n_1(n_2 + 1)} |n_1 - 1, n_2 + 1, \dots\rangle_{\operatorname{Sym}\mathcal{H}}, \\ a(g)^\dagger a(f)\psi &= \sqrt{n_1} a(g)^\dagger |n_1 - 1, n_2, \dots\rangle_{\operatorname{Sym}\mathcal{H}} = \sqrt{n_1(n_2 + 1)} |n_1 - 1, n_2 + 1, \dots\rangle_{\operatorname{Sym}\mathcal{H}}, \end{aligned} \tag{8.37}$$

so $a(f), a(g)^\dagger$ commute. □

PROPOSITION 8.9. *Unless $f = 0$, then $a(f), a(f)^\dagger$ are both unbounded. However, the bound*

$$\|a(f)\psi\|_{\mathrm{Sym}\,\mathcal{H}}, \|a(f)^\dagger\psi\|_{\mathrm{Sym}\,\mathcal{H}} \lesssim \|\sqrt{1+N}\psi\|_{\mathcal{H}} \tag{8.38}$$

holds. Thus, $a(f), a^\dagger(f)$ are continuous maps $\mathcal{D}(N) \to \mathrm{Sym}\,\mathcal{H}$. ■

PROOF. Follows immediately from above. □

1.2. Fermionic case. If ψ is a totally anti-symmetric tensor, then so is $a(f)\psi$. So, $a(f)$ restricts to an operator on the finite-particle subspace $\mathcal{D} \cap \wedge\mathcal{H}$ of the fermionic Fock space.

The *creation* operator

$$a(f)^\dagger : \mathcal{D} \cap \wedge\mathcal{H} \to \mathcal{D} \cap \wedge\mathcal{H} \tag{8.39}$$

is defined by the same formula as in the bosonic case, but switching the symmetric product $\odot$ out for $\wedge$:

$$a(f)^\dagger : \mathcal{D} \ni \psi \mapsto f \wedge (\sqrt{N+1}\psi) = \sqrt{N}(f \wedge \psi). \tag{8.40}$$

Modulo the choice of normalization, this definition is natural.

LEMMA 8.10.

$$a(\phi_1)|n_1, n_2, n_3, \cdots\rangle_{\wedge\mathcal{H}} = \begin{cases} 0 & (n_1 = 0), \\ |0, n_2, n_3, \cdots\rangle_{\wedge\mathcal{H}} & (n_1 = 1), \end{cases} \tag{8.41}$$

$$a(\phi_1)^\dagger|n_1, n_2, n_3, \cdots\rangle_{\wedge\mathcal{H}} = \begin{cases} |1, n_2, n_3, \cdots\rangle_{\wedge\mathcal{H}} & (n_1 = 0), \\ 0 & (n_1 = 1). \end{cases} \tag{8.42}$$

■

PROOF. The cases where the result is 0 are all clear. Also, the identity in eq. (8.42) follows immediately from the definitions:

$$\begin{aligned} a(\phi_1)^\dagger|0, n_2, n_3, \cdots\rangle_{\wedge\mathcal{H}} = a(\phi_1)^\dagger\sqrt{N!}\phi_2^{\otimes n_2}\wedge = \sqrt{(N+1)!}\phi_1 \wedge \phi_2^{\otimes n_2} \wedge \cdots \\ = |1, n_2, n_3, \cdots\rangle_{\wedge\mathcal{H}}, \end{aligned} \tag{8.43}$$

as claimed.

Let $\# = 1 + n_2 + \cdots$. Then:

$$\begin{aligned} a(\phi_1)|1, n_2, n_3, \cdots\rangle_{\wedge\mathcal{H}} = \sqrt{\#!}a(\phi_1)\phi_1 \wedge \phi_2^{\otimes n_2} \wedge \cdots = \sqrt{(\# - 1)!}a(\phi_1)\phi_2^{\otimes n_2} \wedge \cdots \\ = |0, n_2, n_3, \cdots\rangle_{\wedge\mathcal{H}}, \end{aligned} \tag{8.44}$$

where the key point is that when we evaluate $a(\phi_1)\phi_1 \wedge \phi_2^{\otimes n_2} \wedge \cdots$, we get one factor of $\sqrt{\#}$ from the $\sqrt{N+1}$ in eq. (8.19) and another factor of $1/\#$ from when we write the wedge product as the anti-symmetrization of permuted tensor products; the only permutations which give a non-vanishing contribution are those where ϕ_1 appears first, and these comprise $1/\#$ of all permutations. □

PROPOSITION 8.11. *For any $f \in \mathcal{H}$, the (unbounded) operators $a(f), a(f)^\dagger$ are adjoints (as the notation suggests), in the sense that*

$$\langle a(f)\psi, \phi\rangle_{\wedge\mathcal{H}} = \langle \psi, a(f)^\dagger\phi\rangle_{\wedge\mathcal{H}}, \tag{8.45}$$

for all $\phi, \psi \in \mathcal{D} \cap \wedge\mathcal{H}$ and $f \in \mathcal{H}$. ■

PROOF. It suffices to consider $f = \phi_1$, $\psi = |n_1, n_2, \cdots\rangle_{\wedge\mathcal{H}}$, $\phi = |m_1, m_2, \cdots\rangle_{\wedge\mathcal{H}}$. Then, both sides of eq. (8.45) are zero unless $n_1 = 1$, $m_1 = 0$ and $n_j = m_j$ for all $j \geq 2$, in which case both sides are $= 1$. □

PROPOSITION 8.12 (CAR). *For all $f, g \in \mathcal{H}$,*

$$\{a(f), a(g)^\dagger\} = \langle f, g\rangle_{\mathcal{H}}\, \mathrm{id}_{\wedge\mathcal{H}}, \tag{8.46}$$

$$\{a(f), a(g)\} = \{a(f)^\dagger, a(g)^\dagger\} = 0, \tag{8.47}$$

as operators on $\mathcal{D}$. ■

PROOF. The fact that $a(f)^\dagger, a(g)^\dagger$ anti-commute follows immediately from the anti-commutation of the wedge product: $f \wedge g \wedge \phi = -g \wedge f \wedge \phi$, for all $\phi \in \wedge\mathcal{H}$. The fact that $a(f), a(g)$ anti-commute as well follows. Thus, it only remains to check eq. (8.46). As in the proof that the bosonic creation/annihilation operators satisfy the CCR, it suffices to check $f = \phi_1$ and $g \in \{\phi_1, \phi_2\}$. When $g = \phi_1$, then applying $a(f)a(g)^\dagger$, $a(g)^\dagger a(f)$ to $|n_1, n_2, \cdots\rangle_{\wedge\mathcal{H}}$ either kills it or leaves it unchanged. Specifically, if $n_1 = 0$, then $a(g)^\dagger a(f)$ kills it, and $a(f)a(g)^\dagger$ preserves it. If $n_1 = 1$, the roles are reversed. So, in either case,

$$\{a(f), a(g)^\dagger\}|n_1, n_2, \cdots\rangle_{\wedge\mathcal{H}} = |n_1, n_2, \cdots\rangle_{\wedge\mathcal{H}}, \tag{8.48}$$

as desired. On the other hand, if $g = \phi_2$, then $a(f)a(g)^\dagger, a(g)^\dagger a(f)$ both kill $|n_1, n_2, \cdots\rangle_{\wedge\mathcal{H}}$ unless $n_1 = 1$ and $n_2 = 0$. If $n_1 = 1$ and $n_2 = 0$, then the results are

$$a(f)a(g)^\dagger|1, 0, n_3, \cdots\rangle_{\wedge\mathcal{H}} = -a(f)|1, 1, n_3, \cdots\rangle_{\wedge\mathcal{H}} = -|0, 1, n_3, \cdots\rangle_{\wedge\mathcal{H}} \tag{8.49}$$

and

$$a(g)^\dagger a(f)|1, 0, n_3, \cdots\rangle_{\wedge\mathcal{H}} = a(g)^\dagger|0, 0, n_3, \cdots\rangle_{\wedge\mathcal{H}} = |0, 1, n_3, \cdots\rangle_{\wedge\mathcal{H}}. \tag{8.50}$$

respectively. The sign in eq. (8.49) comes from $a(g)^\dagger|1, 0, n_3, \cdots\rangle_{\wedge\mathcal{H}} = \sqrt{(N+1)!}\phi_2 \wedge \phi_1 \wedge \phi_3^{\otimes n_3} \wedge \cdots = -\sqrt{(N+1)!}\phi_1 \wedge \phi_2 \wedge \phi_3^{\otimes n_3} \wedge \cdots = -|1, 1, n_3, \cdots\rangle_{\wedge\mathcal{H}}$. □

Unlike in the bosonic case, fermionic creation/annihilation operators are *bounded*:

PROPOSITION 8.13. *For each $f \in \mathcal{H}$, the fermionic creation/annihilation operators $a(f), a(f)^\dagger$ are bounded operators on $\wedge\mathcal{H}$, with operator norm $\|a(f)\|_{\wedge\mathcal{H}\to\wedge\mathcal{H}}, \|a(f)^\dagger\|_{\wedge\mathcal{H}\to\wedge\mathcal{H}} = \|f\|_{\mathcal{H}}$.* ■

PROOF. Lemma 8.10 gives the case $\|f\|_{\mathcal{H}} = 1$. Linearity finishes the job. □

So, $a(f)^\dagger$ is the adjoint of $a(f)$ in the usual sense.

2. Wick's formula

The careful bookkeeping required to get normalization factors right in the discussion above obscures a basic fact: *the creation/annihilation operators are uniquely determined by their commutation relations or anti-commutation relations* (depending on whether the Fock space is bosonic or fermionic). Indeed, the states in the Fock space of the form

$$a(f_1)^\dagger \cdots a(f_N)^\dagger|\Omega\rangle, \quad N \in \mathbb{N}, \quad f_1, \cdots, f_N \in \mathcal{H} \tag{8.51}$$

are dense in $\mathcal{H}$. This means that any operator on $\mathcal{D}$ that is bounded on each $<N$-particle subspace is uniquely determined by its matrix elements between states of this form.

In particular, this applies to $a(f), a(f)^\dagger$. Their matrix elements can be computed using only

- the CCR/CAR $[a(f), a(g)^\dagger]_\pm = \langle f, g\rangle I$,
- the fact that the vacuum vector Ω is annihilated by all annihilation operators. (This is what we mean in this section when we say that Ω is a vacuum vector.)

Let us consider some simple examples – only in the bosonic case, and only the annihilation operator $a(f)$ (the other possibilities are all similar). Any matrix element of the form $\langle\phi|a(f)|\Omega\rangle$ is zero, so the first nontrivial matrix elements are

$$\begin{aligned}\langle\phi|a(f)a(g)^\dagger|\Omega\rangle &= \langle\phi|[a(f), a(g)^\dagger]|\Omega\rangle + \cancel{\langle\phi|a(g)^\dagger a(f)|\Omega\rangle} \\ &= \langle f, g\rangle\langle\phi|\Omega\rangle.\end{aligned} \tag{8.52}$$

Similarly,

$$\begin{aligned}\langle\phi|a(f)a(g)^\dagger a(g')^\dagger|\Omega\rangle &= \langle\phi|[a(f),a(g)^\dagger]a(g')^\dagger|\Omega\rangle + \langle\phi|a(g)^\dagger a(f)a(g')^\dagger|\Omega\rangle \\ &= \langle f,g\rangle\langle\phi|a(g')^\dagger|\Omega\rangle + \langle\phi|a(g)^\dagger[a(f),a(g')^\dagger]|\Omega\rangle \\ &\qquad + \cancel{\langle\phi|a(g)^\dagger a(g')^\dagger a(f)|\Omega\rangle} \\ &= \langle f,g\rangle\langle\phi|a(g')^\dagger|\Omega\rangle + \langle f,g'\rangle\langle\phi|a(g)^\dagger|\Omega\rangle.\end{aligned} \tag{8.53}$$

Now suppose that $\phi = a(h)^\dagger\psi$, for some ψ. Then, the above is

$$\begin{aligned}&= \langle f,g\rangle\langle\psi|a(h)a(g')^\dagger|\Omega\rangle + \langle f,g'\rangle\langle\psi|a(h)a(g)^\dagger|\Omega\rangle \\ &= (\langle f,g\rangle\langle h,g'\rangle + \langle f,g'\rangle\langle h,g\rangle)\langle\psi|\Omega\rangle.\end{aligned} \tag{8.54}$$

This sort of computation evidently generalizes, proving that $a(f), a(f)^\dagger$ are the only operators on $\mathcal{D}$ satisfying the two properties above.

A slight variant of this idea gives a uniqueness result on representations of the CCR/CAR "in the vacuum sector." By a *representation* of the CCR, we mean a Hilbert space $\mathcal{X}$ and a dense domain $\mathcal{D} \subseteq \mathcal{X}$, and, for each $f \in \mathcal{H}$ (our original Hilbert space) two operators $a(f), a(f)^\dagger \in \mathrm{End}(\mathcal{D})$, such that

- $a(f)^\dagger$ is the adjoint of $a(f)$, meaning that
$$\langle\phi, a(f)^\dagger\psi\rangle = \langle a(f)\phi,\psi\rangle \tag{8.55}$$
for all $\phi,\psi \in \mathcal{D}$,
- the CCR $[a(f),a(g)^\dagger] = \langle f,g\rangle$ is satisfied, for all $f,g \in \mathcal{H}$,
- each of $a(f), a(f)^\dagger$ is bounded on each $<N$-particle subspace.

A *vacuum vector* is a unit vector $\Omega \in \mathcal{H}$ annihilated by all annihilation operators:

$$a(f)|\Omega\rangle = 0 \text{ for all } f \in \mathcal{H}. \tag{8.56}$$

We say that Ω is *cyclic* if the vectors that arise by applying a sequence of creation operators to Ω (as in eq. (8.51)) are dense in $\mathcal{X}$.

PROPOSITION 8.14. *Any representation of the CCR/CAR with a cyclic vacuum vector is unitarily equivalent to the standard one.* ■

PROOF. It suffices to prove that the inner product [Proposition 8.14]

$$\langle a(f_1)^\dagger\cdots a(f_N)^\dagger\Omega, a(g_1)^\dagger\cdots a(g_M)^\dagger\Omega\rangle = \langle\Omega|a(f_N)\cdots a(f_1)a(g_1)^\dagger\cdots a(g_M)^\dagger|\Omega\rangle \tag{8.57}$$

is uniquely determined by the listed properties. For simplicity, we assume that $N \geq M$. (The case $M \leq N$ is just the conjugate computation transpose.) We only consider the CCR case. The CAR case is analogous, except for some extra signs, and commutators should be replaced by anti-commutators.

Now we commute each $a(f_\bullet)$ to the right, starting with $a(f_1)$. The first step is:

$$\begin{aligned}&\langle\Omega|a(f_N)\cdots a(f_1)a(g_1)^\dagger\cdots a(g_M)^\dagger|\Omega\rangle \\ &\quad= \sum_{m=1}^{M}\left\langle\Omega\middle|a(f_N)\cdots a(f_2)\left(\prod_{j=1}^{m-1}a(g_j)^\dagger\right)\overbrace{[a(f_1),a(g_m)^\dagger]}^{\langle f_1,g_m\rangle I}\prod_{j=m+1}^{M}a(g_j)^\dagger\middle|\Omega\right\rangle \\ &\qquad\qquad + \cancel{\langle\Omega|a(f_N)\cdots a(f_2)a(g_1)^\dagger\cdots a(g_M)^\dagger a(f_1)|\Omega\rangle},\end{aligned} \tag{8.58}$$

where the last term is killed by $a(f_1)|\Omega\rangle = 0$.

That is,

$$\langle\Omega|a(f_N)\cdots a(f_1)a(g_1)^\dagger\cdots a(g_M)^\dagger|\Omega\rangle \\ = \sum_{m=1}^{M}\langle f_1, g_m\rangle\Big\langle\Omega\Big|a(f_N)\cdots a(f_2)\Big(\prod_{j=1}^{m-1}a(g_j)^\dagger\Big)\prod_{j=m+1}^{M}a(g_j)^\dagger\Big|\Omega\Big\rangle. \quad (8.59)$$

Now repeat with $a(f_2)$, $a(f_3)$, and so on. In each step, we reduce the number of creation operators by one and the number of annihilation operators by one. If M is *strictly* less than N, we end up with something proportional to

$$\langle\Omega|a(f_N)\cdots a(f_{M+1})|\Omega\rangle = 0. \quad (8.60)$$

Otherwise, if $M = N$, we are left with

$$\Big[\sum_{\sigma\in\mathfrak{S}_N}\prod_{j=1}^{N}\langle f_j, g_{\sigma(j)}\rangle\Big]\langle\Omega|\Omega\rangle = \sum_{\sigma\in\mathfrak{S}_N}\prod_{j=1}^{N}\langle f_j, g_{\sigma(j)}\rangle. \quad (8.61)$$

□

3. Transformation law

PROPOSITION 8.15. *Let $\rho : G \to \mathrm{U}(\mathcal{H})$ denote a unitary representation of some group G. Then, letting $U : G \to \mathrm{U}(\mathsf{Fock}^{\pm}\mathcal{H})$ denote the induced representation on the bosonic/fermionic Fock space $\mathsf{Fock}^{\pm}\mathcal{H}$, the following transformation law holds:*

$$U(g)^{-1}a(f)U(g) = a(\rho(g)f), \quad U(g)^{-1}a(f)^\dagger U(g) = a(\rho(g)f)^\dagger \quad (8.62)$$

for all $g \in G$ and $f \in \mathcal{H}$. ■

PROOF. By Wick's formula. □

4. $\mathfrak{ccr}/\mathfrak{car}$

Given a "one-particle" Hilbert space π, let $\mathfrak{ccr}(\pi)$ denote the algebra of operators on the algebraic bosonic Fock space generated by the creation/annihilation operators, and similarly for $\mathfrak{car}(\pi)$.

Exercises and problems

EXERCISE 8.1: Prove that the symmetrization and anti-symmetrization operators $\mathfrak{S}, \mathfrak{A}$ are orthogonal projections from $\mathcal{TH}$ onto the relevant Fock space $\mathrm{Sym}\,\mathcal{H}, \wedge\mathcal{H}$.

EXERCISE 8.2: If $N = \dim\mathcal{H}$ is finite, what are the dimensions of $\mathrm{Sym}^j\,\mathcal{H}$ and $\wedge^j\mathcal{H}$?

EXERCISE 8.3: Fix $d \in \mathbb{N}^+$, and let $\mathcal{H} = L^2(\mathbb{R}^d)$. To each pure tensor $\phi_1 \otimes \cdots \otimes \phi_j \in \mathcal{H}^{\otimes j}$, associate the function $\iota[\phi_1, \ldots, \phi_j] \in L^2(\mathbb{R}^{dj})$ given by

$$\iota[\phi_1, \ldots, \phi_j](x_1, \ldots, x_j) = \phi(x_1)\ldots\phi(x_j) \quad (8.63)$$

for $x_1, \ldots, x_j \in \mathbb{R}^d$.

(a) Prove that ι extends to unitary $\mathcal{H}^{\otimes j} = L^2(\mathbb{R}^d)^{\otimes j} \to L^2((\mathbb{R}^d)^j) \cong L^2(\mathbb{R}^{dj})$.

Thus, $\mathcal{H}^{\otimes j}$ consists of "wavefunctions" on a higher-dimensional space.

(b) Check that $\mathrm{Sym}^j\,\mathcal{H}$ is identified with the subspace of $L^2((\mathbb{R}^d)^j)$ consisting of symmetric wavefunctions, and $\wedge^j\mathcal{H}$ is identified with the subspace consisting of ant-symmetric wavefunctions.

Exercise 8.4: Complete the proof of Proposition 8.14 by constructing the unitary equivalence between two reps of the CCR/CAR with a cyclic vacuum.

Hint: you can use the norm calculation to show that the map sending $a(f_1)^\dagger \cdots a(f_N)^\dagger|\Omega\rangle$ in one rep to the corresponding vector in the second rep extends to a well-defined linear map on the whole algebraic Fock space.

Problem 8.1: Prove that the number operator N is self-adjoint with the claimed domain.

CHAPTER 9

$\mathfrak{osc}$

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction. [Attributed to Sidney Coleman. The version he says in his recorded lectures is slightly different.]

In our discussion of the CCR, the one-particle Hilbert space π had no structure on it, other than that which it has by virtue of being a Hilbert space. But in physics, we always have some dynamics: suppose that $e^{-itH} : \mathbb{R}_t \to \mathrm{U}(\pi)$ denotes a one-parameter group of unitary operators on π, governing the time-evolution of one-particle states. Hitting this with the Fock functor, we get a one-parameter group $\mathbf{U} : \mathbb{R}_t \to \mathrm{U}(\mathcal{H})$ of unitary operators on the Fock space $\mathcal{H} = \mathrm{Sym}(\pi)$. Under the dynamics thus defined, multi-particle states evolve freely, without seeing what the other particles are doing. It turns out that the generator

$$\mathbf{H} : \mathcal{D}(\mathbf{H}) \to \mathcal{H} \tag{9.1}$$

of the multi-particle dynamics $\mathbf{U}(t) = e^{-it\mathbf{H}}$, together with creation operators and the identity matrix, generates an extension

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook] & \mathfrak{osc}(\pi, H) \arrow[r, two heads] & \mathfrak{ccr}(\pi) \arrow[l, bend left, dashed] \end{tikzcd} \tag{9.2}$$

of the CCR algebra $\mathfrak{ccr}(\pi)$. This is known as the *oscillator* algebra, $\mathfrak{osc}(\pi, H)$. The goal of this chapter is to investigate the structure and representation theory of this object. As the name indicates, it is a highly abstract formulation of the quantum harmonic oscillator, though further development of that analogy will need to wait until next lecture.

1. $\mathfrak{osc}$

The simplest case is $\mathfrak{osc}(\mathbb{C}, 1) = \mathfrak{osc}$. The propagator on the one-particle subspace $\pi = \mathbb{C}$ is just $U(t) = e^{it}$. Consider the n-particle state $|n\rangle = 1^{\otimes n}$. This satisfies

$$\begin{aligned} U(t)|n\rangle = (U(t)1)^{\otimes n} &= (e^{it})^{\otimes n} \\ &= e^{itn}|n\rangle. \end{aligned} \tag{9.3}$$

Thus, the multi-particle Hamiltonian is just the number operator $\mathbf{H}|n\rangle = n|n\rangle$. That is, $\mathbf{H} = N = a^\dagger a$. Thus,

$$[\mathbf{H}, a] = [a^\dagger a, a] = [a^\dagger, a]a = -a \tag{9.4}$$

$$[\mathbf{H}, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] = a^\dagger. \tag{9.5}$$

The commutator of $\mathbf{H} = N$ with $a, a^\dagger$ is therefore a creation/annihilation operator. Thus, defining

$$\mathfrak{osc}_{\mathrm{Lie}} = \mathrm{span}\{I, a, a^\dagger, \mathbf{H} = N\}, \tag{9.6}$$

what we have is a (complex) four-dimensional Lie algebra. The only other non-trivial Lie bracket is the CCR. So, $\mathfrak{osc}_{\mathrm{Lie}}$ can be written:

$$[a, a^\dagger] = I, \quad [I, a] = [I, a^\dagger] = [I, \mathbf{H}] = 0.$$
$$[\mathbf{H}, a] = -a, \quad [\mathbf{H}, a^\dagger] = a^\dagger, \tag{$\mathfrak{osc}_1$}$$

Recalling that the (complex) CCR/Heisenberg algebra $\mathfrak{h}$ is the three-dimensional Lie algebra $\mathfrak{h} = \mathrm{span}\{I, a, a^\dagger\}$, the oscillator algebra is a one-dimensional extension:

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook] & \mathfrak{osc}_{\mathrm{Lie}} \arrow[r, two heads] & \mathfrak{h} \arrow[l, bend left, dashed] \end{tikzcd} \tag{9.7}$$

This splits, as indicated, because $\mathfrak{h} \subset \mathfrak{osc}_{\mathrm{Lie}}$, so $\mathfrak{osc}$ is actually a semidirect product of $\mathfrak{h}$ and the one-dimensional Lie algebra.

We use $\mathfrak{osc}(\pi, H)$ to denote the enveloping algebra generated by $\mathfrak{osc}(\pi, H)$ (endowed with the $*$-operation taking $a \mapsto a^\dagger$ but preserving $\mathbf{H}, I$).

Representations of the oscillator algebra with a cyclic vacuum vector Ω are uniquely determined by two pieces of data:

- the restriction of the representation to the Heisenberg algebra $\mathfrak{h}$,
- the *vacuum energy* $Z \in \mathbb{R}$ (a.k.a. zero point energy), defined by $\mathbf{H}\Omega = Z\Omega$.

Indeed, the relation $[\mathbf{H}, a^\dagger] = a^\dagger$ forces

$$\mathbf{H}|n\rangle = (Z + n)|n\rangle. \tag{9.8}$$

However, different choices of vacuum energy result in identical projective representations.

2. $\mathfrak{osc}(\pi, H)$

We now return to the general (possibly infinite-dimensional) case. Let $H : \pi \supset \mathcal{D}(H) \to \mathcal{H}$ denote the generator of U, so that $U(t) = e^{itH}$, and let

$$\mathcal{D}_\infty = \bigcap_{j=1}^{\infty} \mathcal{D}(H^j) \tag{9.9}$$

denote a Gårding domain thereof.

Let $\mathbf{H}$ denote the corresponding operator on Fock space, with domain

$$\mathcal{D}(\mathbf{H}) = \mathsf{Fock}^{\pm}\mathcal{D}_\infty; \tag{9.10}$$

the $\mathsf{Fock}^{\pm}$ on the right-hand side is the *algebraic* Fock functor, from the category of inner product spaces, so $\mathsf{Fock}^{\pm}\mathcal{D}(H) \subset \mathcal{H}$ consists of finite particles states in which all of the particles lie in the finite-energy subspace $\mathcal{D}(H)$.

Notice that if $f \in D(H)$, then $a(f), a(f)^\dagger$ map $\mathsf{Fock}^{\pm}\mathcal{D}(H) \to \mathsf{Fock}^{\pm}\mathcal{D}(H)$. So, we can consider the Lie algebra

$$\mathfrak{osc}_{\mathrm{Lie}}(\pi, H) = \mathrm{span}_{\mathbb{C}}\{I, a(f), a(f)^\dagger, \mathbf{H} : f \in \mathcal{D}(H)\} \subset \mathrm{End}(\mathsf{Fock}^{\pm}\mathcal{D}_\infty(H)) \tag{9.11}$$

generated by the creation and annihilation operators $a(f), a(f)^\dagger$ together with I and the multi-particle Hamiltonian $\mathbf{H}$. The operator $\mathbf{H}$ is essentially self-adjoint with the domain $\mathcal{D}(\mathbf{H})$.

Let's work out the commutation relations of $\mathbf{H}$ with $a(f), a(f)^\dagger$.

Proposition 9.1. *For any $f \in \mathcal{D}(H)$,*

$$[\mathbf{H}, a(f)^\dagger] = a(Hf)^\dagger, \qquad [\mathbf{H}, a(f)] = -a(Hf). \tag{9.12}$$

■

Proof. The two identities are adjoints, so we prove only the first. Begin with

$$U(t)a(f)^\dagger U(t)^\dagger = a(e^{-itH}f)^\dagger, \tag{9.13}$$

which is a special case of Proposition 8.15. Differentiating both sides with respect to t, and setting $t = 0$,

- the left-hand side gives the commutator $-i[\mathbf{H}, a(f)^\dagger]$, and
- the right-hand side gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} a(e^{-itH}f)^\dagger = a\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{-itH}f\right)^\dagger = -a(iHf)^\dagger = -ia(Hf)^\dagger, \tag{9.14}$$

  using Stone's theorem/the definition of $\mathcal{D}(H)$ and the linearity of $a^\dagger$.

(To make these computations completely rigorous, apply both sides of eq. (9.13) to an arbitrary element of $\mathcal{D}(\mathbf{H})$, and then differentiate. Stone's theorem guarantees that the relevant derivatives exist.) □

Consequently, $\mathbf{H}$, together with $I, a(f), a(f)^\dagger$ for $f \in \mathcal{D}_\infty$, forms a (complex) Lie algebra of operators on $\mathsf{Fock}^+(\mathcal{D}_\infty)$, and this is what we call $\mathfrak{osc}_{\mathrm{Lie}}(\pi, H)$. Drop the subscript to pass to the enveloping algebra $\mathfrak{osc}(\pi, H)$. Evidently, this is an extension

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook] & \mathfrak{osc}(\pi, H) \arrow[r, two heads] & \mathfrak{ccr}(\pi) \arrow[l, dashed, bend left] \end{tikzcd} \tag{9.15}$$

of $\mathfrak{ccr}(\pi)$, just as in the simplest case.

The exact same argument as before shows that a representation of $\mathfrak{osc}(\pi, H)$ with a cyclic vacuum vector is uniquely determined by the $\mathfrak{ccr}$ portion and the vacuum energy.

Exercises and problems

Exercise 9.1: (a) Show that $\mathsf{Z} = \mathbf{H} - a^\dagger a$ is a central element of $\mathfrak{osc}_1$.

(b) (Optional.) Compute the center of the oscillator algebra.

Problem 9.1: Suppose that the one-particle Hilbert space π comes with a unitary representation of $(\mathbb{R}^k, +)$, for some $k \in \mathbb{N}^+$. (This includes the case where π is a Poincaré rep.)

(a) Compute, with proof, the spectrum of the representation on the bosonic Fock space $\operatorname{Sym}\pi$ of the representation induced by the Fock functor.

(b) Repeat for the fermionic Fock space.

Thus, when π is a Poincaré irrep, the Fock representation has exactly the spectrum described back in Chapter 3.

CHAPTER 10

The oscillator representation

Consider the oscillator algebra $\mathfrak{osc}(\pi, H)$ based on some "one-particle" Hilbert space π and one-particle Hamiltonian $H \geq 0$. By Wick's theorem, all nondegenerate representations of $\mathfrak{osc}(\pi, H)$ with a cyclic vacuum vector are unitarily equivalent to a standard Fock representation.

An example where $N = \dim \pi$ is finite is Schrödinger's theory of the quantum harmonic oscillator (QHO) in N spatial dimensions. The ambient Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^N_x)$ and the (multi-particle) Hamiltonian is

$$\mathbf{H} = \frac{1}{2}\Big(- \triangle + V(x) \Big) = \frac{1}{2}\Big(- \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \underbrace{x^{\intercal}\omega^2 x}_{V(x)} \Big), \tag{10.1}$$

$$\omega \in \mathbb{R}^{N\times N}, \quad \omega = \omega^{\intercal} > 0. \tag{10.2}$$

The PDE $i\partial_t \psi = \mathbf{H}\psi$ describes the time-evolution of the wavefunction $\psi(t, -) \in \mathcal{H}$ of a non-relativistic particle under the influence of a restoring force $F = -\nabla V$ proportional to x (Hooke's law). The QHO is one of the few examples in non-relativistic quantum mechanics where the problem is exactly solvable. The source of this "integrability" is the existence of *ladder operators* which, with $\mathbf{H}$, form a representation of the oscillator algebra. Without loss of generality, we restrict attention to the case where $\omega = \operatorname{diag}(\omega_1, \cdots, \omega_N)$. Then,

$$V(x) = \sum_{n=1}^{N} \omega_n^2 x_n^2 \tag{10.3}$$

and the ladder operators are defined as

$$a_n = \frac{1}{\sqrt{2}}\Big(\sqrt{\omega_n} x_n + \frac{1}{\sqrt{\omega_n}} \frac{\partial}{\partial x_n}\Big), \quad a_n^\dagger = \frac{1}{\sqrt{2}}\Big(\sqrt{\omega_n} x_n - \frac{1}{\sqrt{\omega_n}} \frac{\partial}{\partial x_n}\Big). \tag{10.4}$$

Proposition 10.1. *These satisfy the oscillator algebra relations:*

$$[a_n, a_m^\dagger] = \delta_{nm}, \quad [a_n, \mathbf{H}] = -\omega_n a_n, \quad [a_n^\dagger, \mathbf{H}] = \omega_n a_n^\dagger. \tag{10.5}$$

■

Proof. Straightforward computation. □

This is the *oscillator representation* of $\mathfrak{osc}$. Once we know the existence of a cyclic vacuum vector (and the value of the vacuum energy), the spectrum of $\mathbf{H}$, including the explicit form of all eigenfunctions, is uniquely determined. The vacuum vector Ω, the "ground state" of the oscillator, is characterized by being killed by all annihilation operators. This sets up a system of ODEs which is easily solved to yield:

$$\Omega(x) = \left(\frac{\det \omega}{\pi^N}\right)^{1/4} e^{-x^{\intercal}\omega x/2} \tag{10.6}$$

Thus, at the level of dynamics:

All bosonic Fock spaces built from a finite-dimensional one-particle Hilbert space are equivalent to an oscillator.

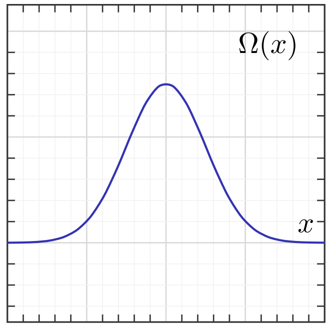


FIGURE 10.1. The ground state of the oscillator.

In this lecture, we review the theory of the oscillator.

The $\dim \pi < \infty$ case suggests an interpretation of the bosonic Fock space $\mathcal{H} = \operatorname{Sym} \pi$ even when π is infinite-dimensional: *it describes an oscillator whose degrees-of-freedom are labeled by complex lines in π.* Perhaps more importantly, we can always approximate, to arbitrary experimental accuracy, an infinite-dimensional π by a space of large but finite dimension. Consequently, the Fock space $\mathcal{H}$, and the multi-particle dynamics on that space, can always be approximated by an oscillator with a large number of degrees-of-freedom.

1. Intertwining the QHO and Fock representations

This section recalls the unitary equivalence

$$U : \operatorname{Sym} \mathbb{C}^N \to L^2(\mathbb{R}^N) \tag{10.7}$$

which intertwines the Fock and oscillator representations of $\mathfrak{osc}_N(\omega)$. For simplicity, we restrict attention to the $N = 1$ case. The general case is essentially verbatim, involving only notational complications. When $N = 1$, different choices of $\omega > 0$ give isomorphic Lie algebras, so we write $\mathfrak{osc}_1 = \mathfrak{h} \rtimes \mathbb{C}$. For concreteness, we will take $\omega = 1$. The generators $\mathsf{Z}, \mathsf{x}, \mathsf{p}, \mathsf{H} \in \mathfrak{osc}_1$ satisfy

$$i[\mathsf{p}, \mathsf{x}] = \mathsf{Z} \tag{CCR}$$

$$i[\mathsf{H}, \mathsf{x}] = \mathsf{p}, \quad i[\mathsf{H}, \mathsf{p}] = -\mathsf{x}, \tag{10.8}$$

the other pairs of generators commuting. The case where ω is general is gotten by rescaling x, H.

In both the Fock and oscillator representations, we use $a^\dagger, a$ to denote the sole pair of creation/annihilation operators, representing $\mathsf{a} = 2^{-1/2}(\mathsf{x} + i\mathsf{p})$, $\mathsf{a}^\dagger = 2^{-1/2}(\mathsf{x} - i\mathsf{p}) \in \mathfrak{osc}_1$, and we use H to denote the representative of H. In either representation, the central charge Z is represented by the identity operator.

Let $|n\rangle \in \operatorname{Sym} \mathbb{C}$, for $n \in \mathbb{N}$, denote the standard orthonormal basis. So $|0\rangle$ is the vacuum vector, and

$$a^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \begin{cases} 0 & (n = 0), \\ \sqrt{n}|n-1\rangle & (\text{otherwise}). \end{cases} \tag{10.9}$$

The energy $\langle n|H|n\rangle$ is just n, and the other matrix elements $\langle m|H|n\rangle$ are of course 0. This completely specifies the Fock representation.

Consider now the oscillator representation, on which $H = \frac{1}{2}(-\partial_x^2 + x^2 - 1)$ and $a, a^\dagger = 2^{-1/2}(x \pm \partial_x)$. We can define a sequence of unit vectors $\phi_0 = \Omega, \phi_1, \cdots \in L^2(\mathbb{R})$ as follows. Let

$$\psi_n(x) = (a^\dagger)^n \Omega(x) = \Big(\frac{x - \partial_x}{\sqrt{2}}\Big)^n \Omega(x), \tag{10.10}$$

where $\Omega(x) = \pi^{-1/4} e^{-x^2/2}$. Let $\phi_n = \psi_n / \sqrt{n!}$. Then, $\phi_0, \phi_1, \phi_2, \cdots$ is an orthonormal set; this is a special case of Wick's theorem. Indeed, we can commute the annihilation operators through creation operators, modulo c-numbers to get

$$\begin{aligned}\langle\psi_n,\psi_m\rangle_{L^2(\mathbb{R})} &= \langle\Omega, a^n(a^\dagger)^m\Omega\rangle_{L^2(\mathbb{R})}\\ &= \sum_{j=0}^{m-1}\langle\Omega, a^{n-1}(a^\dagger)^j[a,a^\dagger](a^\dagger)^{m-j-1}\Omega\rangle_{L^2(\mathbb{R})} + \cancel{\langle\Omega, a^{n-1}(a^\dagger)^m a\Omega\rangle}\\ &= m\langle\Omega, a^{n-1}(a^\dagger)^{m-1}\Omega\rangle_{L^2(\mathbb{R})}.\end{aligned} \tag{10.11}$$

Repeat. If $n > m$ (the $n < m$ case is just the conjugate), then we conclude that $\langle\psi_n,\psi_m\rangle_{L^2} \propto \langle\Omega, a^{n-m}\Omega\rangle = 0$. In the $m = n$ case, we arrive at

$$\|\psi_n\|^2 = n!\langle\Omega,\Omega\rangle = n!, \tag{10.12}$$

so $\|\phi_n\| = 1$.

So, $U : |n\rangle \mapsto \phi_n$ extends to a partial isometry $U : \operatorname{Sym}\mathbb{C} \to L^2(\mathbb{R})$. It is an isometry onto its image, but we do not yet know that it is surjective. (This is the cyclicity of the vacuum.)

The ϕ_n can be written in terms of the *Hermite polynomials* $H_n(x) = (-1)^n e^{x^2}\partial_x^n e^{-x^2}$.

Proposition 10.2. *For each $n \in \mathbb{N}$,*

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}}\Big(\frac{1}{\pi}\Big)^{1/4} e^{-x^2/2}H_n(x). \tag{10.13}$$

■

We will prove this using *Rodrigues' formula*:

$$\begin{aligned}&H_0(x) = 1\\ &H_n(x) = \Big(2x - \frac{\partial}{\partial x}\Big)H_{n-1}(x), \quad n \in \mathbb{N}^+.\end{aligned} \tag{10.14}$$

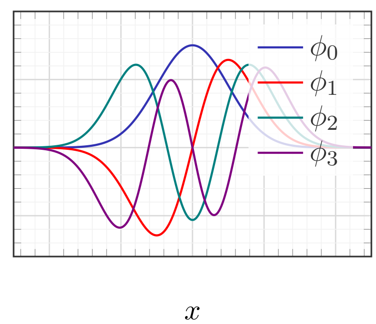


Figure 10.2. The first few ϕ_n.

Proof. The definition of ψ_n is closely related to Rodrigues's formula. Indeed, since Ω is nonvanishing, we can write

$$\Omega^{-1}\psi_n = \Omega^{-1}(a^\dagger)^n\Omega = (M_{\Omega^{-1}}a^\dagger M_\Omega)^n 1, \tag{10.15}$$

where, for clarity, we have written the multiplication map $\bullet \mapsto \Omega^{\pm 1}\bullet$ as $M_{\Omega^{\pm 1}}$. The key point is that, when we take the nth power of $M_{\Omega^{-1}}a^\dagger M_\Omega$, the adjacent powers of Ω cancel.

In order to compute the differential operator $M_{\Omega^{-1}}a^\dagger M_\Omega = M_{e^{x^2/2}}a^\dagger M_{e^{-x^2/2}}$, consider applying it to a function $f \in C^\infty(\mathbb{R})$. The result is

$$M_{e^{x^2/2}}\Big(\frac{x-\partial_x}{\sqrt{2}}\Big)M_{e^{-x^2/2}}f = \Big(\frac{2x-\partial_x}{\sqrt{2}}\Big)f. \tag{10.16}$$

So, $M_{\Omega^{-1}}a^\dagger M_\Omega = (2x - \partial_x)/\sqrt{2}$. Equation (10.15) becomes

$$\Omega^{-1}\psi_n(x) = \frac{1}{2^{n/2}}\underbrace{\Big(2x - \frac{\partial}{\partial x}\Big)^n 1}_{H_n(x),\text{ by Rodrigues}}. \tag{10.17}$$

I.e. $\psi_n(x) = 2^{-n/2}\pi^{-1/4}e^{-x^2/2}H_n(x)$. Remembering that $\phi_n = \psi_n/\sqrt{n!}$ completes the proof. □

Proposition 10.3 (Completeness). $\phi_0, \phi_1, \phi_2, \cdots$ *are an orthonormal basis for* $L^2(\mathbb{R})$. ■

Proof. Let $C = \overline{\operatorname{span}_{\mathbb{C}}\{\phi_n\}_{n=0}^\infty}$. We want to show that $C^\perp = \{0\}$.

Because each Hermite polynomial H_ν has degree exactly ν, we have $x^n \in \operatorname{span}_{\mathbb{C}}\{H_0, \dots, H_n\}$ for each $n \in \mathbb{N}$. Consequently,

$$e^{-x^2/2} x^n \in \operatorname{span}_{\mathbb{C}}\{\phi_0, \dots, \phi_n\} \subset C. \tag{10.18}$$

For each $s \in \mathbb{C}$,

$$\lim_{N\to\infty} \underbrace{e^{-x^2/2} \sum_{n=0}^{N} \frac{s^n x^n}{n!}}_{\in C} = e^{-x^2/2} \sum_{n=0}^{\infty} \frac{s^n x^n}{n!} = e^{-x^2/2+sx}, \tag{10.19}$$

where the limit is in the $L^2(\mathbb{R})$ norm; we will prove this momentarily. Assuming it, it follows that $e^{-x^2/2+sx} \in C$. So, if $\phi \in C^\perp$, then $\phi \perp e^{-x^2/2+sx}$ for every $s \in \mathbb{C}$. That is,

$$\int_{\mathbb{R}} \phi(x) e^{-x^2/2+s^* x} \,\mathrm{d}x = 0 \tag{10.20}$$

for every $s \in \mathbb{C}$. That this holds for all $s \in i\mathbb{R}$ means that the Fourier transform of $\phi(x)e^{-x^2/2} \in L^1(\mathbb{R})$ vanishes identically. Since the Fourier transform $\mathcal{F} : L^1(\mathbb{R}) \to C^0(\mathbb{R})$ is injective, we conclude $\phi = 0$.

Now let us prove eq. (10.19). We want to estimate

$$\begin{aligned} \left\| e^{-x^2/2} \sum_{n=N}^{\infty} \frac{s^n x^n}{n!} \right\|_{L^2(\mathbb{R}_x)} &\leq \|e^{-x^2/4}\|_{L^2(\mathbb{R}_x)} \left\| \sum_{n=N}^{\infty} \frac{s^n x^n}{n!} e^{-x^2/4} \right\|_{L^\infty(\mathbb{R}_x)} \\ &\leq \|e^{-x^2/4}\|_{L^2(\mathbb{R}_x)} \sup_{x\in\mathbb{R}} \sum_{n=N}^{\infty} \frac{|sx|^n}{n!} e^{-x^2/4}. \end{aligned} \tag{10.21}$$

Since $e^{-x^2/4}$ is in $L^2(\mathbb{R}_x)$, it suffices to show that

$$\lim_{N\to\infty} \left[\sup_{x\in\mathbb{R}} \sum_{n=N}^{\infty} \frac{|sx|^n}{n!} e^{-x^2/4} \right] = 0. \tag{10.22}$$

The function $f_n(x) = |x|^n e^{-x^2/4}$ is maximized at $x = \pm\sqrt{2n}$, where it equals $(2n)^{n/2} e^{-n/2}$. So,

$$\sum_{n=N}^{\infty} \frac{|sx|^n}{n!} e^{-x^2/4} \leq \sum_{n=N}^{\infty} \frac{1}{n!} \left(|s| \sqrt{\frac{2n}{e}} \right)^n. \tag{10.23}$$

The series $\sum_{n=1}^{\infty} (n!)^{-1} (|s|\sqrt{2n/e})^n$ is convergent, the $n!$ in the denominator beating the $n^{n/2}$ in the numerator. So, the right-hand side of eq. (10.23) is the tail of a convergent series, which therefore converges to 0 as $N \to \infty$. $\square$

So, the map U, eq. (10.7), is onto $L^2(\mathbb{R})$. This completes the proof of the equivalence of the Fock and oscillator representations of $\mathfrak{osc}_1$. Essentially the same argument applies verbatim to $\mathfrak{osc}_N(\omega)$.

2. The position and momentum operators

In Schrödinger's QHO, the position and momentum operators x_n, $p_n = -i\partial_{x_n}$ can be solved for: $\sqrt{2} x_n = (a_n + a_n^\dagger)$, $\sqrt{2} i p_n = a_n - a_n^\dagger$. This suggests defining on the Fock space $\mathcal{H} = \operatorname{Sym} \pi$, for arbitrary (possibly infinite-dimensional!) π, the "position" and "momentum" operators

$$x(f) = \frac{a(f) + a(f)^\dagger}{\sqrt{2}}, \quad p(f) = \frac{a(f) - a(f)^\dagger}{\sqrt{2} i}. \tag{10.24}$$

These are operators on the finite-particle subspace $\mathcal{D} \subseteq \mathcal{H}$. They are manifestly symmetric on this domain.

Note that $x(f), p(f)$ do *not* depend complex-linearly on f; indeed, $p(f) = x(if)$. Hence, we only consider x below.

PROPOSITION 10.4. *For all $f \in \pi$, the operator $x(f)$ is essentially self-adjoint with domain $\mathcal{D}$.* ■

PROOF. A $\psi \in \mathcal{D}$ is called *analytic* if

$$\sum_{k=0}^{\infty} s^k \|x(f)^k \psi\|/k! < \infty \tag{10.25}$$

for some $s > 0$. One version of *Nelson's analytic vector theorem* [RS75, Thm. X.39] states that a symmetric operator $\mathcal{D} \to \mathcal{D}$ on a dense domain $\mathcal{D}$ is essentially self-adjoint if its domain contains a dense subspace of analytic vectors. (This allows you to directly define the strongly-continuous one-parameter group of unitary operators $s \mapsto e^{isx(t)}$ whose generator ends up being $x(f)$.) In the case at hand, *every* $\psi \in \mathcal{D}$ is analytic. Indeed, if ψ is in the $\leq N$-particle subspace of the Fock space, then

$$\|x(f)^k \psi\| \leq 2^{k/2} \|f\|^k \|\psi\| \sqrt{(N+k)!}, \tag{10.26}$$

as follows from the bounds on $a(f), a(f)^\dagger$ acting on the $\leq (N+j)$-particle subspace for $j \leq k$. The $k!$ in the denominator of the power series in eq. (10.25) beats out the $\leq C^k \sqrt{k!}$ growth of $\|x(f)^k \psi\|$. Consequently, the power series in eq. (10.25) has infinite radius of convergence. □

PROPOSITION 10.5. *The position and momentum operators satisfy the commutation relations*

$$[x(f), x(g)] = 2i\Im\langle f, g\rangle_\pi I \tag{10.27}$$

for all $f, g \in \pi$. ■

PROOF. By the CCR,

$$\begin{aligned} 2[x(f), x(g)] &= [a(f), a(g)^\dagger] - [a(g), a(f)^\dagger] \\ &= (\langle f, g\rangle_\pi - \langle g, f\rangle_\pi) I = 2i\Im\langle f, g\rangle I. \end{aligned} \tag{10.28}$$

□

Consider a vector $\psi \in \mathcal{H} = \operatorname{Sym} \pi$ of the form

$$\psi = a(f_1)^\dagger \cdots a(f_N)^\dagger \Omega, \tag{10.29}$$

$N \in \mathbb{N}$, for $f_1, \cdots, f_N \in \pi$.

3. The Wiener–Itô–Segal isomorphism (⋆)

Suppose that π is infinite-dimensional. What would it mean to have an "oscillator" representation of $\mathfrak{osc}(\pi, H)$? The Hilbert space should take the form

$$\mathcal{H} = L^2(\mathcal{X}, \mu), \qquad \mu : \mathrm{Borel}(\mathcal{X}) \to [0,1] \tag{10.30}$$

for some infinite-dimensional real vector space[1] $\mathcal{X}$ in place of $\mathbb{R}^N$, and the abstract position operators on the Fock space should be realized as concrete position-like operators on $\mathcal{X}$.

A linear coordinate on $\mathcal{X}$ is the same thing as a $\mathbb{R}$-valued linear functional. So, we should have one position operator $Q(f)$ for each $f \in \mathcal{X}^*$.

[1] Let's assume that it is a Hausdorff locally convex topological vector space (LCTVS).

3.1. Discussion of the Hilbert space. When π was finite-dimensional, we could just take $\mathcal{X} = \pi$, since all vector spaces (say, over $\mathbb{R}$) of the same finite dimension are isomorphic. The obvious choice for μ is the Lebesgue measure. Of course the Lebesgue measure is only unique up to scale, but this is unimportant — it corresponds to the freedom to measure the oscillator's displacement in whatever units we wish.

When π is *infinite*-dimensional, matters are less clear. For one thing, there is no "Lebesgue measure" in infinite dimensions:

PROPOSITION 10.6 (Weil '40). *Any locally-finite translation-invariant Radon measure on an infinite-dimensional (Hausdorff) LCTVS $\mathcal{X}$ must be trivial.* ∎

See e.g. [**Yamasaki '85**].

The fix is to incorporate the Gaussian factors present in the QHO eigenfunctions $\propto H_n(x)e^{-x^2/2}$ *into the measure.* Consider the QHO in $N \in \mathbb{N}^+$ dimensions, and let

$$\begin{aligned} M : L^2(\mathbb{R}^N_x) \to L^2(\mathbb{R}^N_x, \mu), \quad \mathrm{d}\mu = e^{-x^\intercal \omega x}\, \mathrm{d}^N x \\ : \phi(x) \mapsto e^{x^\intercal \omega x/2}\phi(x) \end{aligned} \tag{10.31}$$

denote the unitary operation that strips away the Gaussian factors. The new Hamiltonian on $L^2(\mathbb{R}^N, \mu)$ is

$$M\mathbf{H}M^{-1} = \sum_{n=1}^{N} \Big(-\frac{\partial^2}{\partial x_n^2} + 2\omega_n x_n \frac{\partial}{\partial x_n} + \omega_n \Big) \tag{10.32}$$

(the "Ornstein–Uhlenbeck operator"). Understanding the spectrum of the original Hamiltonian $\mathbf{H}$ on $L^2(\mathbb{R}^N)$ is equivalent to understanding the spectrum of $M\mathbf{H}M^{-1}$ on $L^2(\mathbb{R}^N, \mu)$. All we have done is translate the original problem into slightly different language. However, this translation does accomplish something — while the Lebesgue measure does not exist in infinite dimensions, Gaussian measures do. It is the "conjugated" description of the oscillator that generalizes to the $\dim \pi = \infty$ case.

So, when $\dim \pi = \infty$, a natural thing to try is $\mathcal{X} = \pi$ with a Gaussian measure on $\mathcal{X}$, something like

$$\mathrm{d}\mu(f) \propto e^{-\langle f, f\rangle_\pi}, \quad f \in \pi. \tag{10.33}$$

This does not quite work, because in infinite-dimensions Gaussian measures live on a bigger space than the Hilbert space whose inner product determines the covariance matrix. For this reason, one typically embeds

$$\mathcal{X} \hookrightarrow \mathcal{S}' \tag{10.34}$$

into the dual $\mathcal{S}'$ of a nuclear space $\mathcal{S}$. As long as $\langle -, - \rangle_\pi$ is continuous with respect to $\mathcal{S}$'s topology, general theorems, like that of Bochner–Minlos [**Simon**], guarantee that $\mathcal{S}'$ is sufficiently large to host the desired Gaussian measure.

EXAMPLE 10.7. *White noise* on the circle $\mathbb{S}^1 = \mathbb{R}_\theta / 2\pi\mathbb{Z}$ is the Gaussian random variable with covariance matrix $\langle -, - \rangle_{L^2(\mathbb{S}^1)}$, i.e. the random series

$$W = \sum_{n=0}^{\infty} \gamma_n \cos(n\theta) + \sum_{n=1}^{\infty} \tilde{\gamma}_n \sin(n\theta), \tag{10.35}$$

where the $\gamma_\bullet, \tilde{\gamma}_\bullet$ are i.i.d. $\mathbb{R}$-valued standard Gaussian random variables. This almost surely converges unconditionally in $\mathcal{D}'(\mathbb{S}^1)$ to a well-defined distribution. More precisely, it almost surely converges absolutely in the Sobolev space $H^s(\mathbb{S}^1)$ if $s < -1/2$.

However, W almost surely *fails* to lie in $H^s(\mathbb{S}^1)$ if $s \geq -1/2$ and in particular fails to lie in $L^2(\mathbb{S}^1)$. White noise is a random distribution, not a random function. ∎

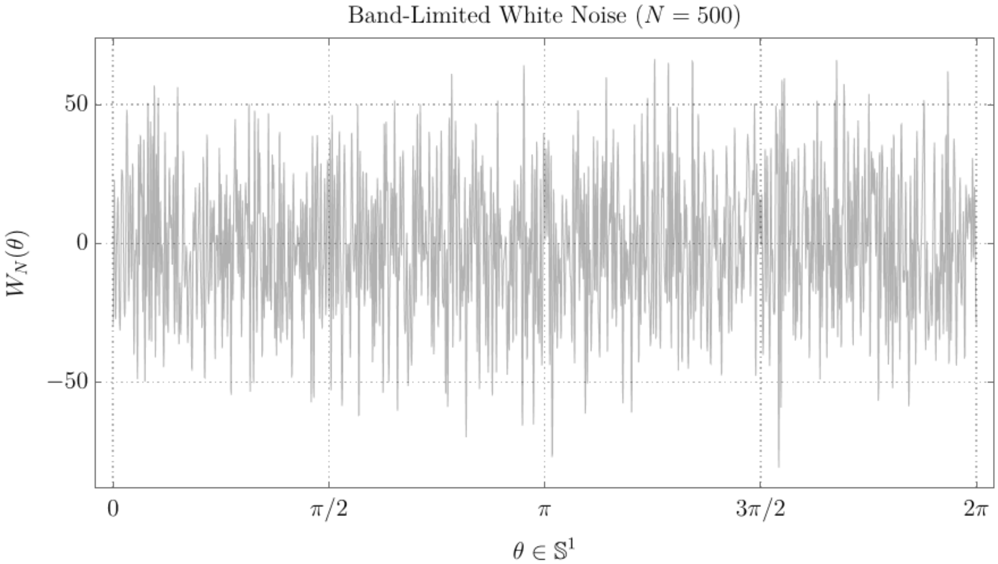


FIGURE 10.3. Band limited white noise W_N is white noise limited to a finite number $N \gg 1$ of terms. That is, $W_N = \sum_{n=0}^{N} \gamma_n \cos(n\theta) + \sum_{n=1}^{N} \tilde{\gamma}_n \sin(n\theta)$. Plotted above is a sample of the random function W_N. Of course, W_N is smooth, but the manifest spikiness illustrates how, as $N \to \infty$, the random function W_N fails to converge in L^2 to a function. The limit W is only a distribution.

Let us now restrict attention to the case when π is a Hilbert subspace of $\mathcal{S}'(\mathbb{R}^d; \mathcal{T})$; for example the Schrödinger picture wave-mechanical realizations of the Poincaré irreps. Then,

$$\mathcal{X} = \mathcal{S}'(\mathbb{R}^d; \mathcal{T}) \tag{10.36}$$

suffices to host whatever Gaussian measures we cook up. Thus, $\mathcal{X}$ hosts a Gaussian measure $\sim e^{-\langle \bullet, \bullet \rangle_\pi / 2}$.

Note that a state of the quantum field is a wavefunction $\Psi : \mathcal{S}'(\mathbb{R}^d; \mathcal{T}) \to \mathbb{C}$ on the very large space $\mathcal{S}'(\mathbb{R}^d; \mathcal{T})$. Thus, it is very hard to visualize. However, we can get a better idea of what Ψ looks like by sampling the measure $|\Psi(u)|^2 \,\mathrm{d}\mu(u)$. The result will be some random distribution $u \in \mathcal{S}'(\mathbb{R}^d; \mathcal{T})$.

3.2. Discussion of the isomorphism. A *monomial* of degree $N \in \mathbb{N}^+$ is an element of $L^2(\mathcal{S}'(\mathbb{R}^d; \mathcal{T}), \mu)$ of the form

$$\mathcal{S}'(\mathbb{R}^d; \mathcal{T}) \ni u \mapsto u(f_1) \cdots u(f_N) \tag{10.37}$$

for some $f_1, \ldots, f_N \in \mathcal{S}(\mathbb{R}^d; \mathcal{T})$, not necessarily distinct. We will denote this $f_1 \boxtimes \cdots \boxtimes f_N$. This is just like a monomial in elementary algebra, except the different variables are labeled by the $f \in \mathcal{S}(\mathbb{R}^d; \mathcal{T})$.

A polynomial of degree N is a linear combination of monomials of degree $\leq N$. We also consider the constant function $u \mapsto c$ to be a monomial (of degree 0), whatever $c \in \mathbb{C}$ is.

A *Wick monomial* of degree $N \in \mathbb{N}^+$, denoted $:f_1 \boxtimes \cdots \boxtimes f_N:$, is the projection of the monomial $f_1 \boxtimes \cdots \boxtimes f_N$ onto the orthogonal complement of the closure of the subspace of $L^2(\mathcal{S}'(\mathbb{R}^d; \mathcal{T}), \mu)$ consisting of all polynomials of degree $< N$.[2]

[2]Also let $:1: = 1$.

Proposition 10.8 (The Wiener–Itô–Segal isomorphism). *There exists a unitary map* $\iota : \operatorname{Sym}\pi \to L^2(\mathcal{S}'(\mathbb{R}^d), \mu)$ *given by*

$$\iota : \phi_1^{\otimes n_1} \odot \phi_2^{\otimes n_2} \odot \cdots \mapsto : \prod_{j=1}^{\infty} \phi_j^{\boxtimes n} : \tag{10.38}$$

on pure tensors. This isomorphism has the following properties.

(a) Spatial isometries all commute with ι.

(b) The map sends the position operator $Q(f)$ *to the multiplication map* $M_f : L^2(\mathcal{S}'(\mathbb{R}^d), \mu) \circlearrowleft$ *defined by* $(M\Psi)(u) = \langle u, f\rangle \Psi(u)$ *for all* $u \in \mathcal{S}'(\mathbb{R}^d; \mathcal{T})$.

■

Proof. This is a generalization of the proof that the eigenfunctions of the one-dimensional QHO generated by applying creation operators to the ground state exhaust $L^2(\mathbb{R})$. [*] □

10.A. The regularity of gaussian noise [*]

In this appendix, we discuss the support properties of the Gaussian measures used in the Wiener–Itô–Segal isomorphism. The setup involves the "one-particle" Hilbert space π, which is sandwiched

$$\mathcal{S}(\mathbb{R}^d) \subset \pi \subseteq \mathcal{S}'(\mathbb{R}^d) \tag{10.39}$$

between $\mathcal{S}, \mathcal{S}'$. (Note that this is not a Gelfand triple, because, unless $\pi = L^2$, the distributional pairing, which is based on the L^2-inner product, does not agree with π's native inner product.)

Remark 10.9. In finite-dimensions, the covariance of a Gaussian measure with density $\propto e^{-x^\intercal \omega x/2}$ is

$$\sigma = \omega^{-1} \tag{10.40}$$

not ω. Thus, if we are attempting to construct a Gaussian measure $\sim e^{-\langle \bullet, \bullet\rangle_\pi/2}$, the covariance matrix is $\mathcal{S}^2 \ni (f,g) \mapsto \langle f, g\rangle_{\pi'}$ where π' is the dual of π with respect to the distributional duality pairing.

For example, when $\pi = H^s(\mathbb{R}^d)$, then $\pi' = H^{-s}(\mathbb{R}^d)$. ■

Example 10.10. The *Brownian bridge* ("pink noise") is the Gaussian random distribution on the circle $\mathbb{S}^1$ whose derivative is white noise:[3]

$$B = \sum_{n=1}^{\infty} \frac{1}{n}(\gamma_n \sin(n\theta) - \tilde{\gamma}_n \cos(n\theta)). \tag{10.41}$$

This is a Gaussian random variable with covariance matrix $\langle \partial^{-1}\bullet, \partial^{-1}\bullet\rangle_{L^2} = \langle -, -\rangle_{\dot{H}^{-1}}$; indeed, the variance of $n^{-1}\gamma$ is n^{-2} for a standard Gaussian γ. Thus, the Brownian bridge "wants" to have finite energy, which would mean $B \in H^1(\mathbb{S}^1)$.

It almost surely lies in the larger space $L^2(\mathbb{S}^1)$. In fact, it almost surely lies in $H^s(\mathbb{S}^1)$ for any $s < 1/2$. But it almost surely fails to lie in $H^s(\mathbb{S}^1)$ for $s \geq 1/2$. ■

[*]

The field configurations of a free quantum field with high spin are almost surely more irregular than the field configurations of a free quantum field with lower spin.

[3]For any $c \in \mathbb{R}$, the shifted Brownian bridge $B + c$ also has the same derivative. The extra condition that uniquely specifies the Brownian bridge is that $\int_0^{2\pi} B(\theta)\,\mathrm{d}\theta = 0$.

APPENDIX A

Notation

Matrices

- σ_1, σ_2, σ_3: the Pauli σ-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

- $\mathscr{P}, \mathscr{T}$: the parity and time-reversal matrices, $\in \mathrm{O}(1,d)$.

$$\mathscr{T} = \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix}.$$

- $\mathscr{C} = \mathscr{P}\mathscr{T}$.

Groups.

- $\mathrm{O}(1,d)$: The full Lorentz group

$$\mathrm{O}(1,d) = \left\{ \Lambda \in \mathbb{R}^{(1+d)\times(1+d)} : \Lambda^\intercal \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix} \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix} \right\},$$

  including parity and time-reversal.
- $\mathrm{SO}(1,d)$: The identity component of the Lorentz group (restricted Lorentz group).
- $\mathrm{P}(1,d)$: The restricted Poincaré group in $1+d$ spacetime dimensions;

$$\mathrm{P}(1,d) \cong \mathbb{R}^{1,d} \rtimes \mathrm{SO}(1,d)$$

  naturally.
- $\mathrm{P}_{\mathrm{full}}(1,d)$: The full Poincaré group, including parity and time-reversal.

Spacetime and Geometry.

- $x = (t, \mathbf{x})$: A spacetime coordinate in Minkowski spacetime $\mathbb{R}^{1,d}$.
- z^2: the squared Minkowski norm of a vector z (mostly plus convention), $z^2 = -(z^0)^2 + \|\mathbf{z}\|^2$.
- η: The Minkowski metric matrix $\mathrm{diag}(-1, 1, \ldots, 1)$.
- $\gamma = (1 - \|\mathbf{v}\|^2)^{-1/2}$: The Lorentz factor associated with velocity $\mathbf{v}$.

Function spaces.

- $\mathcal{S}(\mathbb{R}^N_{\mathbf{x}})$: The space of Schwartz functions on $\mathbb{R}^N$, i.e. functions which, along with all of their derivatives, are decaying as $\|\mathbf{x}\| \to \infty$ faster than any power of $\|\mathbf{x}\|$.
- $\mathcal{S}'(\mathbb{R}^N)$: The space of tempered distributions, i.e. the dual space of $\mathcal{S}(\mathbb{R}^N)$.
- $H^m(\mathbb{R}^N)$, $m \in \mathbb{R}$: The Sobolev space $(1-\Delta)^{-m/2} L^2(\mathbb{R}^N)$.

For each of these spaces, and for any finite-dimensional vector space $\mathcal{T}$, we have $\mathcal{T}$-valued analogues, denoted by adding "$;\mathcal{T}$" between the parentheses.

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