MASSIVE WAVE PROPAGATION NEAR NULL INFINITY

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Abstract. We study, fully microlocally, the propagation of massive waves on the *octagonal compactification*

$$\mathbb{O} = [\overline{\mathbb{R}^{1,d}}; \mathscr{I}; 1/2]$$

of asymptotically Minkowski spacetime, which allows a detailed analysis both at timelike and spacelike infinity (as previously investigated using Parenti–Shubin–Melrose's sc-calculus) and, more novelly, at null infinity, denoted $\mathscr{I}$. The analysis is closely related to Hintz–Vasy's recent analysis of massless wave propagation at null infinity using the "e,b-calculus" on $\mathbb{O}$. We prove several corollaries regarding the Klein–Gordon IVP. Our main technical tool is a fully symbolic pseudodifferential calculus, $\Psi_{\mathrm{de,sc}}(\mathbb{O})$, the "de,sc-calculus" on $\mathbb{O}$. The 'de' refers to the structure ("double edge") of the calculus at null infinity, and the 'sc' refers to the structure ("scattering") at the other boundary faces. We relate this structure to the hyperbolic coordinates used in other studies of the Klein–Gordon equation. Unlike hyperbolic coordinates, the de,sc- boundary fibration structure is Poincaré invariant.

Contents

1. Introduction

The subject of this paper is the propagation of massive waves on Minkowski-like spacetimes (M, g). We will be more precise later regarding the meaning of "Minkowski-like" in the previous sentence – we use the term *admissible* below – but for now we just note that such spacetimes are

- non-trapping, in the sense that null geodesics asymptote in the usual way,
- globally hyperbolic, with t a smooth time function and each $\Sigma_T = \{(T, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$ a Cauchy hypersurface – so that the initial value problem with data specified on Σ_0 is well-posed – and
- asymptotically flat, in both the spacelike and timelike directions, with a classical error (see §7), so that the metric asymptotes to the Minkowski metric at large times and at distances.

The main microlocal estimates below depend only on the asymptotic structure of the metric, but the global hyperbolicity and non-trapping assumptions are required for the stated applications. It must stressed that we do not allow stationary or approximately stationary spacetimes besides the

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exact Minkowski spacetime itself, as these do not asymptote to the Minkowski spacetime at large times. Of course, the exact Minkowski spacetime $(\mathbb{R}^{1,d}, g_{\mathbb{M}})$, with

$$g_{\mathbb{M}} = -\mathrm{d}t^2 + \sum_{j=1}^{d} \mathrm{d}x_j^2 \tag{1}$$

counts as admissible. Our sign convention for Lorentzian metrics is the mostly positive one. While the analysis below goes through for somewhat general manifolds M, we focus on the case when M is $\mathbb{R}^{1,d}$ with an admissible metric.

Fix $\mathsf{m} > 0, d \in \mathbb{N}^+$. Given an admissible Lorentzian metric g on $\mathbb{R}^{1,d}$, let

$$\begin{aligned} \square_g &= -\frac{1}{|g|^{1/2}} \sum_{i,j=0}^{d} \frac{\partial}{\partial x_i}\Big[|g|^{1/2} g^{ij} \frac{\partial}{\partial x_j}\Big] \\ &= -\frac{1}{|g|^{1/2}} \Big(\sum_{i=0}^{d} \frac{\partial}{\partial x_i}\Big[|g|^{1/2} g^{i0} \frac{\partial}{\partial t}\Big] + \sum_{j=1}^{d} \frac{\partial}{\partial x_0}\Big[|g|^{1/2} g^{0j} \frac{\partial}{\partial x_j}\Big]\Big) + \triangle_g \end{aligned} \tag{2}$$

denote the associated d'Alembertian, with the sign convention being such that the Laplace–Beltrami portion $\triangle_g$ is positive semidefinite.

Consider the Klein–Gordon equation

$$\square_g u + Qu + \mathsf{m}^2 u = f, \tag{3}$$

where u is the unknown function or distribution, f is the forcing, and Q is drawn from a subspace of appropriate first order differential operators whose coefficients decay at infinity (more precisely, are short range). For instance, Q can be any Schwartz function, considered as a multiplication operator, so included in this setup is

$$\square + \mathsf{m}^2 + V = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \mathsf{m}^2 + V \tag{4}$$

for $V \in \mathcal{S}(\mathbb{R}^{1,d})$, which governs the evolution of massive waves on the exact Minkowski spacetime in the presence of the potential V. Here, $\square = \square_{g_{\mathbb{M}}} = \partial_t^2 - (\partial_{x_1}^2 + \cdots + \partial_{x_d}^2)$ is the exact flat space d'Alembertian.

The behavior of solutions to the associated IVP

$$\begin{cases} \square_g u + Qu + \mathsf{m}^2 u = f, \\ u|_{t=0} = u^{(0)}, \\ \partial_t u|_{t=0} = u^{(1)}, \end{cases} \tag{5}$$

is a rather classical topic. Nevertheless, it has apparently remained open to establish (beyond the exact Minkowski case) that, e.g. in the case when the forcing and initial data are Schwartz, the solution u admits a full asymptotic expansion at infinity. "At infinity" means at the boundary of the radial compactification

$$\mathbb{M} = \overline{\mathbb{R}^{1,d}} = \overline{\mathbb{R}^{1+d}} = \mathbb{R}^{1+d} \cup \{\infty\mathbb{S}^d\} \tag{6}$$

of the spacetime. Such a result appears below.

For the exact Klein–Gordon operator, a proof of this claim can be found in [H97̈, §7.2]. The proof there, which utilizes the global Fourier transform to produce the solution to the IVP in terms of oscillatory integrals whose asymptotics can be extracted via the method of stationary phase, does not generalize to the case when the PDE has variable coefficients. As is well-known, the Parenti-Shubin-Melrose sc-calculus [Mel94][Vas18] straightforwardly allows us to estimate the solution to the IVP in weighted L^2-based Sobolev spaces, even in the variable coefficient case. The basic estimates are discussed in [Vas18; Vas20], and standard modifications using module regularity

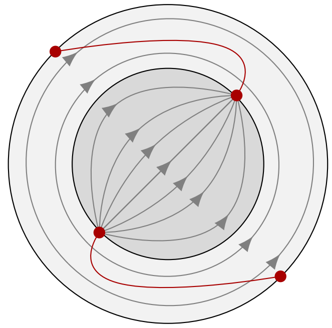

FIGURE 1. The sc-Hamiltonian flow within one sheet ${}^{\mathrm{sc}}\Sigma_{\mathsf{m},+}$ of the sc-characteristic set, depicted in the case $d=1$. The central disk (dark gray) represents one component of ${}^{\mathrm{sc}}\Sigma_{\mathsf{m},+}\cap{}^{\mathrm{sc}}\mathbb{S}^*\mathbb{M}$ (which is disconnected if $d=1$), i.e. one half of the portion of ${}^{\mathrm{sc}}\Sigma_{\mathsf{m},+}$ at fiber infinity. The other half is hidden from view. The lighter gray annulus depicts the portion of ${}^{\mathrm{sc}}\Sigma_{\mathsf{m},+}$ over spatial infinity. The radial sets, at which the properly scaled Hamiltonian vector field vanishes, are colored red.

[HMV04; GR+20; GRHG] allow one to establish asymptotic expansions (cf. [Mel94]) away from *null infinity*,

$$\mathscr{I}=\mathrm{cl}_{\mathbb{M}}\{|t|=r\}\cap\partial\mathbb{M}. \tag{7}$$

Here, $r=\lVert\mathbf{x}\rVert$ is the *spatial* Euclidean radial coordinate. The upshot is that, under appropriate hypotheses, if u solves the IVP, with $f\in\mathcal{S}(\mathbb{R}^d)$, then

- u is Schwartz away from the (closed) past and future caps $\overline{C}_\pm=\mathrm{cl}_{\mathbb{M}}\{\pm t\geq r\}\cap\partial\mathbb{M}$, and
- within $\{|t|>r\}$, we can write
$$u=|t|^{-d/2}e^{-i\mathsf{m}\sqrt{t^2-r^2}}u_-+|t|^{-d/2}e^{+i\mathsf{m}\sqrt{t^2-r^2}}u_+ \tag{8}$$
for $u_\pm\in C^\infty(\mathbb{M}\backslash\mathscr{I})$.

We will assume throughout that the reader is familiar with the sc-calculus. See [Vas18] for an exposition of this theory.

Exactly at null infinity, the notion of module regularity needed to extract asymptotics becomes problematic. The reason for this is that the radial set of the sc-Hamiltonian flow hits fiber infinity there. Relatedly, the phases in eq. (8) become singular at the light cone:

$$\pm\,\mathsf{m}\,\mathrm{d}\sqrt{t^2-r^2}=\pm\mathsf{m}\frac{t\,\mathrm{d}t-r\,\mathrm{d}r}{\sqrt{t^2-r^2}}, \tag{9}$$

and it is these sc- 1-forms that parametrize the radial sets over $\overline{C}_\pm$. Dually, the first order differential operator that one inverts in $C_\pm=\overline{C}_\pm^\circ$ to produce asymptotic expansions (see §3),

$$\frac{1}{\sqrt{t^2-r^2}}\Big(t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}\Big)\mp i\mathsf{m}, \tag{10}$$

which is related to the hyperbolic symmetries of the PDE, becomes singular at the light cone. Multiplying by $(t^2-r^2)^{1/2}$ cures this but causes other problems in the asymptotic extraction process. At first glance, these issues seem like they should be merely technical. This does not appear to be the case. Even if it is, the fact remains that the situation at null infinity requires clarification.

That the radial set hits fiber infinity correctly suggests that the solution to the Klein–Gordon IVP with generic Schwartz initial data and forcing has sc-wavefront set in the corner

$${}^{\mathrm{sc}}\mathbb{S}^*_{\partial\mathbb{M}}\mathbb{M}=\partial\,{}^{\mathrm{sc}}\overline{T}^*_{\partial\mathbb{M}}\mathbb{M}\subseteq{}^{\mathrm{sc}}\overline{T}^*\mathbb{M} \tag{11}$$

of the radially compactified sc-cotangent bundle ${}^{\mathrm{sc}}\overline{T}^*\mathbb{M}$. Such wavefront set is an obstruction to decay. But, as is well-known at least in the exact Minkowski case [Win88][Kla93][Hö97], massive waves (unlike massless waves) do not have an associated "radiation field": the solution to the IVP is rapidly decaying at null infinity, even though there exists sc-wavefront set over it. *What, then, is the sc-wavefront set detecting?* The answer is that it is detecting that any neighborhood of null infinity in $\mathbb{M}$ contains points at timelike infinity, at which solutions to the IVP do not decay rapidly. This suggests that, in order to study massive wave propagation along null geodesics, we should work with a compactification of $\mathbb{R}^{1,d}$ that separates individual null geodesics from timelike infinity.

One compactification that does the trick is the usual Penrose diagram $\mathbb{P} \hookleftarrow \mathbb{R}^{1,d}$ of Minkowski space. But, the Penrose diagram does not offer adequate resolution at timelike infinity, where solutions to Klein–Gordon display their oscillatory asymptotic tails. Rather, as in [BVW15][HV23], we use a third compactification $\mathbb{O} \hookleftarrow \mathbb{R}^{1,d}$ that refines both the radial and Penrose compactifications in the sense that one has compatible blowdown maps $\mathbb{O} \to \mathbb{P}, \mathbb{M}$. The space $\mathbb{O}$ can thus be constructed in two equivalent ways: blowing up $\mathscr{I} \subseteq \mathbb{M}$, in which case we write

$$\mathbb{O} = [\mathbb{M}; \mathscr{I}; 1/2], \tag{12}$$

or blowing up spacelike and timelike infinity in $\mathbb{P}$ in an appropriate way. In the former case, it is convenient to modify the smooth structure at the front faces of the blowup so that the original bdfs ϱ_{Nf} of the front faces of $[\mathbb{M}; \mathscr{I}]$ become the squares

$$\varrho_{\mathrm{Nf}} = \varrho_{\mathrm{nf}}^2 \tag{13}$$

of the new bdfs ϱ_{nf}. This is the '1/2' in "$\mathbb{O} = [\mathbb{M}; \mathscr{I}; 1/2]$." The manifold-with-corners (mwc) $\mathbb{O}$ is depicted in Figure 2, where we have labeled its faces Pf for past timelike infinity, nPf for past null infinity, Sf for spacelike infinity, nFf for future null infinity, and Ff for future timelike infinity. We will refer to $\mathbb{O}$ as the *octagonal* compactification of Minkowski spacetime, as in the $d = 1$ case it is literally an octagon, and the faces nPf, Sf, and nFf are disconnected, each consisting of two components. In this case, it is a slight abuse of terminology to refer to nPf, Sf, and nFf as faces, but it is a harmless one.

For each face f of $\mathbb{O}$, let ϱ_{f} denote a boundary-definition-function (bdf) of f. The statements below will mostly not depend on the particular choices of bdfs. One globally-defined choice of $\varrho_{\mathrm{nPf}}, \varrho_{\mathrm{nFf}}$ is

$$\varrho_{\pm} = \Big(\Big(\frac{t}{\sqrt{1+t^2+r^2}} \mp \frac{1}{\sqrt{2}} \Big)^2 + \frac{1}{1+t^2+r^2} \Big)^{1/4}, \tag{14}$$

where $\varrho_- = \varrho_{\mathrm{nPf}}$ and $\varrho_+ = \varrho_{\mathrm{nFf}}$. Alternatively, the bdfs $\varrho_{\mathrm{Pf}}, \varrho_{\mathrm{nPf}}, \varrho_{\mathrm{nFf}}, \varrho_{\mathrm{Ff}}$ can be chosen such that, near Ff and away from $\mathrm{cl}_{\mathbb{O}}\{r = 0\}$,

$$\varrho_{\mathrm{nFf}} = \sqrt{\frac{t-r}{t+r}}, \qquad \varrho_{\mathrm{Ff}} = \frac{1}{t-r}, \tag{15}$$

and similarly near Pf, with t replaced by $-t$. So, to say that some function is smooth at the corner $\mathrm{Ff} \cap \mathrm{nFf}$ means that it admits a joint Taylor series in these coordinates. Similar statements apply regarding the other corners of $\mathbb{O}$. Near any point in the interior of nPf or nFf, $1/(r + |t|)^{1/2}$ can be taken as a local boundary-defining-function (bdf). Thus, smoothness at $\mathrm{nPf}^\circ \cup \mathrm{nFf}^\circ$ is closely related to the existence of asymptotic expansions with respect to light-cone coordinates.

We discuss $\mathbb{O}$ further in §2.

We can now state our main theorem:

Theorem 1. *Given the setup above, and given any* $\chi \in C^\infty(\mathbb{O})$ *supported in* $\mathrm{cl}_{\mathbb{O}}\{t^2 \geq r^2\}^\circ = \mathrm{cl}_{\mathbb{O}}\{t^2 \geq r^2\} \setminus \mathrm{cl}_{\mathbb{O}}\{t^2 = r^2\}$ *and identically equal to* 1 *in some neighborhood of* $\mathrm{Pf} \cup \mathrm{Ff}$, *u has the form*

$$u = u_0 + \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{-i\mathsf{m}\sqrt{t^2-r^2}} u_- + \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{+i\mathsf{m}\sqrt{t^2-r^2}} u_+ \tag{16}$$

for some $u_0 \in \mathcal{S}(\mathbb{R}^{1,d})$ *and some* $u_\pm \in \varrho_{\mathrm{nPf}}^\infty \varrho_{\mathrm{Sf}}^\infty \varrho_{\mathrm{nFf}}^\infty C^\infty(\mathbb{O}) = \bigcap_{k \in \mathbb{N}} \varrho_{\mathrm{nPf}}^k \varrho_{\mathrm{Sf}}^k \varrho_{\mathrm{nFf}}^k C^\infty(\mathbb{O})$. ■

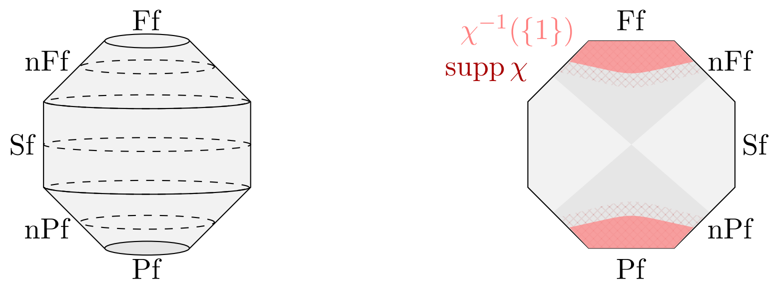

FIGURE 2. A mwc diffeomorphic to $\mathbb{O}$ when $d=2$, with labeled faces (*left*). The union $\mathrm{nPf}\cup\mathrm{nFf}$ is the lift of $\mathscr{I}$ to $\mathbb{O}$. The lift of future infinity is Ff, the lift of past infinity is Pf, and the lift of spacelike infinity is Sf. The support conditions on χ in Theorem 1 (*right*); $\operatorname{supp}\chi$ is denoted in red crosshatch, and $\chi=1$ identically on the solid red region. The set $\mathrm{cl}_{\mathbb{O}}\{t^2>r^2\}^\circ$ is a slightly darker gray.

In particular, solutions to the Klein–Gordon IVP with Schwartz initial data and Schwartz forcing decay rapidly along null geodesics, and the scattering data $u_-|_{C_\pm}, u_+|_{C_\pm}$ are Schwartz functions on $\overline{C}_\pm$. Likewise, the higher order terms in the asymptotic expansions at $C_\pm$ are Schwartz functions on $\overline{C}_\pm$. The support of χ is chosen such that $(t^2-r^2)^{1/2}$ is a (one-step) polyhomogeneous function on a neighborhood of $\operatorname{supp}\chi$. Theorem 1 therefore shows that u is of exponential-polyhomogeneous type on $\mathbb{O}$, which is a precise way of saying that the five boundary hypersurfaces of $\mathbb{O}$ give a complete set of asymptotic regimes. The proof of the theorem is in §7, using the results of the preceding sections.

If the initial data is not rapidly decaying, then the solution to the IVP is not necessarily rapidly decaying at Sf, nor at $\mathrm{nPf}\cup\mathrm{nFf}$. Conversely, if the initial data $(u^{(0)},u^{(1)})$ and forcing f satisfy

$$f\in H^{m-1,s+1}_{\mathrm{sc}}(\mathbb{R}^{1+d})=(1+r^2+t^2)^{-(s+1)/2}H^{m-1}(\mathbb{R}^{1+d}) \tag{17}$$

$$(u^{(0)},u^{(1)})\in H^{m,s+1}_{\mathrm{sc}}(\mathbb{R}^d)\times H^{m-1,s+1}_{\mathrm{sc}}(\mathbb{R}^d) \tag{18}$$

for $m\in\mathbb{N}$ and $s\in\mathbb{R}$, where $H^{m,s}_{\mathrm{sc}}(\mathbb{R}^d)=\langle r\rangle^{-s}H^m(\mathbb{R}^d)$, then one expects $u\in H^{m,s}_{\mathrm{sc}}(\mathbb{R}^{1+d})$ near the interior of Sf. So the amount of decay of u in the spacelike region is controlled by the amount of decay of the initial data and the forcing.

At null infinity, a lack of *regularity* also obstructs decay. For example, using the vector-field method, Klainerman [Kla93, Theorems 2 & 3] shows that, in the exact Minkowski case with zero forcing (and under an assumption about supports), there exists some $c=c_{\mathsf{m},d}>0$ such that

$$|u(t,\mathbf{x})|\leq ct^{-d/2}\begin{cases}(t+r)^{-k/2}I^0_{\mathbb{P},k+\lceil d/2\rceil}(u,\Sigma_0) & (t>0,r\geq t),\\ (t-r+1)^{k/2}(t+r)^{-k/2}\log(t-r+1)I^0_{\mathbb{P},k+\lceil d/2\rceil}(u,H_1) & (t>0,t>r),\end{cases} \tag{19}$$

where

$$I^0_{\mathbb{P},k+\lceil d/2\rceil}(u,\Sigma_0)=O(\lVert u^{(0)}\rVert_{H_{\mathrm{sc}}^{k+\lceil d/2\rceil,k+\lceil d/2\rceil}(\mathbb{R}^d)}+\lVert u^{(1)}\rVert_{H_{\mathrm{sc}}^{k+\lceil d/2\rceil-1,k+\lceil d/2\rceil-1}(\mathbb{R}^d)}) \tag{20}$$

is a quantity depending on the $L^2(\mathbb{R}^d)$ norms of u and its derivatives up to order $k+\lceil d/2\rceil$ on the Cauchy hypersurface $\Sigma_0=\{(t,\mathbf{x}):t=0\}$, and similarly for $I^0_{\mathbb{P},k+\lceil d/2\rceil}(u,H_1)$ on $H_1=\{(t,\mathbf{x}):t^2-r^2=1\}$. So,

$$|u(t,\mathbf{x})|=O\Bigg(\begin{cases}\varrho_{\mathrm{Sf}}^{(d+k)/2}\varrho_{\mathrm{nFf}}^{d+k} & (t\geq 1,r\geq t)\\ \varrho_{\mathrm{nFf}}^{d+k-}\varrho_{\mathrm{Ff}}^{d/2-} & (t\geq 1,t>r)\end{cases}\Bigg). \tag{21}$$

If our initial data only has a finite amount of Sobolev regularity, we can only conclude decay at null infinity to some corresponding finite order, with one extra order of decay for every extra order of regularity.

As an example of what can happen when our forcing is not smooth, consider the advanced and retarded propagators $D_+, D_- \in \mathcal{S}'(\mathbb{R}^{1,d})$ for $\Box + \mathsf{m}^2$. The forward and reverse problems for $\Box + \mathsf{m}^2$ read

$$\begin{cases} \Box u(t,\mathbf{x}) + \mathsf{m}^2 u(t,\mathbf{x}) = \delta(t) f(\mathbf{x}) \\ u(t,\mathbf{x}) = 0 \text{ for } \pm t < 0, \end{cases} \tag{22}$$

for $f \in \mathcal{S}(\mathbb{R}^d_{\mathbf{x}})$, where the positive choice of sign gives the forward problem and the negative choice gives the reverse problem. The distributions $D_\pm$ yield solutions to these problems in the sense that the unique $u \in \mathcal{D}'(\mathbb{R}^{1,d})$ satisfying eq. (22) is $f * D_\pm$. A straightforward (but somewhat nontrivial) calculation reveals that $D_\pm$ is given by

$$D_\pm(t,\mathbf{x}) = \pm \frac{1}{2\pi} \Theta(\pm t) \delta(t^2 - r^2) \mp \frac{\mathsf{m}}{4\pi} \Theta(\pm t) \Theta(t^2 - r^2) \frac{1}{\sqrt{t^2 - r^2}} J_1\big(\mathsf{m}\sqrt{t^2 - r^2}\big) \tag{23}$$

if $d = 3$, where J_1 denotes the Bessel function of the first kind of order one and Θ denotes a Heaviside step function [Sch95, §2.3]. A similar formula holds for other $d \in \mathbb{N}^+$. Note that $D_\pm$ solves the forced Klein–Gordon equation

$$(\Box + \mathsf{m}^2) D_\pm(t,\mathbf{x}) = \delta, \tag{24}$$

where $\delta \in \mathcal{S}'(\mathbb{R}^{1,3})$ is a Dirac δ-function located at the spacetime origin. The Heaviside step function $\Theta(t) = 1_{t \geq 0}$ in eq. (23) guarantees

$$\operatorname{supp} D_\pm(t,\mathbf{x}) \subseteq \{(t,\mathbf{x}) \in \mathbb{R}^{1,d} : \pm t \geq 0\}. \tag{25}$$

We highlight the following features of $D_\pm$:

- Outside of any neighborhood $U \subseteq \mathbb{M}$ of $\mathrm{cl}_{\mathbb{M}}\{|t| \leq r\}$ (and in particular away from null infinity), it follows from the large argument asymptotics of the Bessel function [AS64, §9.2] that
$$D_\pm = |t|^{-3/2} e^{-i\mathsf{m}\sqrt{t^2 - r^2}} d_{\pm,-} + |t|^{-3/2} e^{+i\mathsf{m}\sqrt{t^2-r^2}} d_{\pm,+} \tag{26}$$
for some $d_{\pm,-}, d_{\pm,+} \in C^\infty(\mathbb{M})$, just as in eq. (8). In particular, the nonsmoothness of the forcing does not obstruct decay at timelike infinity, as can already be proven using the sc-calculus.
- Though the convolution of $D_\pm$ with any Schwartz function is rapidly decaying at null infinity, $D_\pm$ are themselves not rapidly decaying there:
$$D_\pm = \mp \sqrt{\frac{\mathsf{m}^2 \varrho^3}{8\pi^3}} \frac{1}{v^{3/4}} \cos\Big(\frac{\mathsf{m} v^{1/2}}{\varrho} - \frac{3\pi}{4}\Big) + o_v(\varrho^{3/2}), \tag{27}$$
$v = |t| - r$ and $\varrho = (|t| + r)^{-1/2}$, where the decay rate of the $o_v(\varrho^{3/2})$ term is uniformly bounded in $v \gg 0$, with a complete asymptotic expansion in ϱ. We therefore have precisely $O(\varrho^{3/2})$ decay at null infinity. It is not at all apparent from eq. (23) why solutions to the forward and reverse problems with $f \in \mathcal{S}(\mathbb{R}^d)$ should be rapidly decaying at null infinity, though this does hold.
- The oscillations in eq. (27) take the form
$$\sim \exp(\pm i \mathsf{m} v^{1/2} \varrho^{-1}), \tag{28}$$
which implies the presence of sc-wavefront set (at finite frequencies) on the Penrose diagram $\mathbb{P}$, as long as we use ϱ as a bdf, rather than $\varrho^2 = 1/(|t| + r)$, this being a bdf of $\partial\mathbb{M}$ in $\mathbb{M}$.
  Note that, as $v \to 0^+$, it is *not* the case that the sc-frequency
$$\mathrm{d}\Big(\frac{v^{1/2}}{\varrho}\Big) = \frac{1}{2} \frac{\mathrm{d}v}{\varrho v^{1/2}} - \frac{v^{1/2}}{\varrho^2} \mathrm{d}\varrho = \zeta \frac{\mathrm{d}v}{\varrho} + \xi \frac{\mathrm{d}\varrho}{\varrho^2} \tag{29}$$
in eq. (28) converges to the zero section of the sc-cotangent bundle. The opposite is true — the sc-frequency approaches fiber infinity. As $v \to 0^+$, the component of the frequency dual

to ϱ, $\xi = -v^{1/2}$, converges to zero in the relevant sense, but the component $\zeta = 1/2v^{1/2}$ dual to v blows up, so the overall effect is that the frequency gets large.

Since $D_\pm$ is identically zero and therefore free of any sort of wavefront set over $\mathrm{cl}_{\mathbb{O}}\{|t| < r - \varepsilon\}$ for each $\varepsilon > 0$, this example suggests a form of propagation over null infinity, in which singularities in the interior of the spacetime travel along null geodesics, hit the corner of the appropriate radially compactified cotangent bundle (see below) over null infinity, and then propagate down into the fibers of that cotangent bundle while propagating forwards along null infinity.

Consider now the form of $\Box$ on $\mathbb{O}$:

- Since the interiors of the timelike and spacelike caps of $\mathbb{M}$ are canonically diffeomorphic with the interiors of $\mathrm{Pf}, \mathrm{Sf}, \mathrm{Ff}$, the operator $\Box + \mathsf{m}^2$ is a sc-differential operator there.
- At the interior of null infinity on the Penrose diagram, $\Box$ has the form $\varrho^2 \Box_0$ for an (unweighted) edge operator $\Box_0$ [HV23]. The same can not be said, as though we can write
$$\Box + \mathsf{m}^2 = \varrho^2(\Box_0 + \varrho^{-2}\mathsf{m}^2), \tag{30}$$
$\varrho^{-2}\mathsf{m}^2$ is too large as $\varrho \to 0$ to be an unweighted edge operator. Nevertheless, $\Box + \mathsf{m}^2$ can be regarded as an unweighted "double edge" (abbreviated "de" for short) operator. The double edge operators were introduced in [LM01], and we refer to this work for a discussion of the double edge calculus in a setting without corners. In the $d = 1$ case, the de-differential operators are just sc-differential operators, but the angular derivatives arising in the $d \geq 2$ case are required to vanish to an extra order. That the angular derivatives in $\Box + \mathsf{m}^2$ have this property is the ultimate reason why one must work with the de-calculus instead of the sc-calculus at null infinity.

  A key point is that the de-calculus is, like the sc-calculus, under symbolic control. This means that de-ΨDOs are controlled via a suitable notion of principal symbol modulo compact errors. Standard symbolic constructions from the theory of Kohn–Nirenberg ΨDOs on compact manifolds go through with straightforward modifications.

This all suggests that, in order to analyze the Klein–Gordon equation everywhere on $\mathbb{O}$, we define a pseudodifferential calculus
$$\Psi_{\mathrm{de,sc}} = \Psi_{\mathrm{de,sc}}(\mathbb{O}) = \bigcup_{m \in \mathbb{R}} \bigcup_{\mathsf{s} \in \mathbb{R}^5} \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}} \tag{31}$$
consisting of pseudodifferential operators that are sc-ΨDOs at $\mathrm{Pf}, \mathrm{Sf}, \mathrm{Ff}$ and de-ΨDOs at $\mathrm{nPf}, \mathrm{nFf}$, being in an appropriate sense both simultaneously at the corners of $\mathbb{O}$. Of course, we have a corresponding algebra $\mathrm{Diff}_{\mathrm{de,sc}}(\mathbb{O})$ of de,sc-differential operators with nice coefficients. A precise definition appears later. In eq. (31),
$$\Psi^{m,(s_{\mathrm{Pf}},s_{\mathrm{nPf}},s_{\mathrm{Sf}},s_{\mathrm{nFf}},s_{\mathrm{Ff}})}_{\mathrm{de,sc}} = \varrho_{\mathrm{Pf}}^{-s_{\mathrm{Pf}}} \varrho_{\mathrm{nPf}}^{-s_{\mathrm{nPf}}} \varrho_{\mathrm{Sf}}^{-s_{\mathrm{Sf}}} \varrho_{\mathrm{nFf}}^{-s_{\mathrm{nFf}}} \varrho_{\mathrm{Ff}}^{-s_{\mathrm{Ff}}} \Psi^{m,\mathsf{0}}_{\mathrm{de,sc}}, \tag{32}$$
so m is the "differential order" and $\mathsf{s} \in \mathbb{R}^5$ measures decay at the five different faces of $\mathbb{O}$. Like the constituent ΨDO calculi, the de,sc-calculus is under symbolic control. The relevant symbols are precisely conormal functions on a compactification
$$ {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O} \hookleftarrow T^*\mathbb{R}^{1,d} \tag{33}$$
of the cotangent bundle of Minkowski space. This is the entire space of a bundle ${}^{\mathrm{de,sc}}\pi : {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O} \to \mathbb{O}$ over $\mathbb{O}$. It is canonically diffeomorphic to ${}^{\mathrm{sc}}\overline{T}^*\mathbb{M}$ away from null infinity and ${}^{\mathrm{de}}\overline{T}^*\mathbb{O}$ way from timelike and spacelike infinity. See §2.

The connection between the geometric setup here and the hyperbolic coordinates employed in [Kla93] is discussed in §3. As far as asymptotic expansions are concerned, the two are not equivalent in general, but for the application to Theorem 1 we need only consider functions decaying rapidly at null infinity, for which the distinction is not important. One selling point of $\mathbb{O}$ is that, like the

Klein–Gordon equation itself, it is Poincaré invariant in the sense that the elements of the Poincaré group lift to diffeomorphisms of $\mathbb{O}$. In contrast, hyperbolic coordinate systems do not interact well with translations. The Poincaré invariance of the approach here is therefore a feature, though we still use hyperbolic coordinates to extract the asymptotic expansions at $\mathrm{Pf}\cup\mathrm{Tf}$.

The d'Alembertian $\Box = \Box_{g_{\mathbb{M}}}$ lies in $\mathrm{Diff}^{2,\mathsf{0}}_{\mathrm{de,sc}} \subseteq \Psi^{2,\mathsf{0}}_{\mathrm{de,sc}}$. This is a consequence of the fact that the Minkowski metric is a de,sc-metric. Later, we check the claim directly. A complication is that

$$\partial_t, \partial_{x_i} \in \mathrm{Diff}^{1,(0,1,0,1,0)}_{\mathrm{de,sc}} \setminus \mathrm{Diff}^{1,\mathsf{0}}_{\mathrm{de,sc}}, \tag{34}$$

and not $\partial_t, \partial_{x_i} \in \mathrm{Diff}^{1,\mathsf{0}}_{\mathrm{de,sc}}$. This is a computation we will do in §2. The particular linear combination of ∂_t^2 and $\partial_{x_i}^2$ appearing in $\Box$ has cancellations at null infinity, and so one gets $\Box \in \Psi^{2,\mathsf{0}}_{\mathrm{de,sc}}$ and not merely

$$\Box \in \Psi^{2,(0,1,0,1,0)}_{\mathrm{de,sc}}. \tag{35}$$

The function $p \in C^\infty(T^*\mathbb{R}^{1,d})$ defined by

$$p : \tau\,\mathrm{d}t + \sum_{i=1}^d \xi_i\,\mathrm{d}x_i \mapsto -\tau^2 + \sum_{i=1}^d \xi_i^2 + \mathsf{m}^2 \tag{36}$$

defines an element of $\sigma^{2,\mathsf{0}}_{\mathrm{de,sc}}(\Box + \mathsf{m}^2)$, where $\sigma^{2,\mathsf{0}}_{\mathrm{de,sc}}$ denotes a to-be-defined "de,sc- (joint) principal symbol map." This will be checked later. Of course, p is the full symbol of $\Box + \mathsf{m}^2$ in the uniform Kohn–Nirenberg calculus, and thus $p \in \sigma^{2,0}_{\mathrm{sc}}(\Box + \mathsf{m}^2)$, but neither of these obviously imply that p is sufficient to represent the principal de,sc-symbol. A priori, it is not even obvious that p is a symbol on the de,sc- phase space. These statements must be checked.

A consequence of the fact that $p \in \sigma^{2,\mathsf{0}}_{\mathrm{de,sc}}(\Box + \mathsf{m}^2)$ is that commutators of $\Box+\mathsf{m}^2$ with de,sc-ΨDOs have de,sc- principle symbols given by Poisson brackets of their symbols with p. We can therefore prove propagation estimates in the usual way, via the construction of a positive commutator, for which one constructs symbols that are monotone along the (appropriately scaled) de,sc-Hamiltonian flow

$$\mathsf{H}_p = \frac{\varrho_{\mathrm{df}} H_p}{\varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}} \in \mathcal{V}_{\mathrm{e}}({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}) \subset \mathcal{V}_{\mathrm{b}}({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}), \tag{37}$$

$$\mathsf{H}_p = \frac{2\varrho_{\mathrm{df}}}{\varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}}\Big[\tau\frac{\partial}{\partial t} - \sum_{i=1}^d \xi_i \frac{\partial}{\partial x_i}\Big] \tag{38}$$

on the de,sc-characteristic set

$$\Sigma_{\mathsf{m}} = \mathrm{Char}^{2,\mathsf{0}}_{\mathrm{de,sc}}(\Box + \mathsf{m}^2) = \tilde{p}^{-1}(\{0\}) \cap (\partial\, {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}). \tag{39}$$

Here, $\tilde{p} \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O};\mathbb{R})$ is the function $\tilde{p} = \varrho_{\mathrm{df}}^2 p$, where ϱ_{df} denotes a defining function of fiber infinity ${}^{\mathrm{de,sc}}\mathbb{S}^*\mathbb{O} \subset {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$. We will study the structure of the Hamiltonian flow

$$\Phi_\lambda = \exp(\mathsf{H}_p\lambda) : {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O} \to {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O} \tag{40}$$

in §4. In the $d = 1$ case, the flow, restricted to one component $\Sigma_{\mathsf{m},+}$ of Σ_{m}, is depicted in Figure 3. More specifically, $\Sigma_{\mathsf{m},\pm} \subseteq {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$ is the sheet of Σ_{m} on which the temporal frequency τ satisfies $\pm\tau > 0$.

As seen in the figure, H_p vanishes at several points on $\Sigma_{\mathsf{m},\pm}$. We split the vanishing set of H_p into several components,

$$\mathcal{R}^\pm_+, \mathcal{R}^\pm_-, \mathcal{N}^\pm_+, \mathcal{N}^\pm_-, \mathcal{C}^\pm_+, \mathcal{C}^\pm_-, \mathcal{K}^\pm_+, \mathcal{K}^\pm_-, \mathcal{A}^\pm_+, \mathcal{A}^\pm_- \subseteq \Sigma_{\mathsf{m},\pm} \tag{41}$$

our *radial sets*. We abbreviate $\mathcal{R} = \mathcal{R}^+_+ \cup \mathcal{R}^-_- \cup \mathcal{R}^+_- \cup \mathcal{R}^-_+$ and likewise for the other radial sets. The radial sets $\mathcal{R}^+_+$, $\mathcal{R}^-_-$, $\mathcal{R}^+_-$, $\mathcal{R}^-_+$ depend on m, but we omit this from the notation. The sign in the superscript denotes which sheet of the characteristic set the component lies in, with a positive sign denoting positive τ component, and the sign in the subscript denotes which half-space $\mathrm{cl}_{\mathbb{O}}\{\pm t > 0\}$ the component lies in. The interpretation of the different radial sets is as follows:

- The radial sets $\mathcal{R}_+^\pm, \mathcal{R}_-^\pm$, located over the timelike caps, are where the de,sc-wavefront set associated with the temporal tails of massive waves lives,
- $\mathcal{N}_+^\pm, \mathcal{N}_-^\pm$ are the endpoints of the Hamiltonian flow along which singularities (including singularities in initial data) propagate, thus are the entryway for singularities in the interior to the fibers over null infinity,
- $\mathcal{C}_+^\pm, \mathcal{C}_-^\pm, \mathcal{K}_+^\pm, \mathcal{K}_-^\pm$ are the parts of the corners of the de,sc-phase space lying in the characteristic set Σ_{m}, with zero momentum in the directions dual to the angular coordinates, and not already in one of the $\mathcal{N}$'s, with $\mathcal{C}$ being over the corner with timelike infinity and $\mathcal{K}$ being over the corner with spacelike infinity, and
- $\mathcal{A}_+^\pm, \mathcal{A}_-^\pm$ are additional radial sets that show up only in the $(1+d)$-dimensional case for $d \geq 2$ and are therefore not depicted in Figure 3 (but see Figure 5 and Figure 6). These can be probed via families of null geodesics with large angular momentum.

The simplest of the radial sets to define are $\mathcal{R}_-^\pm, \mathcal{R}_+^\pm$. Identifying ${}^{\mathrm{de,sc}}T^*_{\mathrm{Pf}^\circ \cup \mathrm{Tf}^\circ}\mathbb{O}$ with ${}^{\mathrm{sc}}T^*_{C_- \cup C_+}\mathbb{M}$, where $C_\pm$ are the (open) timelike caps of $\partial\mathbb{M}$,

$$\begin{aligned} \mathcal{R}_-^\pm &= \mathrm{cl}_{{}^{\mathrm{de,sc}}T^*\mathbb{O}}(\mathcal{R}_0^\pm \cap {}^{\mathrm{sc}}\pi^{-1}(C_-)) \\ \mathcal{R}_+^\pm &= \mathrm{cl}_{{}^{\mathrm{de,sc}}T^*\mathbb{O}}(\mathcal{R}_0^\pm \cap {}^{\mathrm{sc}}\pi^{-1}(C_+)), \end{aligned} \tag{42}$$

where

$$\mathcal{R}_0^\pm = \mathrm{Graph}(\pm \mathsf{m}\, \mathrm{d}\sqrt{t^2 - r^2}|_{C_- \cup C_+}) \tag{43}$$

are the two (disconnected) radial sets of the usual sc-Hamiltonian flow on ${}^{\mathrm{sc}}\pi : {}^{\mathrm{sc}}\overline{T}^*\mathbb{M} \to \mathbb{M}$, one in each sheet, depicted in Figure 1. In contrast to $\mathcal{R}_0$, the radial set $\mathcal{R}$ *does not hit fiber infinity*. See Figure 3, where this is indicated. Consequently, we have well-behaved notions of module regularity associated with $\mathcal{R}$. These are discussed below, in §3.2.

When studying the scattering problem, we propagate control through the radial sets in the following order:

$$\mathcal{R}_-^+, \mathcal{N}_-^+ \backslash {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Sf} \cap \mathrm{nPf}), \mathcal{C}_-^+, \mathcal{K}_-^+, \mathcal{N}_-^+, \mathcal{A}_-^+, \mathcal{A}_+^+, \mathcal{N}_+^+ \backslash {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Tf} \cap \mathrm{nFf}), \mathcal{K}_+^+, \mathcal{C}_+^+, \mathcal{N}_+^+, \mathcal{R}_+^+, \tag{44}$$

on the $\tau > 0$ sheet, and

$$\mathcal{R}_+^-, \mathcal{N}_+^- \backslash {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Sf} \cap \mathrm{nPf}), \mathcal{C}_+^-, \mathcal{K}_+^-, \mathcal{N}_+^-, \mathcal{A}_+^-, \mathcal{A}_-^-, \mathcal{N}_-^- \backslash {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Tf} \cap \mathrm{nFf}), \mathcal{K}_-^-, \mathcal{C}_-^-, \mathcal{N}_-^-, \mathcal{R}_-^- \tag{45}$$

on the other. Note the flow segments — see Figure 3, the black arrows — between the two endpoints of each component of $\mathcal{N}$. Consequently, we are forced to prove two separate radial point estimates at $\mathcal{N}$: one in which control is propagated into a proper portion (a "ray," beginning over spacelike or timelike infinity, stopping short of the other corner) and another in which the whole is controlled altogether. For unsurprising technical reasons, the former is somewhat subtle. We only prove the estimates needed here, though we do not rule out that more can be said.

Hintz and Vasy [HV23] have recently investigated massless wave propagation near null infinity using fully microlocal tools very similar to those used here. In contrast to the de,sc-calculus employed below, their e,b-calculus is not symbolic, for the same reason that the e- ("edge") and b- ("boundary") calculi are not symbolic. While this is necessary when studying mass*less* wave propagation (for which radiation must be understood), this means that the authors do not study propagation at finite frequencies, which, for the reasons sketched above, is important in understanding massive wave propagation. The purely symbolic de,sc-calculus turns out to be well-suited for this purpose, not only away from null infinity as previously understood, but also at null infinity.

The radial sets $\mathcal{N}, \mathcal{C}, \mathcal{K}, \mathcal{A}$ previously appeared in [HV23] under different aliases. The inclusion $\mathbb{S}^*\mathbb{R}^{1,d} \hookrightarrow {}^{\mathrm{de,sc}}\mathbb{S}^*\mathbb{O}$ extends to a diffeomorphism

$${}^{\mathrm{e,b}}\mathbb{S}^*\mathbb{O} \to {}^{\mathrm{de,sc}}\mathbb{S}^*\mathbb{O}, \tag{46}$$

so fiber infinity of the de,sc-cotangent bundle is canonically identifiable with fiber infinity of the e,b-cotangent bundle. So, besides the fiber radial direction, the situation at fiber infinity is the same

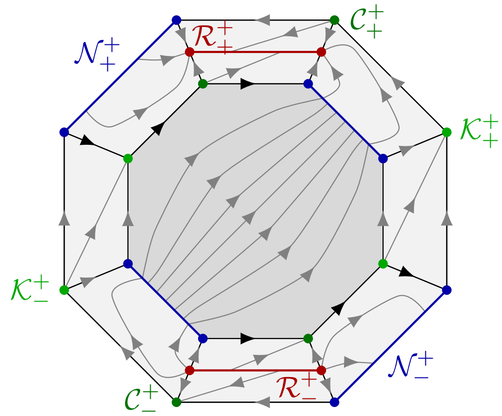

FIGURE 3. The sc,sc-Hamiltonian flow within one sheet of the sc,sc-characteristic set $\Sigma_{\mathsf{m},+}$, when $d = 1$ (with one half of the portion at fiber infinity hidden from view). As in the previous figure, fiber infinity is depicted in dark gray. Each quadrilateral panel depicts the portion of $\Sigma_{\mathsf{m},+}$ over one of the faces $\mathrm{f} \in \{\mathrm{Pf}, \mathrm{nPF}, \mathrm{Sf}, \mathrm{nFf}, \mathrm{Ff}\}$. (Since $d = 1$, $\mathrm{nPf}, \mathrm{Sf}, \mathrm{nFf}$ each consist of two connected components.) The radial sets – $\mathcal{R}_\pm^+$, $\mathcal{C}_\pm^+$, $\mathcal{N}_\pm^+$, $\mathcal{K}_\pm^+$ – are depicted in various colors. The source of the flow is $\mathcal{R}_-^+$, and, correspondingly, the sink is $\mathcal{R}_+^+$. The other six radial sets (each of which, because $d = 1$, consists of two connected components, only one of which is labeled above) are saddle points.

here as in [HV23]. In [HV23], the authors use terminology which, while fitting for the analysis at fiber infinity, is misleading when the flow in the de,sc-fiber radial directions is considered. From their e,b-perspective, the components of $\mathcal{N}$ are global sources and sinks. From the de,sc-perspective, this role is instead played by $\mathcal{R}$. More confusingly, de,sc-singularities can propagate from $\mathcal{N} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Sf})$ through the interior of the fibers of the de,sc-cotangent bundle back up to $\mathcal{K}$ (as can be seen in Figure 3), so $\mathcal{N}$ is not even a source/sink for the flow between the the radial sets in [HV23] once one considers the fiber radial component of the flow. So, some terminological change is necessary.

Finally, we point out recent work [GRHG] of Gell-Redman, Gomes, and Hassell on the nonlinear Schrödinger equation. Their regularity theory bears some similarities to the test modules used below, but it does not appear possible to straightforwardly apply their approach to the Klein–Gordon equation.

2. The octagonal compactification $\mathbb{O}$

We now discuss the octagonal compactification of $\mathbb{R}^{1,d}$. In §2.1, we describe the compactification itself. In §2.2, we briefly discuss the de,sc-phase space — here, we will describe some coordinate systems which will be used in §4. In §2.3, we outline the construction and features of the symbolic ΨDO calculus whose symbols live on that phase space. This includes a discussion of de,sc-based Sobolev spaces and associated wavefront sets, with respect to which the analysis in §5 will be phrased.

2.1. The Base Space.

Let

$$S_\pm = \partial\mathbb{M} \cap \mathrm{cl}_{\mathbb{M}}\{(t, \mathbf{x}) \in \mathbb{R}^{1,d} : \pm t = r\}, \tag{47}$$

$C_\pm = (\partial\mathbb{M} \cap \mathrm{cl}_{\mathbb{M}}\{(t, \mathbf{x}) \in \mathbb{R}^{1,d} : \pm t > r\}) \backslash S_\pm$, and $C_0 = (\partial\mathbb{M} \cap \mathrm{cl}_{\mathbb{M}}\{(t, \mathbf{x}) \in \mathbb{R}^{1,d} : t^2 < r^2\}) \backslash (S_- \cup S_+)$. (Note that $C_0, C_\pm$ are relatively open subsets of $\partial\mathbb{M}$.) "Null infinity," $\mathscr{I}$, when referring to a subset of $\mathbb{M}$, then refers to $S_- \cup S_+$, and timelike infinity refers to $C_- \cup C_+$. Spacelike infinity is C_0.

The sets S_-, S_+ are Poincaré invariant in the sense that, if $\Lambda : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}$ is an element of the Poincaré group, then Λ extends to a diffeomorphism of $\mathbb{M}$ the action of which $S_\pm$ are closed under. The *octagonal compactification* $\mathbb{R}^{1+d} \hookrightarrow \mathbb{O} = \mathbb{O}^{1,d}$ is defined by

$$\mathbb{O} = \Big[\mathbb{M}; \{S_-, S_+\}; \frac{1}{2}\Big] = \Big[\mathbb{M}; \mathscr{I}; \frac{1}{2}\Big], \tag{48}$$

i.e. we first perform a polar blowup of the boundary p-submanifolds S_-, S_+ – in either order – and then modify the smooth structure at the front face(s) of the blowups using the coordinate change $\varrho \mapsto \varrho^{1/2}$. That is, if we set

$$\mathbb{O}_0 = [\mathbb{M}; \{S_-, S_+\}], \tag{49}$$

then $\mathbb{O} = \mathbb{O}_0$ at the level of sets, and if ϱ denotes a bdf of the front face (or a front face) of this blowup, then $\varrho^{1/2}$ denotes a bdf of the corresponding face of $\mathbb{O}$. We will only write the "$1, d$" label on $\mathbb{O}^{1,d}$ when necessary. Otherwise, d should be assumed to be arbitrary. The space $\mathbb{O}$ is a mwc with corners of codimension two.

Proposition 2.1. *If Λ is any element of the Poincaré group, then Λ extends to an automorphism of $\mathbb{O}$ under which each boundary hypersurface is a closed set.* ■

Proof. We observe, first of all, that it suffices to prove the proposition for $\mathbb{O}_0$ in place of $\mathbb{O}$. Indeed, suppose that Λ extends to an automorphism $\Lambda_{\mathrm{ext}} : \mathbb{O}_0 \to \mathbb{O}_0$ fixing each boundary hypersurface. We only need to check that this map is actually smooth with respect to the smooth structure of $\mathbb{O}$. Then, after applying the same reasoning to the inverse, we can conclude that $\Lambda_{\mathrm{ext}} \in \mathrm{Aut}(\mathbb{O})$. To see this, note that, letting $\varrho_{\mathrm{f},0}$ denote any bdf for $\mathrm{f} \in \{\mathrm{Pf}, \mathrm{nPf}, \mathrm{Sf}, \mathrm{nFf}, \mathrm{Ff}\}$ *in* $\mathbb{O}_0$,

$$\varrho_{\mathrm{f},0} \circ \Lambda_{\mathrm{ext}} \in \varrho_{\mathrm{f},0} C^\infty(\mathbb{O}_0; \mathbb{R}^+). \tag{50}$$

Since we can take $\varrho_{\mathrm{f},0} = \varrho_{\mathrm{f}}$ unless $\mathrm{f} \in \{\mathrm{nPf}, \mathrm{nFf}\}$, $\varrho_{\mathrm{f}} \circ \Lambda_{\mathrm{ext}} \in \varrho_{\mathrm{f}} C^\infty(\mathbb{O}_0; \mathbb{R}^+) \subseteq C^\infty(\mathbb{O})$ for each $\mathrm{f} \in \{\mathrm{Pf}, \mathrm{Sf}, \mathrm{Ff}\}$. Taking square roots of eq. (50),

$$\varrho_{\mathrm{f}} \circ \Lambda_{\mathrm{ext}} \in \varrho_{\mathrm{f}} C^\infty(\mathbb{O}_0; \mathbb{R}^+) \subseteq \varrho_{\mathrm{f}} C^\infty(\mathbb{O}) \tag{51}$$

for $\mathrm{f} \in \{\mathrm{nPf}, \mathrm{nFf}\}$. So, indeed, it suffices to prove the claim made in the proposition for "$\mathbb{O}_0$" in place of "$\mathbb{O}$."

We recall that any invertible affine transformation extends to a diffeomorphism of $\mathbb{M}$ [Mel94] and that any translation extends to a diffeomorphism under which $\partial\mathbb{M}$ is invariant. In particular, the Poincaré group can be considered as a group of diffeomorphisms of $\mathbb{M}$. Of course, each of $C_0, C_\pm, S_\pm$ is closed under the action of any element of the Lorentz group. The claim of the proposition then follows from the lemma (which we apply with $X = \mathbb{M}$) that any diffeomorphism of any mwb X fixing (but not necessarily acting as the identity on) a submanifold $S \subseteq \partial X$ and each of the components of $\partial X \backslash S$ lifts to a diffeomorphism of the mwc $[X; S]$ (with the lift fixing each boundary hypersurface). □

The lemma mentioned in the previous proposition follows from the fact that $[X; S]$ is, in a neighborhood of the lift of S, diffeomorphic to the outward pointed normal bundle ${}^+N^*S$ of S [Mel].

Let $\mathrm{bd} : \mathbb{O} \to \mathbb{M}$ denote the blowdown map. For convenience, we can take $\mathbb{O}^\circ = \mathbb{M}^\circ = \mathbb{R}^{1+d}$, with $\mathrm{bd}|_{\mathbb{O}^\circ} = \mathrm{id}_{\mathbb{R}^{1+d}}$, along with

$$\mathbb{O} \backslash \mathrm{bd}^{-1}(S_- \cup S_+) = \mathbb{M} \backslash (S_- \cup S_+), \tag{52}$$

this all just making literal some conventional abuses of notation. Away from null infinity, $\mathbb{O}$ and $\mathbb{M}$ are "canonically diffeomorphic." In particular, if $\psi \in C^\infty(\mathbb{R}^{1+d})$ satisfies $\mathrm{cl}_{\mathbb{M}} \operatorname{supp} \psi \cap (S_- \cup S_+) = \varnothing$, then ψ extends to a smooth function on $\mathbb{O}$ if and only if extends to a smooth function on $\mathbb{M}$. We will use observations like this without comment below.

For the most part, we work away from $\mathrm{cl}_{\mathbb{M}}\{r = 0\}$. This allows us to work with (spatial) polar coordinates. Let $\dot{\mathbb{O}} = \mathbb{O} \backslash \mathrm{cl}_{\mathbb{O}}\{r = 0\}$. This is canonically diffeomorphic to $\hat{\mathbb{O}} \times \mathbb{S}^{d-1}_{\mathbf{x}/r}$, where

$$\hat{\mathbb{O}} = \mathbb{O}^{1,1} \backslash \mathrm{cl}_{\mathbb{O}^{1,1}}\{(t, r) \in \mathbb{R}^{1,1}_{t,r} : r \leq 0\}. \tag{53}$$

This mwc is noncompact (we do not add a boundary face corresponding to $r = 0$). The interior is equal to $\{(t,r) \in \mathbb{R}^{1,1} : r > 0\}$. Then, we have a diffeomorphism

$$\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{d-1} \to \dot{\mathbb{O}}^\circ = \mathbb{R}^{1,d} \backslash \{r = 0\}, \tag{54}$$

$(t,r,\theta) \mapsto (t, r\theta)$, which extends to a diffeomorphism $\hat{\mathbb{O}} \times \mathbb{S}^{d-1} \to \dot{\mathbb{O}}$. We will abuse notation below and conflate $\hat{\mathbb{O}} \times \mathbb{S}^{d-1}$ with $\dot{\mathbb{O}}$.

We will make use of the following coordinate charts for $\hat{\mathbb{O}}$:

- For each $T > 0$, let

$$\hat{\Omega}_{\mathrm{nfTf},\pm,T} = (\mathrm{cl}_{\mathbb{O}}\{|t| + T > r, \pm t > 0\})^\circ, \tag{55}$$

  and let $\varrho_{\mathrm{nf}} : \hat{\Omega}_{\mathrm{nfTf},\pm,T} \to [0,\infty)$ and $\varrho_{\mathrm{Tf}} : \hat{\Omega}_{\mathrm{nfTf},\pm,T} \to [0,\infty)$ be defined by $\varrho_{\mathrm{nf}} = (|t| - r + R)^{1/2}/(|t| + r + T)^{1/2}$ and $\varrho_{\mathrm{Tf}} = (|t| - r + T)^{-1}$. Then, $(\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Tf}}) : \hat{\Omega}_{\mathrm{nfTf},\pm,T} \to [0,\infty)^2$ is a coordinate chart on $\hat{\mathbb{O}}$. Solving for r,t in terms of $\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Tf}}$,

$$\begin{aligned} t &= \pm((2\varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Tf}})^{-1}(1 + \varrho_{\mathrm{nf}}^2) - T) \\ r &= (2\varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Tf}})^{-1}(1 - \varrho_{\mathrm{nf}}^2). \end{aligned} \tag{56}$$

  For later use in computing coordinate changes, we record the partial derivatives

$$\frac{\partial \varrho_{\mathrm{Tf}}}{\partial t} = \mp \varrho_{\mathrm{Tf}}^2, \quad \frac{\partial \varrho_{\mathrm{nf}}}{\partial t} = \pm \frac{1}{2}(1 - \varrho_{\mathrm{nf}}^2) \varrho_{\mathrm{nf}} \varrho_{\mathrm{Tf}}, \tag{57}$$

$$\frac{\partial \varrho_{\mathrm{Tf}}}{\partial r} = \varrho_{\mathrm{Tf}}^2, \quad \frac{\partial \varrho_{\mathrm{nf}}}{\partial r} = -\frac{1}{2}(1 + \varrho_{\mathrm{nf}}^2) \varrho_{\mathrm{nf}} \varrho_{\mathrm{Tf}}. \tag{58}$$

- For each $R > 0$, let

$$\hat{\Omega}_{\mathrm{nfSf},\pm,R} = (\mathrm{cl}_{\mathbb{O}}\{|t| < r + R, \pm t > 0\})^\circ \tag{59}$$

  and let $\varrho_{\mathrm{nf}} : \hat{\Omega}_{\mathrm{nfSf},\pm,R} \to [0,\infty)$ and $\varrho_{\mathrm{Sf}} : \hat{\Omega}_{\mathrm{nfSf},\pm,R} \to [0,\infty)$ be defined by $\varrho_{\mathrm{nf}} = (r - |t| + R)^{1/2}/(r + |t| + R)^{1/2}$ and $\varrho_{\mathrm{Sf}} = (r - |t| + R)^{-1}$. Then, $(\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Sf}}) : \hat{\Omega}_{\mathrm{nfSf},\pm,R} \to [0,\infty)^2$ is a coordinate chart on $\hat{\mathbb{O}}$. Solving for r,t in terms of $\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Sf}}$, we have

$$r = (2\varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Sf}})^{-1}(1 + \varrho_{\mathrm{nf}}^2) - R \tag{60}$$

$$t = \pm(2\varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Sf}})^{-1}(1 - \varrho_{\mathrm{nf}}^2). \tag{61}$$

  The partial derivatives are

$$\frac{\partial \varrho_{\mathrm{Sf}}}{\partial t} = \pm \varrho_{\mathrm{Sf}}^2, \quad \frac{\partial \varrho_{\mathrm{nf}}}{\partial t} = \mp \frac{1}{2}(1 + \varrho_{\mathrm{nf}}^2), \varrho_{\mathrm{nf}} \varrho_{\mathrm{Sf}} \tag{62}$$

$$\frac{\partial \varrho_{\mathrm{Sf}}}{\partial r} = -\varrho_{\mathrm{Sf}}^2, \quad \frac{\partial \varrho_{\mathrm{nf}}}{\partial r} = \frac{1}{2}(1 - \varrho_{\mathrm{nf}}^2) \varrho_{\mathrm{nf}} \varrho_{\mathrm{Sf}}. \tag{63}$$

In defining $\hat{\Omega}_{\mathrm{nfTf},\pm,T}$ and $\hat{\Omega}_{\mathrm{nfSf},\pm,R}$, we are using the topological notion of interior rather than the slightly different manifold-theoretic notion. Thus, both sets contain points of $\partial \hat{\mathbb{O}}$.

From the computations above, we see that, for any $m, s \in \mathbb{R}$,

$$\mathrm{Diff}_{\mathrm{sc}}^{m,s}(\mathbb{M}) \subseteq \mathrm{Diff}_{\mathrm{de,sc}}^{m,(s,2s-m,s,2s-m,s)}(\mathbb{O}). \tag{64}$$

Let $\Omega_\bullet = \hat{\Omega}_\bullet \times \mathbb{S}^{d-1}$.

Proposition 2.2. *On $\hat{\mathbb{O}}$, the d'Alembertian $\Box$ is given by the following:*

- *in $\Omega_{\mathrm{nfTf},\pm,T}$,*

$$\Box = -\varrho_{\mathrm{nf}}^4 \varrho_{\mathrm{Tf}}^2 \frac{\partial^2}{\partial \varrho_{\mathrm{nf}}^2} + 2\varrho_{\mathrm{nf}}^3 \varrho_{\mathrm{Tf}}^3 \frac{\partial^2}{\partial \varrho_{\mathrm{nf}} \partial \varrho_{\mathrm{Tf}}} + 2\varrho_{\mathrm{nf}}^3 \varrho_{\mathrm{Tf}}^3 \frac{\partial}{\partial \varrho_{\mathrm{nf}}}, \tag{65}$$

  and

- *in* $\Omega_{\mathrm{nfSf},\pm,R}$,

$$\Box = +\varrho_{\mathrm{nf}}^4\varrho_{\mathrm{Sf}}^2\frac{\partial^2}{\partial\varrho_{\mathrm{nf}}^2} - 2\varrho_{\mathrm{nf}}^3\varrho_{\mathrm{Sf}}^3\frac{\partial^2}{\partial\varrho_{\mathrm{nf}}\partial\varrho_{\mathrm{Sf}}} - 2\varrho_{\mathrm{nf}}^3\varrho_{\mathrm{Sf}}^3\frac{\partial}{\partial\varrho_{\mathrm{nf}}}, \tag{66}$$

where we are identifying the coordinate patches $\hat{\Omega}_{\mathrm{nfTf},\pm,T}, \hat{\Omega}_{\mathrm{nfSf},\pm,R}$ *and their images in* $\mathbb{R}^2$ *under the coordinate charts above.* ■

Proof. The first formula is the result of replacing

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial\varrho_{\mathrm{nf}}}{\partial t}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} + \frac{\partial\varrho_{\mathrm{Tf}}}{\partial t}\frac{\partial}{\partial\varrho_{\mathrm{Tf}}} = \pm\frac{1}{2}(1-\varrho_{\mathrm{nf}}^2)\varrho_{\mathrm{nf}}\varrho_{\mathrm{Tf}}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} \mp \varrho_{\mathrm{Tf}}^2\frac{\partial}{\partial\varrho_{\mathrm{Tf}}}\\ \frac{\partial}{\partial r} &= \frac{\partial\varrho_{\mathrm{nf}}}{\partial r}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} + \frac{\partial\varrho_{\mathrm{Tf}}}{\partial r}\frac{\partial}{\partial\varrho_{\mathrm{Tf}}} = -\frac{1}{2}(1+\varrho_{\mathrm{nf}}^2)\varrho_{\mathrm{nf}}\varrho_{\mathrm{Tf}}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} + \varrho_{\mathrm{Tf}}^2\frac{\partial}{\partial\varrho_{\mathrm{Tf}}}\end{aligned} \tag{67}$$

in $\Box = \partial_t^2 - \partial_r^2$. The second is the result of replacing

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial\varrho_{\mathrm{nf}}}{\partial t}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} + \frac{\partial\varrho_{\mathrm{Sf}}}{\partial t}\frac{\partial}{\partial\varrho_{\mathrm{Sf}}} = \mp\frac{1}{2}(1+\varrho_{\mathrm{nf}}^2)\varrho_{\mathrm{nf}}\varrho_{\mathrm{Sf}}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} \pm \varrho_{\mathrm{Sf}}^2\frac{\partial}{\partial\varrho_{\mathrm{Sf}}}\\ \frac{\partial}{\partial r} &= \frac{\partial\varrho_{\mathrm{nf}}}{\partial r}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} + \frac{\partial\varrho_{\mathrm{Sf}}}{\partial r}\frac{\partial}{\partial\varrho_{\mathrm{Sf}}} = +\frac{1}{2}(1-\varrho_{\mathrm{nf}}^2)\varrho_{\mathrm{nf}}\varrho_{\mathrm{Sf}}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} - \varrho_{\mathrm{Tf}}^2\frac{\partial}{\partial\varrho_{\mathrm{Sf}}}.\end{aligned} \tag{68}$$

□

We can conclude that $\Box \in \mathrm{Diff}^{2,\mathsf{0}}_{\mathrm{de,sc}}(\mathbb{O})$.

Recall that if M is a compact mwc, $\mathcal{F}(M)$ is the set of its faces, and ϱ_{f} denotes a bdf of $\mathrm{f} \in \mathcal{F}(M)$, then we have an LCTVS

$$\mathcal{S}(M) = \bigcap_{s\in\mathbb{R}}\Big[\prod_{\mathrm{f}\in\mathcal{F}(M)}\varrho_{\mathrm{f}}^s\Big]C^\infty(M) = \bigcap_{s\in\mathbb{R}}\bigcap_{k\in\mathbb{N}}\Big[\prod_{\mathrm{f}\in\mathcal{F}(M)}\varrho_{\mathrm{f}}^s\Big]C^k(M) \tag{69}$$

of "Schwartz" functions. When $M = \mathbb{M}$, then this is just the usual set of Schwartz functions on $\mathbb{R}^{1,d} = \mathbb{R}^{1+d}$. Identifying smooth functions on mwcs with their restrictions to interiors, $\mathcal{S}(\mathbb{R}^{1,d}) = \mathcal{S}(\mathbb{O})$, and this holds at the level of TVSs. Indeed, the blowdown map $\mathbb{O} \to \mathbb{M}$ is smooth, so any Schwartz function on $\mathbb{R}^{1,d}$ extends to a Schwartz function on $\mathbb{O}$, and the inclusion $\mathcal{S}(\mathbb{R}^{1,d}) \hookrightarrow \mathcal{S}(\mathbb{O})$ is continuous. The converse follows straightforwardly from the formulas above.

Consequently, a tempered distribution on $\mathbb{O}$, meaning an element of $\mathcal{S}'(\mathbb{O}) = \mathcal{S}(\mathbb{O})^*$, is just a tempered distribution on $\mathbb{R}^{1,d}$.

2.2. **The de,sc-Cotangent Bundle.** We now define the de,sc-tangent bundle $\pi_{\mathrm{de,sc}} : {}^{\mathrm{de,sc}}T\mathbb{O} \to \mathbb{O}$. As an indexed set, this is ${}^{\mathrm{de,sc}}T\mathbb{O} = \{{}^{\mathrm{de,sc}}T_p\mathbb{O}\}_{p\in\mathbb{O}}$, whose elements are the vector spaces

$${}^{\mathrm{de,sc}}T_p\mathbb{O} = \mathcal{V}_{\mathrm{de,sc}}(\mathbb{O};\mathbb{R})/\mathcal{I}_p\mathcal{V}_{\mathrm{de,sc}}(\mathbb{O};\mathbb{R}), \tag{70}$$

where $\mathcal{I}_p \subset C^\infty(\mathbb{O};\mathbb{R})$ is the ideal of smooth real-valued functions on $\mathbb{O}$ vanishing at p, and $\pi_{\mathrm{de,sc}} : {}^{\mathrm{de,sc}}T_p\mathbb{O} \ni \alpha \mapsto p$. Naturally, we can regard

$$\pi_{\mathrm{de,sc}} : {}^{\mathrm{de,sc}}T\mathbb{O} \to \mathbb{O} \tag{71}$$

as a real vector bundle over $\mathbb{O}$. The entire space ${}^{\mathrm{de,sc}}T\mathbb{O}$ is a mwc diffeomorphic to $\mathbb{O}\times\mathbb{R}^{1+d}$. Then, the de,sc-cotangent bundle

$${}^{\mathrm{de,sc}}\pi : {}^{\mathrm{de,sc}}T^*\mathbb{O} \to \mathbb{O} \tag{72}$$

is just defined to be the dual vector bundle to $\pi_{\mathrm{de,sc}} : {}^{\mathrm{de,sc}}T\mathbb{O} \to \mathbb{O}$. For convenience, we can arrange that ${}^{\mathrm{de,sc}}T^*_{\mathbb{R}^{1,d}}\mathbb{O} = T^*\mathbb{R}^{1,d}$ at the level of indexed sets (in which case this is identification is a bundle isomorphism).

Let ${}^{\mathrm{sc}}\pi : {}^{\mathrm{sc}}T^*\mathbb{M} \to \mathbb{M}$ denote the sc-cotangent bundle — see [Mel94][Mel95]. It can be shown that there exists a diffeomorphism

$${}^{\mathrm{sc}}\mathrm{plr} : \hat{\mathbb{M}} = \mathbb{M}^{1,1}\backslash\mathrm{cl}_{\mathbb{M}^{1,1}}\{(t,x)\in\mathbb{R}^{1,1} : x \le 0\}\times\mathbb{R}\times\mathbb{R}\times T^*\mathbb{S}^{d-1} \to {}^{\mathrm{sc}}T^*\mathbb{M}\backslash{}^{\mathrm{sc}}\pi^{-1}\mathrm{cl}_{\mathbb{M}}\{r=0\} \tag{73}$$

such that, for all $t, \tau, \Xi \in \mathbb{R}$, $r \in \mathbb{R}^+$, and $\eta_{\mathrm{sc}} \in T^*\mathbb{S}^{d-1}$,

$$ {}^{\mathrm{sc}}\mathrm{plr}((t,r),\tau,\Xi,\eta_{\mathrm{sc}}) = \tau\,\mathrm{d}t + \Xi\,\mathrm{d}r + r\,\mathrm{eulr}^*(\eta_{\mathrm{sc}}), \tag{74} $$

where $\mathrm{eulr} : \mathbb{R}^{1,d}_{t,\mathbf{x}} \backslash \{r=0\} \to \mathbb{S}^{d-1}_\theta$ is the map $(t,\mathbf{x}) \mapsto \mathbf{x}/r$. For the comparison with the de,sc-cotangent bundle, it is slightly better to work with $\mu = \tau + \Xi$ and $\nu = \tau - \Xi$, in terms of which $\tau\,\mathrm{d}t + \Xi\,\mathrm{d}r = \mu(\mathrm{d}t + \mathrm{d}r) + \nu(\mathrm{d}t - \mathrm{d}r)$.

Similarly, it can be shown that there exists a diffeomorphism

$$ {}^{\mathrm{de,sc}}\mathrm{plr} : \hat{\mathbb{O}} \times \mathbb{R} \times \mathbb{R} \times T^*\mathbb{S}^{d-1} \to {}^{\mathrm{de,sc}}T^*\mathbb{O} \backslash {}^{\mathrm{de,sc}}\pi^{-1}\,\mathrm{cl}_{\mathbb{O}}\{r=0\} \tag{75} $$

such that, for all $t, \mu, \nu \in \mathbb{R}$, $r \in \mathbb{R}^+$, and $\eta_{\mathrm{sc}} \in T^*\mathbb{S}^{d-1}$,

$$ {}^{\mathrm{de,sc}}\mathrm{plr}((t,r),\mu,\nu,\eta_{\mathrm{sc}}) = \frac{\varrho_{\mathrm{nFf}}}{\varrho_{\mathrm{nPf}}}\mu(\mathrm{d}t + \mathrm{d}r) + \frac{\varrho_{\mathrm{nPf}}}{\varrho_{\mathrm{nFf}}}\nu(\mathrm{d}t - \mathrm{d}r) + r\,\mathrm{eulr}^*(\eta_{\mathrm{sc}}). \tag{76} $$

As the subscript indicates, the coordinate η_{sc} should be thought of as keeping track of spatial angle and of the angular component of sc-frequency. We will drop the subscript 'sc' in later sections. With this diffeomorphism in mind, we set

$$ {}^{\mathrm{de,sc}}T^*\hat{\mathbb{O}} = \hat{\mathbb{O}} \times \mathbb{R}_\mu \times \mathbb{R}_\nu. \tag{77} $$

So, away from ${}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{cl}_{\mathbb{O}}\{r=0\})$, ${}^{\mathrm{de,sc}}T^*\hat{\mathbb{O}} \times (T^*\mathbb{S}^{d-1})_{\eta_{\mathrm{sc}}} \cong {}^{\mathrm{de,sc}}T^*\mathbb{O}$ "canonically."

As can be seen from the discussion above, ${}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$ is, away from null infinity, canonically diffeomorphic to ${}^{\mathrm{sc}}\overline{T}^*\mathbb{M}$.

In §4, we will use coordinates ξ, ζ, which, over $\hat{\Omega}_{\mathrm{nfTf},\pm,T}$, are associated to points in ${}^{\mathrm{de,sc}}T^*\hat{\mathbb{O}}$ via

$$ (\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Tf}}, \xi, \zeta) \mapsto \frac{\xi\,\mathrm{d}\varrho_{\mathrm{nf}}}{\varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Tf}}} + \frac{\zeta\,\mathrm{d}\varrho_{\mathrm{Tf}}}{\varrho_{\mathrm{nf}} \varrho_{\mathrm{Tf}}^2}. \tag{78} $$

Over $\hat{\Omega}_{\mathrm{nfSf},\pm,R}$, we use ξ, ζ to denote the coordinates

$$ (\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Sf}}, \xi, \zeta) \mapsto \frac{\xi\,\mathrm{d}\varrho_{\mathrm{nf}}}{\varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Sf}}} + \frac{\zeta\,\mathrm{d}\varrho_{\mathrm{Sf}}}{\varrho_{\mathrm{nf}} \varrho_{\mathrm{Sf}}^2} \tag{79} $$

on ${}^{\mathrm{de,sc}}T^*\hat{\mathbb{O}}$.

Proposition 2.3. *The function*

$$ p_0 : \tau\,\mathrm{d}t + \sum_{i=1}^d \xi_i\,\mathrm{d}x_i \mapsto \tau^2 + \sum_{i=1}^d \xi_i^2 \in C^\infty(T^*\mathbb{R}^{1,d}) \tag{80} $$

is a representative of $\sigma^{2,0}_{\mathrm{de,sc}}(\Box)$. ■

Proof. We already know that p_0 is a representative for the sc-principal symbol of $\Box$, so it suffices to work near null infinity. Passing to polar coordinates, it suffices to consider the $d=1$ case, working on $\hat{\mathbb{O}}$.

- In terms of the coordinates $(\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Tf}}, \xi, \zeta)$ on $\hat{\Omega}_{\mathrm{nfTf},\pm,T}$, solving for ξ and ζ in $\varrho_{\mathrm{nf}}^{-2}\varrho_{\mathrm{Tf}}^{-1}\xi\,\mathrm{d}\varrho_{\mathrm{nf}} + \varrho_{\mathrm{nf}}^{-1}\varrho_{\mathrm{Tf}}^{-2}\zeta\,\mathrm{d}\varrho_{\mathrm{Tf}} = \tau\,\mathrm{d}t + \Xi\,\mathrm{d}r$ yields

$$ \pm\tau = \frac{1}{2\varrho_{\mathrm{nf}}}(1-\varrho_{\mathrm{nf}}^2)\xi - \frac{\zeta}{\varrho_{\mathrm{nf}}}, \tag{81} $$

$$ \Xi = -\frac{1}{2\varrho_{\mathrm{nf}}}(1+\varrho_{\mathrm{nf}}^2)\,\xi + \frac{\zeta}{\varrho_{\mathrm{nf}}}. \tag{82} $$

Thus, the symbol $\Xi^2 - \tau^2$ is given by $\xi^2 - 2\xi\zeta$ with respect to this coordinate system.

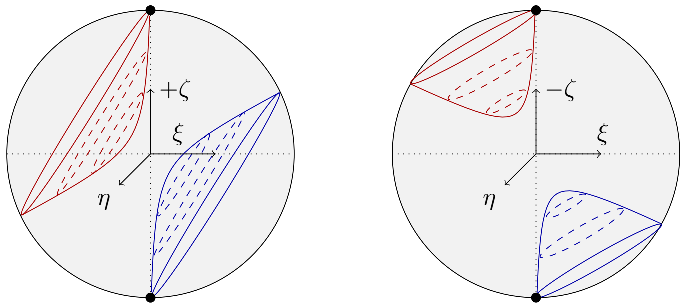

FIGURE 4. The characteristic set $\Sigma_{\mathsf{m}} = \Sigma_{\mathsf{m},-} \cup \Sigma_{\mathsf{m},+}$, over $\alpha \in \mathrm{nFf}$, depicted using the momenta dual to $(\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Sf}})$ (*left*, if $\alpha \in \Omega_{\mathrm{nfSf},+,R}$) and the momenta dual to $(\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Tf}})$ (*right*, if $\alpha \in \Omega_{\mathrm{nfTf},\pm,T}$). The vertical axis is oriented so that page-up corresponds to positive timelike momentum. The '$\bullet$' marks the submanifolds $\mathcal{N}^+_+, \mathcal{N}^-_+$. Over nPf, the situation is similar.

- In terms of the coordinates $(\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Sf}}, \xi, \zeta)$ on $\hat{\Omega}_{\mathrm{nfSf},\pm,R}$, solving for ξ and ζ in $\varrho_{\mathrm{nf}}^{-2} \varrho_{\mathrm{Sf}}^{-1} \xi \,\mathrm{d}\varrho_{\mathrm{nf}} + \varrho_{\mathrm{nf}}^{-1} \varrho_{\mathrm{Sf}}^{-2} \zeta \,\mathrm{d}\varrho_{\mathrm{Sf}} = \tau \,\mathrm{d}t + \Xi \,\mathrm{d}r$ yields

$$\pm\tau = -\frac{1}{2\varrho_{\mathrm{nf}}}(1+\varrho_{\mathrm{nf}}^2)\xi + \frac{\zeta}{\varrho_{\mathrm{nf}}}, \tag{83}$$

$$\Xi = \frac{1}{2\varrho_{\mathrm{nf}}}(1-\varrho_{\mathrm{nf}}^2)\xi - \frac{\zeta}{\varrho_{\mathrm{nf}}}. \tag{84}$$

  Thus, $\Xi^2 - \tau^2$ is given by $-\xi^2 + 2\xi\zeta$ with respect to this coordinate system.

From these formulas, we see that $\Xi^2 - \tau^2$ is an element of $S^{2,0}_{\mathrm{de,sc}}$.

Comparing with those in Proposition 2.2, it can be concluded that $\Xi^2 - \tau^2$ is a representative of the de,sc-principal symbol of the d'Alembertian. □

Define ${}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$ to be the radial compactification of ${}^{\mathrm{de,sc}}T^*\mathbb{O}$. Going forwards, let

$${}^{\mathrm{de,sc}}\pi : {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O} \to \mathbb{O} \tag{85}$$

denote the extension of ${}^{\mathrm{de,sc}}\pi$ to the radial compactified bundle. So, e.g. ${}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf})$ will denote the set of compactified fibers over null infinity (or over one component of null infinity, depending on context). Let $\varrho_{\mathrm{df}} \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}; \mathbb{R}^+)$ denote a bdf for the new face at fiber infinity, which we label df. (We will also consider the bdfs ϱ_{f} of the faces of $\mathbb{O}$ as bdfs of their lifts to the de,sc-cotangent bundle and the radial compactification thereof. That is, we conflate ϱ_{f} and $\varrho_{\mathrm{f}} \circ {}^{\mathrm{de,sc}}\pi$.)

The diffeomorphisms discussed above extend to radial compactifications. They (and their extensions) will be left implicit below.

Given $m \in \mathbb{R}$ and $\mathsf{s} = (s_{\mathrm{Pf}}, s_{\mathrm{nPf}}, s_{\mathrm{Sf}}, s_{\mathrm{nFf}}, s_{\mathrm{Ff}}) \in \mathbb{R}^5$, Let

$$S^{m,\mathsf{s}}_{\mathrm{de,sc}} = S^{m,\mathsf{s}}_{\mathrm{de,sc}}({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}) = \varrho_{\mathrm{df}}^{-m} \varrho^{-\mathsf{s}} S^{0,0}_{\mathrm{de,sc}}, \tag{86}$$

where $S^{0,0}_{\mathrm{de,sc}}$ is the Fréchet space of conormal functions on ${}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$. These are "de,sc-symbols," and, as usual, the space

$$S_{\mathrm{de,sc}} = \bigcup_{m\in\mathbb{R},\mathsf{s}\in\mathbb{R}^5} S^{m,\mathsf{s}}_{\mathrm{de,sc}} \tag{87}$$

has the structure of a multigraded Fréchet algebra.

2.3. **The de,sc-calculus.** Here, we summarize the basic properties of the calculus $\Psi_{\mathrm{de,sc}}$. The details are analogous to those in the construction of the sc-calculus, so we concentrate on the main points. (So, for instance, we will not talk about the topologies of de,sc-pseudodifferential operators, nor about uniform families of operators.) Since the relevant calculi end up being coordinate invariant, and since the de- and sc-calculi are constructed and exposited in [LM01] and [Mel94] respectively, the main order of business is to construct the calculus near the corners of $\mathbb{O}$, which we model (using local coordinates $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{d-1})$ on $\mathbb{S}^{d-1}_{\boldsymbol{\theta}}$) by

$$\mathbb{R}^{d+1}_2 = [0,\infty)_{\varrho_{\mathrm{nf}}} \times [0,\infty)_{\varrho_{\mathrm{Of}}} \times \mathbb{R}^{d-1}_{\boldsymbol{\theta}}, \tag{88}$$

with the face $\{\varrho_{\mathrm{nf}} = 0\}$ of the right-hand side corresponding to null infinity. Here 'Of' stands for "other face," meaning any of $\mathrm{Pf}, \mathrm{Sf}, \mathrm{Ff}$, depending on which corner of $\mathbb{O}$ is under consideration. Thus, we discuss the construction of

$$\Psi_{\mathrm{de,sc,c}}(\mathbb{R}^{d+1}_2) = \bigcup_{m,s,\varsigma \in \mathbb{R}} \Psi^{m,(s,\varsigma)}_{\mathrm{de,sc,c}}(\mathbb{R}^{d+1}_2), \tag{89}$$

where s is the "de-decay order" at $\{\varrho_{\mathrm{nf}} = 0\}$ and ς is the "sc-decay order" at $\{\varrho_{\mathrm{Of}} = 0\}$. The extra 'c' denotes that these operators will have properly supported Schwartz kernels K, so that $K(-,\chi) \in \mathcal{E}'(\mathbb{R}^{d+1}_2)$ whenever $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^{d+1}_2)$. Roughly speaking, this local de,sc-calculus is the result of quantizing

$$\mathcal{V}_{\mathrm{de,sc}}(\mathbb{R}^{d+1}_2) = \operatorname{span}_{C^\infty_{\mathrm{c}}(\mathbb{R}^{d+1}_2)} \Big\{ \varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Of}} \frac{\partial}{\partial \varrho_{\mathrm{nf}}}, \varrho_{\mathrm{nf}} \varrho_{\mathrm{Of}}^2 \frac{\partial}{\partial \varrho_{\mathrm{Of}}}, \varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Of}} V : V \in \mathcal{V}(\mathbb{R}^{d-1}_{\boldsymbol{\theta}}) \Big\}. \tag{90}$$

From this Lie algebra, we get the coball-bundle ${}^{\mathrm{de,sc}}\overline{T}^* \mathbb{R}^{d+1}_2$.

For (conormal) symbols a on ${}^{\mathrm{de,sc}}\overline{T}^* \mathbb{R}^{d+1}_2$ of sufficiently low order, we can define an element $\mathrm{Op}(a)$ of $\Psi_{\mathrm{de,sc}}(\mathbb{R}^{d+1}_2)$ via its Schwartz kernel,

$$K_a \in \mathcal{S}'(\mathbb{R}^{d+1}_2 \times \mathbb{R}^{d+1}_2), \tag{91}$$

given by

$$K_a(x_{\mathrm{L}}, x_{\mathrm{R}}) = \frac{\chi}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \Big[\exp\Big(\pm i \frac{\zeta}{\varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Of}}} (\varrho_{\mathrm{nf}} - \varrho'_{\mathrm{nf}}) \pm i \frac{\xi}{\varrho_{\mathrm{nf}} \varrho_{\mathrm{Of}}^2} (\varrho_{\mathrm{Of}} - \varrho'_{\mathrm{Of}}) \pm \sum_{j=1}^{d-1} \frac{i\eta_j}{\varrho_{\mathrm{nf}}^2 \varrho_{\mathrm{Of}}} (\theta_j - \theta'_j) \Big) \\ \times a(\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Of}}, \boldsymbol{\theta}, \zeta, \xi, \eta) \Big] \,\mathrm{d}\zeta \,\mathrm{d}\xi \,\mathrm{d}^{d-1}\eta, \tag{92}$$

where $\chi \in C^\infty((\mathbb{R}^{d+1}_2)^2_{\mathrm{b}})$ is identically equal to 1 near the diagonal of the b-double space

$$(\mathbb{R}^{d+1}_2)^2_{\mathrm{b}} \cong ([0,\infty)^2_{\mathrm{b}})^2 \times \mathbb{R}^{2d-2} \tag{93}$$

and identically 0 near boundary faces disjoint from the diagonal. Here, $x_{\mathrm{L}} = (\varrho_{\mathrm{nf}}, \varrho_{\mathrm{Of}}, \boldsymbol{\theta})$ and $x_{\mathrm{R}} = (\varrho'_{\mathrm{nf}}, \varrho'_{\mathrm{Of}}, \boldsymbol{\theta}')$. The choice of sign in the exponent in eq. (92) is to be fixed as a convention. Actually, in order to establish the basic properties of the calculus, it is useful to introduce spaces of symbols which depend on both x_{L} and x_{R}, these being quantized in the same manner. In either case, these definitions are extended to symbols of arbitrary order using slightly modified versions of the standard estimates for oscillatory integrals. The initial restriction to a of sufficiently low order (meaning, sufficiently decaying) is to guarantee that the integral above converges, but standard estimates show this restriction to be unnecessary. For each $m, s, \varsigma \in \mathbb{R}$ and (compactly supported)

$$a \in S^{m,s,\varsigma}({}^{\mathrm{de,sc}}\overline{T}^* \mathbb{R}^{d+1}_2), \tag{94}$$

let $\mathrm{Op}(a) = K_a$ as above, and let $\Psi^{m,(s,\varsigma)}_{\mathrm{de,sc,c}}(\mathbb{R}^{d+1}_2)$ denote the set of operators whose Schwartz kernels have the form $K_a + R$ for some properly supported remainder kernel

$$R \in \mathcal{S}(\mathbb{R}^{d+1}_2 \times \mathbb{R}^{d+1}_2). \tag{95}$$

Elements of $\Psi_{\mathrm{de,sc,c}}(\mathbb{R}^{d+1}_2)$ are initially defined as maps $C^\infty_{\mathrm{c}}(\mathbb{R}^{d+1}_2) \to \mathcal{S}'(\mathbb{R}^{d+1}_2)$, but they extend (uniquely) to maps

$$\mathcal{S}'(\mathbb{R}^{d+1}_2) \to \mathcal{S}'(\mathbb{R}^{d+1}_2), \tag{96}$$

and elements of $\Psi_{\mathrm{de,sc,c}}(\mathbb{R}^{d+1}_2)$ can be identified with the corresponding maps. This completes our sketch of the definition of $\Psi_{\mathrm{de,sc,c}}(\mathbb{R}^{d+1}_2)$.

We now return to the discussion of the properties of the calculus on $\mathbb{O}$. The calculus $\Psi_{\mathrm{de,sc}} = \Psi_{\mathrm{de,sc}}(\mathbb{O})$ behaves very similarly to the sc-calculus. This is because we have *principal symbol maps*

$$\sigma^{m,\mathsf{s}}_{\mathrm{de,sc}} : \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}(\mathbb{O}) \to S^{[m],[\mathsf{s}]}_{\mathrm{de,sc}} = S^{m,\mathsf{s}}_{\mathrm{de,sc}}(\mathbb{O})/S^{m-1,\mathsf{s}-1}_{\mathrm{de,sc}}(\mathbb{O}) \tag{97}$$

fitting into a short exact sequence

$$0 \to \Psi^{m-1,\mathsf{s}-1}_{\mathrm{de,sc}} \hookrightarrow \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}} \to S^{[m],[\mathsf{s}]}_{\mathrm{de,sc}} \to 0 \tag{98}$$

of vector spaces. This interacts with Op in the expected way: $\sigma^{m,\mathsf{s}}_{\mathrm{de,sc}}(\mathrm{Op}(a)+R) = a \bmod S^{m-1,\mathsf{s}-1}_{\mathrm{de,sc}}$, whenever R is as above.

The set $\bigcup_{m\in\mathbb{R},\mathsf{s}\in\mathbb{R}^5} \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}$ is a a multi-graded algebra. In particular, this means that

$$AB \in \Psi^{m+m',\mathsf{s}+\mathsf{s}'}_{\mathrm{de,sc}} \tag{99}$$

whenever $A \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}$ and $B \in \Psi^{m',\mathsf{s}'}_{\mathrm{de,sc}}$. The principal symbol map is an algebra homomorphism to leading order, in the sense that

$$\sigma^{m+m',\mathsf{s}+\mathsf{s}'}_{\mathrm{de,sc}}(AB) = \sigma^{m,\mathsf{s}}_{\mathrm{de,sc}}(A)\sigma^{m',\mathsf{s}'}_{\mathrm{de,sc}}(B) \overset{\mathrm{def}}{=} ab \bmod S^{m+m'-1,\mathsf{s}+\mathsf{s}'-1}_{\mathrm{de,sc}}, \tag{100}$$

where a,b are any representatives of $\sigma^{m,\mathsf{s}}_{\mathrm{de,sc}}(A)$ and $\sigma^{m',\mathsf{s}'}_{\mathrm{de,sc}}(B)$, respectively (this equivalence class not depending on the choice of a,b).

We also have a notion of "de,sc–essential support:"

$$\mathrm{WF}'_{\mathrm{de,sc}}(A) \overset{\mathrm{def}}{=} \operatorname{esssupp}(a) \cap \partial\, {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O} \tag{101}$$

whenever $A = \mathrm{Op}(a) + R$ for R as above. (That is, $\mathrm{WF}'_{\mathrm{de,sc}}(A)$ is the closure of the set of points on the boundary of the radially-compactified de,sc-cotangent bundle at which a is not rapidly decaying.) This is well-defined, meaning that if we have $\mathrm{Op}(a) + R = \mathrm{Op}(a') + R'$ for some a', R', then a, a' have the same essential support. We have

$$\mathrm{WF}'_{\mathrm{de,sc}}(AB) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(A) \cap \mathrm{WF}'_{\mathrm{de,sc}}(B) \tag{102}$$

$$\mathrm{WF}'_{\mathrm{de,sc}}(A+B) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(A) \cup \mathrm{WF}'_{\mathrm{de,sc}}(B) \tag{103}$$

for any $A, B \in \Psi_{\mathrm{de,sc}}$. In particular, $\mathrm{WF}'_{\mathrm{de,sc}}([A,B]) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(A) \cap \mathrm{WF}'_{\mathrm{de,sc}}(B)$.

The *de,sc-characteristic set* $\mathrm{Char}^{m,\mathsf{s}}_{\mathrm{de,sc}}(A)$ of $A \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}$ is defined as

$$\mathrm{Char}^{m,\mathsf{s}}_{\mathrm{de,sc}}(A) = \mathrm{char}^{m,\mathsf{s}}_{\mathrm{de,sc}}(a) = \partial\, {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O} \backslash \operatorname{ell}^{m,\mathsf{s}}_{\mathrm{de,sc}}(a), \tag{104}$$

where the elliptic set is defined in the usual way. We mainly care about the case when a is a classical symbol. Considering the case $m = 0$ and $\mathsf{s} = 0$, this means that a is a smooth function on the radially-compactified cotangent bundle, in which case the de,sc-characteristic set of $A = \mathrm{Op}(a)$ is

$$\mathrm{Char}^{0,0}_{\mathrm{de,sc}}(A) = a^{-1}(\{0\}) \cap (\partial\, {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}). \tag{105}$$

An operator $A \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}$ is said to be *elliptic* (with respect to the de,sc-calculus!) if $\mathrm{Char}^{m,\mathsf{s}}_{\mathrm{de,sc}}(A) = \varnothing$. (Note that this depends implicitly on m, s.) It should be emphasized (though this is likely not necessary for the reader familiar with the analogous fact about the sc-calculus) that this is stronger notion than the usual notion of ellipticity in $\mathbb{O}^\circ = \mathbb{R}^{1,d}$, which is equivalent to

$$\mathrm{Char}^{m,\mathsf{s}}_{\mathrm{de,sc}}(A) \cap \overline{T}^*\mathbb{R}^{1,d} = \varnothing. \tag{106}$$

I.e., the ordinary notion of ellipticity in the interior is ellipticity at fiber infinity over the interior. De,sc-ellipticity requires ellipticity in the fibers of the de,sc-cotangent bundle over the boundary of $\mathbb{O}$, as well as at fiber infinity over the boundary.

It follows from the principal symbol short exact sequence and the leading order commutativity of the principal symbol map that

$$[A,B]\in\Psi_{\mathrm{de,sc}}^{m+m'-1,\mathsf{s}+\mathsf{s}'-1}. \tag{107}$$

It can be shown (using e.g. a reduction formula for the full symbol calculus) that

$$\sigma_{\mathrm{de,sc}}^{m+m'-1,\mathsf{s}+\mathsf{s}'-1}([A,B])=\pm i\{a,b\}\bmod S_{\mathrm{de,sc}}^{m+m'-2,\mathsf{s}+\mathsf{s}'-2}, \tag{108}$$

where the sign depends on our sign convention in defining Op, in eq. (92). The right-hand side is just the usual Poisson bracket for $T^*\mathbb{R}^{1,d}$. (It must, of course, be checked that $\{a,b\}$ is actually a de,sc-symbol of the claimed orders.)

It can be shown directly, that, if

$$R\in\Psi_{\mathrm{de,sc}}^{-\infty,-\infty}=\cap_{m,\mathsf{s}}\Psi_{\mathrm{de,sc}}^{m,\mathsf{s}}, \tag{109}$$

then R defines a bounded linear map on $L^2(\mathbb{R}^{1,d})$. (Indeed, $R\in\Psi_{\mathrm{sc}}^{-\infty,-\infty}(\mathbb{R}^{1,d})$.)

Hörmander's square root trick can be used to extend this to $A\in\Psi_{\mathrm{de,sc}}^{0,0}$.

The quantization procedure yields, for each $m\in\mathbb{R},\mathsf{s}\in\mathbb{R}^5$, a plethora of elliptic elements of $\Psi_{\mathrm{de,sc}}^{m,\mathsf{s}}$. Pick one and call it $\Lambda^{m,\mathsf{s}}$. For $m=0$, we can take

$$\Lambda^{0,\mathsf{s}}=\varrho_{\mathrm{Pf}}^{-s_1}\varrho_{\mathrm{nPf}}^{-s_2}\varrho_{\mathrm{Sf}}^{-s_3}\varrho_{\mathrm{nFf}}^{-s_4}\varrho_{\mathrm{Ff}}^{-s_5}=\varrho^{-\mathsf{s}}. \tag{110}$$

We now define $H_{\mathrm{de,sc}}^{0,0}=L^2(\mathbb{R}^{1,d})$ and, for each $\mathsf{s}\in\mathbb{R}^5$, $H_{\mathrm{de,sc}}^{0,\mathsf{s}}=\varrho^{\mathsf{s}}H_{\mathrm{de,sc}}^{0,0}$. We can now define Sobolev spaces $H_{\mathrm{de,sc}}^{m,\mathsf{s}}$ as follows:

- if $m>0$, then we define
$$\begin{aligned}H_{\mathrm{de,sc}}^{m,\mathsf{s}}&=\{u\in H_{\mathrm{de,sc}}^{0,0}:Au\in H_{\mathrm{de,sc}}^{0,0}\text{ for all }A\in\Psi_{\mathrm{de,sc}}^{m,\mathsf{s}}\} &(111)\\ &=\{u\in H_{\mathrm{de,sc}}^{0,0}:\Lambda^{m,\mathsf{s}}u\in L^2(\mathbb{R}^{1,d})\}, &(112)\end{aligned}$$
with norm $\|u\|_{H_{\mathrm{de,sc}}^{m,\mathsf{s}}}=\|u\|_{L^2}+\|\Lambda^{m,\mathsf{s}}u\|_{L^2}$, and
- if $m<0$, then we define
$$\begin{aligned}H_{\mathrm{de,sc}}^{m,\mathsf{s}}&=\{\Lambda^{-m,-\mathsf{s}}u+v:u,v\in L^2(\mathbb{R}^{1,d})\} &(113)\\ &=\{Au:u\in L^2(\mathbb{R}^{1,d}),A\in\Psi_{\mathrm{de,sc}}^{-m,-\mathsf{s}}\}, &(114)\end{aligned}$$
with a corresponding norm.

Each element of $\Psi_{\mathrm{de,sc}}^{m,\mathsf{s}}$ defines a bounded map $H_{\mathrm{de,sc}}^{m',\mathsf{s}'}\to H_{\mathrm{de,sc}}^{m'-m,\mathsf{s}'-\mathsf{s}}$ for any $m\in\mathbb{R}$ and $\mathsf{s}'\in\mathbb{R}^5$.

The failure of a $u\in\mathcal{S}'$ to lie in $H_{\mathrm{de,sc}}^{m,\mathsf{s}}$ is measured by a notion of de,sc-wavefront set,

$$\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u)=\bigcap_{A\in\Psi_{\mathrm{de,sc}}^{0,0}\text{ s.t. }Au\in H_{\mathrm{de,sc}}^{m,\mathsf{s}}}\mathrm{Char}_{\mathrm{de,sc}}^{m,\mathsf{s}}(A). \tag{115}$$

Thus, $u\in H_{\mathrm{de,sc}}^{m,\mathsf{s}}\iff\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u)=\varnothing$. (The $\Rightarrow$ direction is trivial, and the $\Leftarrow$ direction follows via the standard patching argument.) Also, let

$$\mathrm{WF}_{\mathrm{de,sc}}(u)=\mathrm{cl}_{{}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}}\Big[\bigcup_{m,\mathsf{s}}\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u)\Big]. \tag{116}$$

It can be shown that $u\in\mathcal{S}\iff\mathrm{WF}_{\mathrm{de,sc}}(u)=\varnothing$, so de,sc-wavefront set measures microlocal obstructions to being Schwartz on $\mathbb{R}^{1,d}$, as $\mathrm{WF}_{\mathrm{sc}}(u)$ does.

De,sc-ΨDOs are microlocal, which means that

$$\mathrm{WF}_{\mathrm{de,sc}}^{m-m',\mathsf{s}-\mathsf{s}'}(Au)\subseteq\mathrm{WF}'_{\mathrm{de,sc}}(A)\cap\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \tag{117}$$

for any $u \in \mathcal{S}'$ and $A \in \Psi^{m',\mathsf{s}'}_{\mathrm{de,sc}}$. On the other hand, the de,sc-version of microlocal elliptic regularity states that

$$\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \subseteq \mathrm{WF}^{m-m',\mathsf{s}-\mathsf{s}'}_{\mathrm{de,sc}}(Au) \cup \mathrm{Char}^{m',\mathsf{s}'}_{\mathrm{de,sc}}(A). \tag{118}$$

3. Asymptotics, module regularity, and the Poincaré cylinder

We now discuss module regularity at $\mathcal{R}$ (which really means additional regularity *outside* of $\mathcal{R}$) and its relation to asymptotic expansions at timelike infinity. In §3.1, we discuss the Poincaré cylinder, which is the geometrization of the hyperbolic coordinate system. In §3.2, we discuss the test modules relevant to the rest of the paper. In §3.3, we prove the main result of this section, which states that if $u \in \mathcal{S}'$ solves the Klein–Gordon equation $Pu = f$ for $f \in \mathcal{S}'$ with sufficient module regularity, then, if u has sufficient module regularity as well (which at this stage of our analysis is still a hypothesis), then u admits an asymptotic expansion to some specified order on $\mathbb{O}$.

3.1. The Poincaré Cylinder.

By the (punctured) *Poincaré cylinder*, we mean the mwc

$$\overline{\mathbb{R}} \setminus \{0\} \times \mathbb{B}^d = ([-\infty, 0) \cup (0, +\infty])_\tau \times \mathbb{B}^d_{\mathbf{y}}, \tag{119}$$

equipped with the Lorentzian metric $g_{\mathrm{e,sc}} = -\,\mathrm{d}\tau^2 + \tau^{-2} g_{\mathbb{H}}$, where $g_{\mathbb{H}} \in {}^0\mathrm{Sym}^2\, T^*\mathbb{B}$ is the metric

$$g_{\mathbb{H}}(y_1, \dots, y_d) = \frac{4 \sum_{j=1}^d \mathrm{d}y_j^2}{(1 - \sum_{j=1}^d y_j^2)^2} \tag{120}$$

on the unit ball $\mathbb{B}^d_{\mathbf{y}}$, which makes $\mathbb{B}^d$ into the Poincaré ball model of hyperbolic space. It is also useful to refer to the unpunctured cylinder $\overline{\mathbb{R}} \times \mathbb{B}^d$. As the subscript indicates, $g_{\mathrm{e,sc}}$ is an "e,sc-metric" on $\overline{\mathbb{R}} \times \mathbb{B}^d$, where

- the "e" (for "edge") refers to the structure of the metric at $\overline{\mathbb{R}} \times \partial\mathbb{B}^d$, for which $1 - \sum_{j=1}^d y_j^2$ is a bdf, and
- the "sc" refers to the structure at $\partial\overline{\mathbb{R}} \times \mathbb{B}^d$, for which $\langle \tau \rangle^{-1}$ is a bdf.

The set of *e,sc-vector fields* on the unpunctured cylinder is defined by

$$\mathcal{V}_{\mathrm{e,sc}} = \mathrm{span}_{C^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d)}[\{\partial_\tau\} \cup \langle \tau \rangle^{-1} \mathcal{V}_0(\mathbb{B}^d)] = \mathrm{span}_{C^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d)}[\{\partial_\tau\} \cup \{\langle \tau \rangle^{-1}(1 - y^2)\partial_{y_j}\}_{j=1}^d], \tag{121}$$

where $\mathcal{V}_0(\mathbb{B}^d)$ is the set of vector fields on $\mathbb{B}^d$ that vanish at $\partial\mathbb{B}^d$ (considered as vector fields on the Poincaré cylinder that are constant in τ). Likewise for $\mathcal{V}_{\mathrm{e,sc}}((\overline{\mathbb{R}} \setminus \{0\}) \times \mathbb{B}^d)$.

Let $\mathbb{X} \subseteq \mathbb{O}$ denote the relatively open subset $(\mathrm{cl}_{\mathbb{O}}\{(t, \mathbf{x}) \in \mathbb{R}^{1,d} : t^2 \geq r^2\})^\circ \subseteq \mathbb{O}$. This is a non-compact sub-mwc (it includes part of the boundary of $\mathbb{O}$, but it is disjoint from $\mathrm{cl}_{\mathbb{O}}\{t^2 = r^2\}$). Then, $\mathbb{X}^\circ = \{(t, \mathbf{x}) \in \mathbb{R}^{1,d} : t^2 > r^2\}$. Consider the map

$$\iota : \mathbb{X}^\circ_{t,\mathbf{x}} \to \mathbb{R}_\tau \backslash \{0\} \times \mathbb{B}^{d\circ}_{\mathbf{y}} \tag{122}$$

given by

$$\begin{aligned} \tau &= (t^2 - r^2)^{1/2} \operatorname{sign}(t) \in \mathbb{R} \backslash \{0\}, \\ \mathbf{y} &= \mathbf{x}(|t| + (t^2 - r^2)^{1/2})^{-1} \operatorname{sign}(t) \in \mathbb{B}^{d\circ}. \end{aligned} \tag{123}$$

The set-theoretic inverse $\iota^{-1} : \mathbb{R}_\tau \backslash \{0\} \times \mathbb{B}^{d\circ}_{\mathbf{y}} \to \mathbb{X}^\circ_{t,\mathbf{x}}$ of ι is given by

$$\mathbb{R}_\tau \backslash \{0\} \times \mathbb{B}^{d\circ}_{\mathbf{y}} \ni (\tau, \mathbf{y}) \mapsto \Big(\tau \Big(\frac{1 + y^2}{1 - y^2} \Big), \frac{2\tau \mathbf{y}}{1 - y^2} \Big), \tag{124}$$

where $y^2 = \sum_{j=1}^d y_j^2$.

Proposition 3.1. *The map $\iota : \mathbb{X}^\circ_{t,\mathbf{x}} \to \mathbb{R}_\tau \backslash \{0\} \times \mathbb{B}^{d\circ}_{\mathbf{y}}$, is a diffeomorphism, and it extends to a smooth map $\mathbb{X} \to \overline{\mathbb{R}} \times \mathbb{B}^d$.* ■

Proof. It is apparent from the explicit formulas eq. (123), eq. (124) that ι and and its set-theoretic inverse are both smooth, so ι is a diffeomorphism between $\mathbb{X}^\circ$ and $\mathbb{R}\backslash\{0\}\times\mathbb{B}^{d\circ}$. In order to show that it extends to a smooth map $\mathbb{X}\to\overline{\mathbb{R}}\times\mathbb{B}^d$, it suffices to work in local coordinate charts near $\partial\mathbb{O}$:

(I) For $c>0$ and $\sigma\in\{-1,+1\}$, in the set $\{t^2>(1+c)r^2,\sigma t>+1\}$ we can use the coordinates $\rho=1/|t|$ and $\hat{\mathbf{x}}=\mathbf{x}/|t|$, in terms of which

$$\tau^{-1}=\sigma\rho(1-\|\hat{\mathbf{x}}\|^2)^{-1/2}, \tag{125}$$

$$\mathbf{y}=\sigma\hat{\mathbf{x}}(1+(1-\|\hat{\mathbf{x}}\|^2)^{1/2})^{-1}. \tag{126}$$

In $\{t^2>(1+c)r^2,|t|>+1\}$, we have $\rho<1$ and $\|\hat{\mathbf{x}}\|<(1+c)^{-1/2}$, so τ^{-1} and $\mathbf{y}$ are smooth functions of ρ and $\hat{\mathbf{x}}$, all the way down to $\rho=0$.

(II) In the set $\{t^2<(1+c)r^2,\sigma t>+1\}$, we instead work with the coordinates ϱ_{Ff}, ϱ_{nFf}, along with $\theta=\mathbf{x}/r\in\mathbb{S}^{d-1}$. In terms of these,

$$\tau^{-1}=\sigma\varrho_{\mathrm{nf}}\varrho_{\mathrm{nFf}} \tag{127}$$

$$\mathbf{y}=\sigma\theta(1-\varrho_{\mathrm{nf}})(1+\varrho_{\mathrm{nf}})^{-1}. \tag{128}$$

Again, we see that, locally, τ^{-1} and $\mathbf{y}$ are smooth functions of ϱ_{nf} and ϱ_{nFf}, now all the way to $\partial([0,\infty)_{\varrho_{\mathrm{nf}}}\times[0,\infty)_{\varrho_{\mathrm{nFf}}})$.

□

We denote the extension of ι using the same symbol.

As a corollary of the previous proposition,

$$f\in C^\infty(\overline{\mathbb{R}}\setminus\{0\}\times\mathbb{B}^d)\Rightarrow\iota^*f\in C^\infty(\mathbb{X}). \tag{129}$$

For such f, ι^*f is constant on each component of null infinity. Smoothness on $(\overline{\mathbb{R}}\setminus\{0\})\times\mathbb{B}^d$ is therefore stronger than smoothness on $\mathbb{X}$. If $f\in C^\infty(\mathbb{X}^\circ)$, then even though we may not have $f=\iota^*f_0$ for some $f_0\in C^\infty((\overline{\mathbb{R}}\backslash\{0\})\times\mathbb{B}^{d\circ})$ we can still form

$$\iota_*f\in C^\infty((\mathbb{R}\backslash\{0\})\times\mathbb{B}^{d\circ}), \tag{130}$$

since ι is a diffeomorphism in the interior.

We read off of the proof of the previous proposition that

$$\iota^*\langle\tau\rangle^{-1}\in\varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}C^\infty(\mathbb{X};\mathbb{R}^+). \tag{131}$$

Moreover, from the computation $\iota^*(1-y^2)=4\varrho_{\mathrm{nf}}(1+\varrho_{\mathrm{nf}})^{-2}$, we get

$$\iota^*(1-y^2)\in\varrho_{\mathrm{nPf}}\varrho_{\mathrm{nFf}}C^\infty(\mathbb{X};\mathbb{R}^+). \tag{132}$$

Consequently, letting ${}^{\mathrm{e,sc}}\Omega^1(\overline{\mathbb{R}}\times\mathbb{B}^d)$ denote the set of e,sc- one-forms (constructed via dualizing $\mathcal{V}_{\mathrm{e,sc}}$ via $g_{\mathrm{e,sc}}$) on the Poincaré cylinder:

Proposition 3.2. *If* $\omega\in{}^{\mathrm{e,sc}}\Omega^1(\overline{\mathbb{R}}\times\mathbb{B}^d)$*, then* $\iota^*\omega\in{}^{\mathrm{de,sc}}\Omega^1(\mathbb{X})$*.* ■□

Since $\iota:\mathbb{X}^\circ_{t,\mathbf{x}}\to\mathbb{R}_\tau\backslash\{0\}\times\mathbb{B}^{d\circ}_{\mathbf{y}}$ is a diffeomorphism, given any vector field V on $\mathbb{R}_\tau\backslash\{0\}\times\mathbb{B}^{d\circ}_{\mathbf{y}}$ we can pull back V by ι to form a vector field ι^*V on $\mathbb{X}^\circ_{t,\mathbf{x}}$. likewise for differential operators. We record for use below that

$$\iota^*\frac{\partial}{\partial\tau}=\frac{t}{(t^2-r^2)^{1/2}}\partial_t+\frac{1}{(t^2-r^2)^{1/2}}\sum_{j=1}^d x_j\partial_{x_j}, \tag{133}$$

$$\iota^*\frac{\partial}{\partial y_j}=x_j\Big(\frac{t+(t^2-r^2)^{1/2}}{(t^2-r^2)^{1/2}}\Big)\partial_t+(t+(t^2-r^2)^{1/2})\partial_{x_j}+\sum_{k=1}^j\frac{x_jx_k}{(t^2-r^2)^{1/2}}\partial_{x_k}. \tag{134}$$

Consequently, as is well-known [H97], $\Box = \iota^* \Box_{g_{\mathrm{e,sc}}}$, where

$$\Box_{g_{\mathrm{e,sc}}} = \partial_\tau^2 + d\tau^{-1}\partial_\tau + \tau^{-2}\triangle_{\mathbb{H}^d} \in \mathrm{Diff}^{2,0,0}_{\mathrm{e,sc}}(\overline{\mathbb{R}}\backslash\{0\} \times \mathbb{B}^d) \tag{135}$$

on $\overline{\mathbb{R}}\backslash\{0\} \times \mathbb{B}^d$, which is the d'Alembertian of $g_{\mathrm{e,sc}}$.

On the other hand, in $\Omega_{\mathrm{nfTf},\pm,0}$,

$$\iota^* \frac{\partial}{\partial \tau} = -\varrho_{\mathrm{nf}} \varrho_{\mathrm{Tf}}^2 \frac{\partial}{\partial \varrho_{\mathrm{Tf}}} \tag{136}$$

$$\iota^* \frac{\partial}{\partial y} = -\frac{(1+\varrho_{\mathrm{nf}})^2}{2} \frac{\partial}{\partial \varrho_{\mathrm{nf}}} + \frac{(1+\varrho_{\mathrm{nf}})^2}{2} \frac{\varrho_{\mathrm{Tf}}}{\varrho_{\mathrm{nf}}} \frac{\partial}{\partial \varrho_{\mathrm{Tf}}}, \tag{137}$$

where ∂_y is shorthand for $\partial_y = y^{-1}\sum_{j=1}^d y_j \partial_{y_j}$. From these formulas and the observation that angular derivatives in $\overline{\mathbb{R}} \times \mathbb{B}^d$ pull back to angular derivatives on $\mathbb{X}$, we read:

Proposition 3.3. *If $V \in \mathcal{V}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$, then $\iota^* V \in \mathcal{V}_{\mathrm{de,sc}}(\mathbb{X})$, and the elements of $\{\iota^* V : V \in \mathcal{V}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)\}$ generate $\mathcal{V}_{\mathrm{de,sc}}(\mathbb{X})$ as a $C^\infty(\mathbb{X})$-module.* ■□

Let

$$\mathrm{Diff}^{m,s,\varsigma}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d) = \langle \tau \rangle^s (1-y^2)^{-\varsigma} \,\mathrm{Diff}^{m,0,0}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d) \tag{138}$$

denote the set of e,sc-differential operators of order at most m, weighted by $\langle \tau \rangle^s$ and $(1-y^2)^{-\varsigma}$. In addition, let

$$\mathrm{Diff}^{m,(s,s+\varsigma,s+\varsigma,s)}_{\mathrm{de,sc}}(\mathbb{X}) = \mathrm{span}_{C^\infty_{\mathrm{c}}(\mathbb{X})} \mathrm{Diff}^{m,(s,s+\varsigma,\infty,s+\varsigma,s)}_{\mathrm{de,sc}}(\mathbb{O}). \tag{139}$$

Proposition 3.4. *Given any $L \in \mathrm{Diff}^{m,s,\varsigma}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$,*

$$\chi \iota^* L \in \mathrm{Diff}^{m,(s,s+\varsigma,-\infty,s+\varsigma,s)}_{\mathrm{de,sc}}(\mathbb{O}), \tag{140}$$

for any $\chi \in C^\infty_{\mathrm{c}}(\mathbb{X})$. Moreover, the differential operators on $\mathbb{X}$ of this form generate the $C^\infty_{\mathrm{c}}(\mathbb{X})$-module $\mathrm{Diff}^{m,(s,s+\varsigma,s+\varsigma,s)}_{\mathrm{de,sc}}(\mathbb{X})$. ■

Proof. The subset of $\mathrm{Diff}^{\infty,\infty,\infty}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$ consisting of L such that $\chi\iota^* L \in \mathrm{Diff}^{m,(s,s+\varsigma,s+\varsigma,s)}_{\mathrm{de,sc}}(\mathbb{X})$ whenever $L \in \mathrm{Diff}^{m,s,\varsigma}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$ is a subring of $\mathrm{Diff}^{\infty,\infty,\infty}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$. So, in order to prove the first part of the proposition, it suffices to check the case

$$L \in C^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d) \cup \{\langle \tau \rangle^{-s}(1-y^2)^{-\varsigma} : s,\varsigma \in \mathbb{R}\} \cup \{\partial_\tau\} \cup \{\langle \tau \rangle^{-1}(1-y^2)\partial_{y_j}\}_{j=1}^d, \tag{141}$$

as the elements of the set on the right-hand side generate $\mathrm{Diff}^{\infty,\infty,\infty}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$ as a ring. Each of these cases we have already checked, as recorded in eq. (129), eq. (131), eq. (132), and Proposition 3.3. Likewise, the second part of the proposition follows from the second clause of Proposition 3.3. □

For each $m \in \mathbb{N}$ and $s,\varsigma \in \mathbb{R}$, let

$$H^{m,s,\varsigma}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d) = \{u \in L^2(\overline{\mathbb{R}} \times \mathbb{B}^d) : Lu \in L^2(\overline{\mathbb{R}} \times \mathbb{B}^d) \text{ for all } L \in \mathrm{Diff}^{m,s,\varsigma}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)\}. \tag{142}$$

Also, let $H^{m,(s,s+\varsigma,s+\varsigma,s)}_{\mathrm{de,sc,loc}}(\mathbb{X})$ denote the set of distributions which lie in $H^{m,(s,s+\varsigma,-\infty,s+\varsigma,s)}_{\mathrm{de,sc}}(\mathbb{X})$ upon multiplication by an element of $C^\infty_{\mathrm{c}}(\mathbb{X})$.

Proposition 3.5. *For any $m \in \mathbb{N}$ and $s,\varsigma \in \mathbb{R}$,*

$$\iota^* H^{m,s,\varsigma}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d) \subseteq H^{m,(s,s+\varsigma,s+\varsigma,s)}_{\mathrm{de,sc,loc}}(\mathbb{X}). \tag{143}$$

Conversely, if $\chi \in C^\infty_{\mathrm{c}}(\mathbb{X})$, then $u \in H^{m,(s,s+\varsigma,\infty,s+\varsigma,s)}_{\mathrm{de,sc}}(\mathbb{O}) \Rightarrow \iota_* \chi u \in H^{m,s,\varsigma}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$. ■

Proof. We begin with the $m,s,\varsigma = 0$ case. By definition, $H^{0,0,0}_{\mathrm{e,sc}}(\overline{\mathbb{R}}\times\mathbb{B}^d) = L^2_{\mathrm{e,sc}}(\overline{\mathbb{R}}\times\mathbb{B}^d) = L^2(\mathbb{R}\times\mathbb{B}^d, g_{\mathrm{e,sc}})$, so

$$\iota^* H^{0,0,0}_{\mathrm{e,sc}}(\overline{\mathbb{R}}\times\mathbb{B}^d) \subseteq L^2(\mathbb{X}^\circ, \iota^* g_{\mathrm{e,sc}}). \tag{144}$$

By the computations above, $\iota^* g_{\mathrm{e,sc}} \in C^\infty(\mathbb{X};\mathbb{R}^+) g_{\mathrm{de,sc}}$, which implies that

$$L^2(\mathbb{X}^\circ, \iota^* g_{\mathrm{e,sc}}) \subseteq L^2_{\mathrm{loc}}(\mathbb{X}, g_{\mathrm{de,sc}}) = H^{0;0}_{\mathrm{de,sc,loc}}(\mathbb{X}). \tag{145}$$

So, $\iota^* H^{0,0,0}_{\mathrm{e,sc}}(\overline{\mathbb{R}}\times\mathbb{B}^d) \subseteq H^{0;0}_{\mathrm{de,sc,loc}}(\mathbb{X})$. Conversely, suppose that $u \in H^{0;0}_{\mathrm{de,sc,loc}}(\mathbb{X})$. For any $\chi \in C^\infty_{\mathrm{c}}(\mathbb{X})$,

$$\|\chi\iota_* u\|^2_{H^{0,0,0}_{\mathrm{e,sc}}} = \int_{\overline{\mathbb{R}}\times\mathbb{B}^d} |\iota_*\chi u|^2 \,\mathrm{dVol}_{\mathrm{e,sc}} = \int_{\mathbb{X}} |\chi u|^2 J \,\mathrm{dVol}_{\mathrm{de,sc}}, \tag{146}$$

where $J = \mathrm{dVol}_{\mathrm{e,sc}}/\,\mathrm{dVol}_{\mathrm{de,sc}} \in C^\infty(\mathbb{X};\mathbb{R}^+)$. Since $\operatorname{supp}\chi$ is compact,

$$\int_{\mathbb{X}} |\chi u|^2 J \,\mathrm{dVol}_{\mathrm{de,sc}} \leq (\sup\nolimits_{\operatorname{supp}\chi} J) \int_{\mathbb{X}} |\chi u|^2 \,\mathrm{dVol}_{\mathrm{de,sc}} \leq (\sup\nolimits_{\operatorname{supp}\chi} J) \|\chi u\|^2_{H^{0;0}_{\mathrm{de,sc}}(\mathbb{X})} < \infty. \tag{147}$$

Thus, $\iota_*\chi u \in H^{0,0,0}_{\mathrm{e,sc}}(\overline{\mathbb{R}}\times\mathbb{B}^d)$.

The case of general $m \in \mathbb{N}$ and $s,\varsigma \in \mathbb{R}$ follows from the already considered case along with Proposition 3.4. □

3.2. **Test Modules.** There are six test modules $\mathfrak{M}^\varsigma_\sigma, \mathfrak{N}_\sigma \subseteq \Psi^{1,1}_{\mathrm{de,sc}}$ that we use, where $\varsigma,\sigma \in \{-,+\}$. These are defined by

$$\mathfrak{M}^\varsigma_\sigma = \{A \in \Psi^{1,1}_{\mathrm{de,sc}} : \mathrm{char}^{1,1}_{\mathrm{de,sc}}(A) \supseteq \mathcal{R}^\varsigma_\sigma\}, \tag{148}$$

$$\mathfrak{N}_\sigma = \{A \in \Psi^{1,1}_{\mathrm{de,sc}} : \mathrm{char}^{1,1}_{\mathrm{de,sc}}(A) \supseteq \operatorname{span}_{\mathbb{R}} \mathcal{R}^\varsigma_\sigma\} \subseteq \mathfrak{M}^-_\sigma \cap \mathfrak{M}^+_\sigma \tag{149}$$

at the level of sets, and we consider them as $\Psi^{0,0}_{\mathrm{de,sc}}$-bimodules. In eq. (149),

$$\operatorname{span}_{\mathbb{R}} \mathcal{R}^\varsigma_\sigma = \mathbb{R}\iota^* \,\mathrm{d}\tau(\mathrm{Tf}) \tag{150}$$

is the line subbundle of ${}^{\mathrm{de,sc}}T^*_{\mathrm{Tf}}\mathbb{O}$ over the relevant timelike cap containing $\mathcal{R}^\varsigma_\sigma$, where $\mathrm{Tf} = \mathrm{Ff}$ or $\mathrm{Tf} = \mathrm{Pf}$, depending on σ. Here, we are interpreting the de,sc-1-form $\iota^*\,\mathrm{d}\tau$ as a function $\mathbb{O} \to {}^{\mathrm{de,sc}}T^*\mathbb{O}$.

From $\sigma^{1,1}_{\mathrm{de,sc}}([A,B]) \propto \{\sigma^{1,1}_{\mathrm{de,sc}}(A), \sigma^{1,1}_{\mathrm{de,sc}}(B)\}$:

Proposition 3.6. *Each $\mathfrak{M}^\varsigma_\sigma$ and $\mathfrak{N}_\sigma$ is closed under the taking of commutators.* ■□

Recall that the sets $\mathcal{R}^\varsigma_\sigma$ were defined in the introduction. A more concrete definition can be found in the next section. Equation (150) is consistent with these:

Proposition 3.7. *The radial set $\mathcal{R}^\varsigma_\sigma$ is given by $\pm\mathsf{m}\iota^*\mathrm{d}\tau(\mathrm{Tf})$.* ■

Proof. It suffices to check only the $\mathcal{R}^+_+$ case, since the others follow by symmetry. We consider the situation over the interior of Ff. There, if we parametrize ${}^{\mathrm{sc}}T^*\mathbb{M}$ using

$$(t,r,\theta,p,q,\eta) \mapsto p\,\mathrm{d}t + q\,\mathrm{d}r + r\eta\,\mathrm{d}\theta, \tag{151}$$

the radial set $\mathcal{R}^\pm_0$ from the sc-dynamics is given by ${}^{\mathrm{sc}}T^*_{\partial\mathbb{M}}\mathbb{M} \cap \{\eta = 0, p^2 - q^2 = \mathsf{m}^2, \pm p > 0, (p,-q) \parallel (t,r)\}$, except at null infinity, where $\mathcal{R}^\pm_0$ hits fiber infinity. We have, for $\tau > 0$,

$$\mu\,\mathrm{d}\tau + \frac{\Upsilon\tau\,\mathrm{d}y}{1-y^2} = p\,\mathrm{d}t + q\,\mathrm{d}r \tag{152}$$

for $\mu = (p(1+y^2)+2qy)(1-y^2)^{-1} = \tau^{-1}(pt+qr)$ and $\Upsilon = 2(1-y^2)^{-1}(2py+q(1+y^2)) = 2\tau^{-1}(pr+qt)$. Consequently, on $\mathcal{R}^\pm_0$, Υ vanishes, and μ is just $+\mathsf{m}$. We conclude that

$$\mathsf{m}\iota^*\,\mathrm{d}\tau(\mathrm{Tf}^\circ) = \mathcal{R}^+_0 \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Tf}^\circ). \tag{153}$$

Since $\mathrm{d}\tau$ is a smooth e,sc-form on the Poincaré cylinder, its pullback via ι is a smooth de,sc-form, so $\mathsf{m}\iota^*\,\mathrm{d}\tau(\mathrm{Tf})$ is the graph of a continuous function over the closed set Tf. It is therefore a closed subset of ${}^{\mathrm{de,sc}}T^*\mathbb{O}$.

Since $\mathsf{m}\iota^*\,\mathrm{d}\tau(\mathrm{Tf}^\circ)$ is dense in this graph, we deduce that $\mathcal{R}_+^+ = \mathsf{m}\iota^*\mathrm{d}\tau(\mathrm{Tf})$. □

We have $\Psi^{0,0}_{\mathrm{de,sc}} \subset \mathfrak{N}_\sigma$, so $1 \in \mathfrak{N}_\sigma, \mathfrak{M}^\varsigma_\sigma$.

Let $\mathfrak{N}_0, \mathfrak{M}_0$ denote the $C^\infty(\overline{\mathbb{R}}\times\mathbb{B}^d)$-modules of elements of $\mathrm{Diff}^{1,1,0}_{\mathrm{e,sc}}(\overline{\mathbb{R}}\times\mathbb{B}^d)$ given by

$$\mathfrak{N}_0 = \mathrm{span}_{C^\infty(\overline{\mathbb{R}}\times\mathbb{B}^d)}[\{\partial_\tau\}\cup\{(1-y^2)\partial_{y_j}\}_{j=1}^d]. \tag{154}$$

$$\mathfrak{M}_0 = \mathrm{span}_{C^\infty(\overline{\mathbb{R}}\times\mathbb{B}^d)}[\{\tau\partial_\tau\}\cup\{(1-y^2)\partial_{y_j}\}_{j=1}^d]. \tag{155}$$

Fix $\chi \in C^\infty_{\mathrm{c}}(\mathbb{X})$ that is identically equal to 1 near timelike infinity.

From the computations in the previous subsection, we get:

Proposition 3.8. *For each choice of sign $\sigma \in \{-,+\}$, the $\Psi^{0,0}_{\mathrm{de,sc}}$-module $\mathfrak{N}_\sigma$ is generated as both a left and right module by the elements of*

$$\{1\}\cup\{(1-1_{\sigma t>0}\chi)V : V\in \mathrm{Diff}^{1,1}_{\mathrm{de,sc}}\}\cup\{1_{\sigma t>0}\chi\iota^* V_0 : V_0\in\mathfrak{N}_0\}. \tag{156}$$

Consequently, $\mathfrak{N}_\sigma$ admits a finite generating set consisting of differential operators. For each choice of pair of signs $\sigma,\varsigma \in \{-,+\}$, $\mathfrak{M}^\varsigma_\sigma$ is generated (as both a left and right module) by the elements of $\mathfrak{N}_\sigma$ along with

$$V_\pm = \chi\tau(\partial_\tau \mp i\mathsf{m}), \tag{157}$$

where the sign depends on ς. ■□

By eq. (133), $V_\pm$ agrees with the vector field in eq. (10) where $\chi = 1$.

We fix a finite generating set of $\mathfrak{N}_\sigma$ consisting of differential operators. Let

$$\begin{aligned} A_0 &= 1 \\ A_j &= 1_{\sigma t>0}\chi\iota^*(1-y^2)\partial_{y_j} \text{ for } j=1,\ldots,d, \\ A_{1+d} &= 1_{\sigma t>0}\chi\iota^*\partial_\tau. \end{aligned} \tag{158}$$

In addition, taking $N\in\mathbb{N}$ sufficiently large, let $A_{2+d},\ldots,A_N$ be elements of $\mathrm{Diff}^{1,1}_{\mathrm{de,sc}}$ supported away from Tf that together with $A_0,\ldots,A_{1+d}$ generate $\mathfrak{N}_\sigma$. We notationally suppress the σ dependence.

Let P denote an arbitrary de,sc-ΨDO of the form $P = \Box + \lambda + R$ for some $R \in \Psi^{2,-1}_{\mathrm{de,sc}}$ and $\lambda\in\mathbb{C}$. We now show that the module $\mathfrak{N}_\pm$ is "P-critical" at $\mathcal{R}_\pm = \mathcal{R}^+_\pm \cup \mathcal{R}^-_\pm$:

Proposition 3.9. *There exists, for each $\sigma\in\{-,+\}$, a collection $\{C_{j,k}\}_{j,k=1}^N \subset \Psi^{1,0}_{\mathrm{de,sc}}$, depending on R, such that*

- *$i\varrho^{-1}[P,A_j] = \sum_{k=0}^N C_{j,k}A_k$,*
- *if $k\neq 0$, $\sigma^{1,0}_{\mathrm{de,sc}}(C_{j,k}) = 0$ on $\mathcal{R}_\sigma$,*

where $\varrho = \varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}$. ■

Remark. This version of "P-criticality" is a slightly weaker statement than the one offered in [GR+20] in the context of the Schrödinger-Helmholtz equation, as they require $\sigma^{1,0}_{\mathrm{de,sc}}(C_{j,0}) = 0$ on $\mathcal{R}_\pm$, but this is unnecessary for the proof of radial point and propagation estimates involving module regularity in later sections. Proving only this weaker statement allows us to require slightly less of R.

Proof. We only consider the '+' case explicitly.

It suffices to consider the case $R = 0$. Indeed:

- If we have found $C_{j,k}(\Box)$ satisfying the conclusion of the proposition when $P = \Box + \lambda$, then for general $P = \Box + \lambda + R$,

$$i\varrho^{-1}[P, A_j] = i\varrho^{-1}[\Box, A_j] + i\varrho^{-1}[R, A_j] = R_j + \sum_{k=0}^{N} C_{j,k}(\Box)A_k, \tag{159}$$

where $R_j = i\varrho^{-1}[R, A_j] \in \Psi^{2,0}_{\mathrm{de,sc}}$. Choose $G \in \Psi^{-\infty,0}_{\mathrm{de,sc}}$ satisfying

$$\mathrm{WF}'_{\mathrm{de,sc}}(1 - G) \cap \mathcal{R}^+_+ = \varnothing, \tag{160}$$

which we can do because $\mathcal{R}^+_+$ is disjoint from fiber infinity. Let $\Lambda \in \Psi^{1,0}_{\mathrm{de,sc}}$ be elliptic and $\Lambda_{-1} \in \Psi^{-1,0}_{\mathrm{de,sc}}$ be a parametrix for Λ, so that $1 = \Lambda\Lambda_{-1} + E$ for $E \in \Psi^{-\infty,-\infty}_{\mathrm{de,sc}}$. Then, we can rewrite eq. (159) as

$$\begin{aligned} i\varrho^{-1}[P, A_j] &= (1 - G)\Lambda\Lambda_{-1}(1 - G)R_j \\ &\quad + (1 - (1 - G)\Lambda\Lambda_{-1}(1 - G))R_j + \sum_{k=0}^{N} C_{j,k}(\Box)A_k. \end{aligned} \tag{161}$$

Since $\Lambda_{-1}(1 - G)R_j \in \Psi^{1,0}_{\mathrm{de,sc}} \subseteq \mathfrak{N}_+$, we can write

$$\Lambda_{-1}(1 - G)R_j = \sum_{k=0}^{N} D_{j,k}A_k \tag{162}$$

for some $D_{j,k} \in \Psi^{0,0}_{\mathrm{de,sc}}$. Also,

$$\begin{aligned} (1 - (1 - G)\Lambda\Lambda_{-1}(1 - G))R_j &= ((1 - G)E(1 - G) + (1 - G)G + G(1 - G) - G^2)R_j \\ &\in \Psi^{-\infty,0}_{\mathrm{de,sc}}. \end{aligned} \tag{163}$$

By eq. (161) and eq. (162), $i\varrho^{-1}[P, A_j] = \sum_{k=0}^{N} C_{j,k}(P)A_k$ holds for

$$C_{j,k}(P) = \begin{cases} C_{j,0}(\Box) + (1 - G)\Lambda D_{j,0} + (1 - (1 - G)\Lambda\Lambda_{-1}(1 - G))R_j & (k = 0) \\ C_{j,k}(\Box) + (1 - G)\Lambda D_{j,k} & (k \neq 0). \end{cases} \tag{164}$$

Since the essential support of $1-G$ is disjoint from $\mathcal{R}_\sigma$, we have, if $k \neq 0$, $\sigma^{1,0}_{\mathrm{de,sc}}(C_{j,k}(P)) = 0$ on $\mathcal{R}_\sigma$.

Suppose now that $P = \Box + \lambda$. We focus on the situation for $\mathfrak{N}_+$ near $\mathrm{Ff} \cap \mathrm{nFf}$, with the situation in the other regions either similar or strictly easier.

- First consider $A \in \{A_{2+d}, \ldots, A_N\}$. Then, $\varrho^{-1}[P, A] \in \mathrm{Diff}^{2,1}_{\mathrm{de,sc}}$ and is supported away from $\mathcal{R}_+$. We can write this as

$$\Lambda(\Lambda_{-1}\varrho^{-1}[P, A]) + E\varrho^{-1}[P, A]. \tag{165}$$

As $\Lambda_{-1}\varrho^{-1}[P, A] \in \Psi^{1,1}_{\mathrm{de,sc}}$ and is microsupported away from $\mathcal{R}_+$, it lies in $\mathfrak{N}_+$. Thus, we can write

$$\Lambda_{-1}\varrho^{-1}[P, A] = \sum_{k=0}^{N} C^{(1)}_k A_k \tag{166}$$

for some $C^{(1)}_k \in \Psi^{0,0}_{\mathrm{de,sc}}$ which we can choose to be microsupported away from $\mathcal{R}_+$. Thus, $i\varrho^{-1}[P, A] = \sum_{k=0}^{N} C_k A_k$ for $C_k \in \Psi^{1,0}_{\mathrm{de,sc}}$ given by

$$C_k = \begin{cases} i\Lambda C^{(1)}_0 + iE\varrho^{-1}[P, A] & (k = 0), \\ i\Lambda C^{(1)}_k & (k \neq 0), \end{cases} \tag{167}$$

and these are microsupported away from $\mathcal{R}_+$.

- We now check $A_1,\ldots,A_d$. Using eq. (135) and the fact that $\triangle_{\mathbb{H}^d}$ is a 0-operator (and the fact that $A_1,\ldots,A_d$ are all 0-operators on the Poincaré ball as well),
$$\varrho^{-1}[P,A_j]\in\varrho^{-1}\chi_0\iota^*\tau^{-2}\operatorname{Diff}^2_0(\mathbb{B}^d)\subset\varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}^\infty\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}\chi_0\iota^*\operatorname{Diff}^2_0(\mathbb{B}^d) \tag{168}$$
for each $j\in\{1,\ldots,d\}$, where $\chi_0\in C^\infty_{\mathrm{c}}(\mathbb{X})$ is identically 1 on the support of χ. Observe that $\chi_0\iota^*\mathcal{V}_0(\mathbb{B}^d)\subset\mathfrak{N}_\sigma$. Via the same argument using Λ,Λ_{-1} as above, this implies that, for $j\in\{1,\ldots,d\}$, $i\varrho^{-1}[P,A_j]=\sum_{j=0}^N C_{j,k}A_k$ for
$$C_{j,k}\in\varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}^\infty\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}\chi_0\Psi^{1,0}_{\mathrm{de,sc}}\subseteq\Psi^{1,-1}_{\mathrm{de,sc}}, \tag{169}$$
which therefore has principal symbol vanishing on $\mathcal{R}_\sigma$.
- On the other hand,
$$[P,\iota^*\partial_\tau]=\iota^*[\Box_{g_{\mathrm{e,sc}}},\partial_\tau]=\chi\iota^*\Big[\frac{d}{\tau^2}\frac{\partial}{\partial\tau}+\frac{2}{\tau^2}\triangle_{\mathbb{H}^d}\Big], \tag{170}$$
which implies that $\varrho^{-1}[P,A_{1+d}]$ has the desired form.

□

For each $k,\kappa\in\mathbb{N}$,

- let $\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$ denote the $\Psi^{0,0}_{\mathrm{de,sc}}$-bimodule generated by the de,sc-ΨDOs of the form $L_1\cdots L_{k+\kappa}$, where κ of the $L_\bullet$'s are in $\mathfrak{M}_{\varsigma,\sigma}$ and the remainder are in $\mathfrak{N}_\sigma$, and
- let $\mathfrak{N}^k_\sigma$ denote the $\Psi^{0,0}_{\mathrm{de,sc}}$-bimodule generated by the k-fold products of members of $\mathfrak{N}_\sigma$.

Products of the form $B_1\cdots B_k$ for $B_\bullet\in\{A_0,\ldots,A_N\}$ generate $\mathfrak{N}^k_\sigma$. Similarly, products of the form $B_1\cdots B_{\kappa+k}$ for $B_\bullet\in\{A_0,\ldots,A_N,V_\varsigma\}$ with at most κ of the $B_\bullet$ being V_ς generate $\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$. Let
$$\mathfrak{N}^k=\mathfrak{N}^k_-\cap\mathfrak{N}^k_+. \tag{171}$$
Conventionally, $\mathfrak{N}^0=\Psi^{0,0}_{\mathrm{de,sc}}$.

From the formula for $\Box$ in hyperbolic coordinates:

Proposition 3.10.
$$\Box+\mathsf{m}^2=(\iota^*\tau^{-2})[V_\pm V_\mp+(d-1)V_\pm]+\varrho^2R_\pm \tag{172}$$
for some $R_\pm\in\mathfrak{N}^2$. ■□

Proof. Let $\chi\in C^\infty_{\mathrm{c}}(\mathbb{X})$ be identically 1 near timelike infinity, and decompose
$$\Box+\mathsf{m}^2=\chi(\Box+\mathsf{m}^2)+(1-\chi)(\Box+\mathsf{m}^2). \tag{173}$$
The second satisfies $(1-\chi)(\Box+\mathsf{m}^2)\in\varrho^2\mathfrak{N}^2$. On the other hand, $\chi(\Box+\mathsf{m}^2)=\chi\iota^*(\Box_{g_{\mathrm{e,sc}}}+\mathsf{m}^2)$, and the right-hand side is computed as
$$\chi\iota^*(\Box_{g_{\mathrm{e,sc}}}+\mathsf{m}^2)(\iota^*\tau^{-2})\chi\Big[V_+V_-+(d-1)V_++\iota^*\triangle_{\mathbb{H}^d}\pm i\mathsf{m}(d-2)\Big]. \tag{174}$$
Using the computations in the previous subsection, $(\iota^*\tau^{-2})\chi(\iota^*\triangle_{\mathbb{H}^d}\pm i\mathsf{m}(d-2))\in\varrho^2\mathfrak{N}^2$ as well. We conclude that eq. (172) holds with $R_\pm=\varrho^{-2}((1-\chi)(\Box+\mathsf{m}^2)+(\iota^*\tau^{-2})\chi(\iota^*\triangle_{\mathbb{H}^d}\pm i\mathsf{m}(d-2)))$. □

Proposition 3.11. *Fix* $\varsigma,\sigma\in\{-,+\}$. *If* $A\in\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$ *and* $B\in\mathfrak{M}^{\varkappa,j}_{\varsigma,\sigma}$, *then*
$$[A,B]\in 1_{k+j>0}\mathfrak{M}^{\kappa+\varkappa,\max\{k+j-1,0\}}_{\varsigma,\sigma}+1_{\kappa+\varkappa>0}\mathfrak{M}^{\max\{\kappa+\varkappa-1,0\},k+j}_{\varsigma,\sigma}+\Psi^{0,0}_{\mathrm{de,sc}}. \tag{175}$$
■

Proof. We proceed inductively:

- If all of $k,\kappa,j,\varkappa$ are 0, then the result just states the fact that $\Psi^{0,0}_{\mathrm{de,sc}}$ is closed under the taking of commutators.

- Suppose that κ, k are both 0 and $\varkappa + j = 1$ (and the case where $\varkappa, j$ are both 0 and $\kappa + k = 1$ is similar). In this case, the result states that $[A, B] \in \Psi^{0,0}_{\mathrm{de,sc}}$ for all $A \in \Psi^{0,0}_{\mathrm{de,sc}}$ and $B \in \mathfrak{M}^{\varsigma}_{\sigma}$. This just holds because
$$[\Psi^{1,1}_{\mathrm{de,sc}}, \Psi^{0,0}_{\mathrm{de,sc}}] \subseteq \Psi^{0,0}_{\mathrm{de,sc}}. \tag{176}$$
- Suppose that $\kappa + k = 1$ and $\varkappa + j = 1$. There are three essentially different cases to consider.
  - If $\kappa, \varkappa = 1$, then the result states that $[A, B] \in \mathfrak{M}^{\varsigma}_{\sigma}$ for all $A, B \in \mathfrak{M}^{\varsigma}_{\sigma}$, which is part of Proposition 3.6.
  - Likewise, if $k, j = 1$, then the result states that $[A, B] \in \mathfrak{N}_{\sigma}$ for all $A, B \in \mathfrak{N}_{\sigma}$, which is also part of Proposition 3.6.
  - If $\kappa, j = 1$ (with the case $\varkappa, k = 1$ being similar), then the result states that
$$[A, B] \in \mathfrak{M}^{1,0}_{\varsigma,\sigma} + \mathfrak{M}^{0,1}_{\varsigma,\sigma} = \mathfrak{M}^{1,0}_{\varsigma,\sigma} \tag{177}$$
for all $A \in \mathfrak{M}^{\varsigma}_{\sigma}$ and $B \in \mathfrak{N}_{\sigma}$. This is a weaker statement than the result in the $\kappa, \varkappa = 1$ case.
- We now handle the case when $\kappa + k \geq 2$ or $\varkappa + j \geq 2$, proceeding inductively on $\kappa + \varkappa + k + j$. We discuss the case $\kappa + k \geq 2$, and the (overlapping) case $\varkappa + j \geq 2$ is similar.

  Assuming that $\kappa + k \geq 2$, if $A \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$, we can write
$$A = 1_{\kappa>0} A_0 A' + 1_{k>0} A_1 A'' \tag{178}$$
for $A_0 \in \mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma}$, $A' \in \mathfrak{M}^{\varsigma}_{\sigma}$, $A_1 \in \mathfrak{M}^{\kappa,\max\{k-1,0\}}_{\varsigma,\sigma}$, and $A'' \in \mathfrak{N}_{\sigma}$. Then,
$$[A, B] = 1_{\kappa>0}(A_0[A', B] + [A_0, B]A') + 1_{k>0}(A_1[A'', B] + [A_1, B]A''). \tag{179}$$
These satisfy
$$\begin{aligned}
1_{\kappa>0} A_0[A', B] &\in 1_{\kappa>0}\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma}(1_{j>0}\mathfrak{M}^{1+\varkappa,\max\{j-1,0\}}_{\varsigma,\sigma} + \mathfrak{M}^{\varkappa,j}_{\varsigma,\sigma}) \\
&\subseteq 1_{k+j>0}\mathfrak{M}^{\kappa+\varkappa,\max\{k+j-1,0\}}_{\varsigma,\sigma} + 1_{\kappa+\varkappa>0}\mathfrak{M}^{\max\{\kappa+\varkappa-1,0\},k+j}, \\
1_{\kappa>0}[A_0, B]A' &\in 1_{\kappa>0}(1_{k+j>0}\mathfrak{M}^{\max\{\kappa-1,0\}+\varkappa,\max\{k+j-1,0\}}_{\varsigma,\sigma} + 1_{\kappa+\varkappa>1}\mathfrak{M}^{\max\{\kappa+\varkappa-2,0\},k+j}_{\varsigma,\sigma})\mathfrak{M}^{\varsigma}_{\sigma} \\
&\subseteq 1_{k+j>0}\mathfrak{M}^{\kappa+\varkappa,\max\{k+j-1,0\}} + 1_{\kappa+\varkappa>0}\mathfrak{M}^{\max\{\kappa+\varkappa-1,0\},k+j}_{\varsigma,\sigma}, \\
1_{k>0} A_1[A'', B] &\in 1_{k>0}\mathfrak{M}^{\kappa,\max\{k-1,0\}}_{\varsigma,\sigma}(\mathfrak{M}^{\varkappa,j}_{\varsigma,\sigma} + 1_{\varkappa>0}\mathfrak{M}^{\max\{\varkappa-1,0\},j+1}_{\varsigma,\sigma}) \\
&\subseteq 1_{k+j>0}\mathfrak{M}^{\kappa+\varkappa,\max\{k+j-1,0\}}_{\varsigma,\sigma} + 1_{\kappa+\varkappa>0}\mathfrak{M}^{\max\{\kappa+\varkappa-1,0\},k+j}_{\varsigma,\sigma} \\
1_{k>0}[A_1, B]A'' &\in 1_{k>0}(1_{k+j>1}\mathfrak{M}^{\kappa+\varkappa,\max\{k+j-2,0\}}_{\varsigma,\sigma} + 1_{\kappa+\varkappa>0}\mathfrak{M}^{\max\{\kappa+\varkappa-1,0\},\max\{k+j-1,0\}}_{\varsigma,\sigma})\mathfrak{N}_{\sigma} \\
&\subseteq 1_{k+j>0}\mathfrak{M}^{\kappa+\varkappa,\max\{k+j-1,0\}}_{\varsigma,\sigma} + 1_{\kappa+\varkappa>0}\mathfrak{M}^{\max\{\kappa+\varkappa-1,0\},k+j}_{\varsigma,\sigma}.
\end{aligned} \tag{180}$$
So, we can conclude that eq. (175) holds.

□

Let
$$\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma} = \operatorname{span}_{\mathbb{C}}\{AB : A \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}, B \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}\}, \tag{181}$$
and analogously
$$\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}} = \operatorname{span}_{\mathbb{C}}\{BA : A \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}, B \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}\}. \tag{182}$$

Proposition 3.12.

- *If* $A \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}$ *and* $B \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$, *then* $[A, B] \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}(1_{\kappa>0}\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma} + 1_{k>0}\mathfrak{M}^{\kappa,\max\{k-1,0\}}_{\varsigma,\sigma} + \Psi^{0,0}_{\mathrm{de,sc}})$.
- $\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma} = \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}$.

■

Proof. We prove the result via simultaneous induction on $\kappa, k \in \mathbb{N}$. The case $\kappa + k \leq 1$ is an immediate consequence of the algebraic properties of the de,sc-calculus.

Suppose now that $\kappa, k \in \mathbb{N}$ satisfy $\kappa + k \geq 2$, and suppose that we have proven the result for all pairs κ_0, k_0 such that $\kappa_0 + k_0 < \kappa + k$.

- For $B \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$, we can write, like eq. (178),

$$B = 1_{\kappa>0} B_0 B' + 1_{k>0} B_1 B'' \tag{183}$$

for $B_0 \in \mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma}$, $B' \in \mathfrak{M}^{\varsigma}_{\sigma}$, $B_1 \in \mathfrak{M}^{\kappa,\max\{k-1,0\}}_{\varsigma,\sigma}$, and $B'' \in \mathfrak{N}_\sigma$. For $A \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}$,

$$[A,B] = 1_{\kappa>0}([A,B_0]B' + B_0[A,B']) + 1_{k>0}([A,B_1]B'' + B_0[A,B']). \tag{184}$$

The terms on the right-hand side satisfy

$$\begin{aligned} 1_{\kappa>0}[A,B_0]B' &\in 1_{\kappa>0}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}(1_{\kappa>1}\mathfrak{M}^{\max\{\kappa-2,0\},k}_{\varsigma,\sigma} + 1_{k>0}\mathfrak{M}^{\max\{\kappa-1,0\},\max\{k-1,0\}}_{\varsigma,\sigma})\mathfrak{M}^{\varsigma}_{\sigma} \\ &\subseteq 1_{\kappa>0}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma} + 1_{k>0}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\kappa,\max\{k-1,0\}}_{\varsigma,\sigma}, \\ 1_{\kappa>0}B_0[A,B'] &\in 1_{\kappa>0}\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}} = 1_{\kappa>0}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma}, \\ 1_{k>0}[A,B_1]B'' &\in 1_{k>0}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}(1_{\kappa>0}\mathfrak{M}^{\max\{\kappa-1,0\},\max\{k-1,0\}}_{\varsigma,\sigma} + 1_{k>1}\mathfrak{M}^{\kappa,\max\{k-2,0\}}_{\varsigma,\sigma})\mathfrak{N}_\sigma \\ &\subseteq 1_{\kappa>0}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma} + 1_{k>0}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\kappa,\max\{k-1,0\}}_{\varsigma,\sigma}, \\ 1_{k>0}B_0[A,B'] &\in 1_{k>0}\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}} = 1_{k>0}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma}, \end{aligned} \tag{185}$$

where we used the inductive hypothesis. So,

$$[A,B] \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}(1_{\kappa>0}\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma} + 1_{k>0}\mathfrak{M}^{\kappa,\max\{k-1,0\}}_{\varsigma,\sigma} + \Psi^{0,0}_{\mathrm{de,sc}}). \tag{186}$$

- If $A \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}$ and $B \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$, then, using eq. (186),

$$AB = BA + [A,B] \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}} + \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}(\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma} + \mathfrak{M}^{\kappa,\max\{k-1,0\}}_{\varsigma,\sigma}) = \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}} \tag{187}$$

$$BA = AB - [A,B] \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma} + \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}(\mathfrak{M}^{\max\{\kappa-1,0\},k}_{\varsigma,\sigma} + \mathfrak{M}^{\kappa,\max\{k-1,0\}}_{\varsigma,\sigma}) = \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}, \tag{188}$$

which together show that

$$\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma} = \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}\Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}. \tag{189}$$

□

Consequently:

Corollary 3.13. *If $A \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}$, $B \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$, and $C \in \mathfrak{M}^{\varkappa,j}_{\varsigma,\sigma}$, then*

$$[AB,C] = A[B,C] + [A,C]B \in \Psi^{m,\mathsf{s}}_{\mathrm{de,sc}}(1_{k+j>0}\mathfrak{M}^{\kappa+\varkappa,\max\{k+j-1,0\}}_{\varsigma,\sigma} + 1_{\kappa+\varkappa>0}\mathfrak{M}^{\max\{\kappa+\varkappa-1,0\},k+j}_{\varsigma,\sigma} + \Psi^{0,0}_{\mathrm{de,sc}}). \tag{190}$$

■□

For each $m \in \mathbb{R}$, $\mathsf{s} \in \mathbb{R}^5$, and $\kappa, k \in \mathbb{N}$, let

$$H^{m,\mathsf{s};\kappa,k}_{\mathrm{de,sc};\varsigma,\sigma} = H^{m,\mathsf{s};\kappa,k}_{\mathrm{de,sc};\varsigma,\sigma}(\mathbb{O}) \tag{191}$$

denote the Sobolev space consisting of elements of $H^{m,\mathsf{s}}_{\mathrm{de,sc}}$ which remain in this space under the application of any element of $\mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$.

Correspondingly, for $m, s, \varsigma \in \mathbb{R}$ and $\kappa, k \in \mathbb{N}$, let

$$H^{m,s,\varsigma;\kappa,k}_{\mathrm{e,sc}} = H^{m,s,\varsigma;\kappa,k}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d) \tag{192}$$

be the set of elements $u \in H^{m,s,\varsigma}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$ such that Au is also in this space when A is a product of κ elements of $\mathfrak{M}_0$ and k elements of $\mathfrak{N}_0$.

Proposition 3.14. *Fix a sign $\varepsilon \in \{-,+\}$. If $\chi \in C^\infty_{\mathrm{c}}(\mathbb{X})$ has support in $\mathrm{cl}_{\mathbb{X}}\{\varepsilon t > 0\}$ and $u \in \mathcal{S}'$, then, for any $s, \varsigma, \sigma \in \mathbb{R}$,*

$$\chi u \in H^{m,(s,s+\varsigma,\sigma,s+\varsigma,s);\kappa,k}_{\mathrm{de,sc};\pm,\varepsilon}(\mathbb{O}) \tag{193}$$

if and only if $e^{\mp i\mathsf{m}\tau}\iota_*\chi u \in H^{m,s,\varsigma;\kappa,k}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$.

In particular, if $u \in H^{m,(s,\infty,\infty,\infty,s);\infty,\infty}_{\mathrm{de,sc};\pm,\varepsilon}(\mathbb{O})$, then $e^{\mp i\mathsf{m}\tau}\iota_*\chi u \in H^{m,s,\infty;\infty;\infty}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$. ■

Proof. The $\kappa = k = 0$ case of this follows from Proposition 3.5 and the observation that multiplication by $e^{\mp i\mathsf{m}\tau}$ defines an isomorphism of e,sc-Sobolev spaces. This latter fact follows from the $m, s, \varsigma = 0$ case and the observation that if L is an e,sc-differential operator,

$$[L, e^{\mp i\mathsf{m}\tau}] \in \mathrm{Diff}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d) \tag{194}$$

is an e,sc-differential operator of uniformly lower order.

To deduce the case where $\kappa > 0$ or $k > 0$, it suffices to note that the elements of $\mathfrak{N}_\varepsilon$ push forward to elements of $\mathfrak{N}_0$ via ι, and elements of $\mathfrak{M}_{-,\varepsilon}, \mathfrak{M}_{+,\varepsilon}$, when conjugated by $\exp(\mp i\mathsf{m}\tau)$, push forward to elements of $\mathfrak{M}_0$. For example, if

$$\chi u \in H^{m,(s,s+\varsigma,\sigma,s+\varsigma,s);1,0}_{\mathrm{de,sc};\pm,\varepsilon}(\mathbb{O}), \tag{195}$$

then, considering a generic element

$$L = a\tau\partial_\tau + \sum_{j=1}^d a_j(1-y^2)\partial_{y_j} \tag{196}$$

of $\mathfrak{M}_0$, where $a, a_j \in C^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d)$,

$$\begin{aligned}(a_0 + L)e^{\mp i\mathsf{m}\tau}\iota_*\chi u &= \iota_*\Big[\Big(\iota^*a_0 + (\iota^*a)\iota^*\Big(\tau\frac{\partial}{\partial\tau}\Big) + \sum_{j=1}^d(\iota^*a_j)\iota^*\Big((1-y^2)\frac{\partial}{\partial y_j}\Big)\Big)e^{\mp i\mathsf{m}\iota^*\tau}\chi u\Big] \\ &= e^{\mp i\mathsf{m}\tau}\iota_*\Big[\Big(\iota^*a_0 + (\iota^*a)V_\pm + \sum_{j=1}^d(\iota^*a_j)\iota^*\Big((1-y^2)\frac{\partial}{\partial y_j}\Big)\Big)\chi u\Big],\end{aligned} \tag{197}$$

so that

$$(a_0 + L)e^{\mp i\mathsf{m}\tau}\iota_*\chi u \in e^{\mp i\mathsf{m}\tau}\iota_* H^{m,(s,s+\varsigma,\sigma,s+\varsigma,s)}_{\mathrm{de,sc};\pm,\varepsilon} \subseteq e^{\mp i\mathsf{m}\tau}H^{m,s,\varsigma}_{\mathrm{e,sc}} \subseteq H^{m,s,\varsigma}_{\mathrm{e,sc}} \tag{198}$$

for any $a_0 \in C^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d)$, so we conclude that $e^{\mp i\mathsf{m}\tau}\iota_*\chi u \in H^{m,s,\varsigma;1,0}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$. Conversely, if

$$e^{\mp i\mathsf{m}\tau}\iota_*\chi u \in H^{m,s,\varsigma;1,0}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d), \tag{199}$$

then, for any $a_0, a, a_1, \ldots, a_d \in C^\infty(\mathbb{X})$,

$$\begin{aligned}\Big[a_0 + aV_\pm + \sum_{j=1}^d a_jA_j\Big]\chi u &= \Big[a_0\iota^* + a\iota^* \circ \Big(\tau\frac{\partial}{\partial\tau} \mp i\tau\mathsf{m}\Big) + \sum_{j=1}^d a_j\iota^* \circ (1-y^2)\partial_{y_j}\Big]\iota_*\chi u\Big] \\ &= e^{\pm i\mathsf{m}\iota^*\tau}\Big[a_0\iota^* + a\iota^* \circ \Big(\tau\frac{\partial}{\partial\tau}\Big) + \sum_{j=1}^d a_j\iota^* \circ (1-y^2)\partial_{y_j}\Big]e^{\mp i\mathsf{m}\tau}\iota_*\chi u\Big],\end{aligned} \tag{200}$$

so that

$$\Big[a_0 + aV_\pm + \sum_{j=1}^d a_jA_j\Big]\chi u \in e^{\pm i\mathsf{m}\iota^*\tau}C^\infty(\mathbb{X})\iota^*H^{m,s,\varsigma}_{\mathrm{e,sc}} \subseteq H^{m,(s,s+\varsigma,\sigma,s+\varsigma,s)}_{\mathrm{de,sc}}. \tag{201}$$

So, we conclude that $\chi u \in H^{m,(s,s+\varsigma,\sigma,s+\varsigma,s);1,0}_{\mathrm{de,sc};\pm,\varepsilon}(\mathbb{O})$. The case for general κ, k is analogous. □

3.3. **Asymptotics from Module Regularity.** We apply standard notation to refer to spaces of extendable distributions on $\overline{\mathbb{R}}\times\mathbb{B}^d$ with partial polyhomogeneous expansions. So,

$$\mathcal{A}^{0,\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)=\{u\in L^\infty : Lu\in(1-y^2)^\varsigma L^\infty(\overline{\mathbb{R}}\times\mathbb{B}^d)\;\forall\, L\in\operatorname{Diff}_{\mathrm{b}}(\overline{\mathbb{R}}\times\mathbb{B}^d),\varsigma\in\mathbb{R}\} \tag{202}$$

$$\mathcal{A}^{s,\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)=\langle\tau\rangle^{-s}\mathcal{A}^{0,\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d), \tag{203}$$

with similar notations employed when u has partial polyhomogeneous expansions at the boundary faces of the Poincaré cylinder. In general, if $\mathcal{E}\subseteq\mathbb{C}\times\mathbb{N}$ is an index set, we write

$$\mathcal{A}^{(\mathcal{E},\alpha),\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d) \tag{204}$$

to denote the Fréchet space of distributions which have partial polyhomogeneous expansions with index set $\mathcal{E}$ at $(\partial\overline{\mathbb{R}})\times\mathbb{B}^d$ (with a conormal remainder of order α), the terms of which are Schwartz at $\overline{\mathbb{R}}\times\partial\mathbb{B}^d$. We can also work with LCTVSs of functions lying locally in one of these spaces, in specified open sets. For example, it is often convenient to exclude $\{\tau=0\}$, so we write

$$\mathcal{A}^{(\mathcal{E},\alpha),\infty}_{\mathrm{loc}}((\overline{\mathbb{R}}\backslash\{0\})\times\mathbb{B}^d) \tag{205}$$

to denote the set of functions $u:(\overline{\mathbb{R}}\backslash\{0\})\times\mathbb{B}^d\to\mathbb{C}$ such that $\chi f\in\mathcal{A}^{(\mathcal{E},\alpha),\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)$ for any $\chi\in C^\infty_{\mathrm{c}}(\overline{\mathbb{R}}\backslash\{0\})\times\mathbb{B}^d)$.

Lemma 3.15. *If $f\in C^\infty_{\mathrm{c}}(\mathbb{X})$ and $u\in\mathcal{A}^{((0,0),\alpha),\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)$ for some $\alpha>0$, then $(\iota_*f)u\in\mathcal{A}^{((0,0),\alpha),\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)$.* ■

Proof. We write $u=\langle\tau\rangle^{-\alpha}r+\sum_{n\in\mathbb{N},n<\alpha}\langle\tau\rangle^{-n}u_n$ for $uf_n\in\mathcal{A}^{(0,0),\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)$ and $r\in\mathcal{A}^{0,\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)$.

Via Leibniz and eq. (136) and eq. (137), if $L\in\operatorname{Diff}_{\mathrm{b}}(\overline{\mathbb{R}}\times\mathbb{B}^d)$, then, for any $\varsigma\in\mathbb{R}$,

$$(1-y^2)^\varsigma L(r\iota_*f)\in L^\infty(\overline{\mathbb{R}}\times\mathbb{B}^d), \tag{206}$$

so $r\iota_*f\in\mathcal{A}^{0,\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)$. Similarly, for any $L\in\operatorname{Diff}_{\mathrm{E}}(\overline{\mathbb{R}}\times\mathbb{B}^d)$,

$$(1-y^2)^\varsigma L(r\iota_*u_n)\in L^\infty(\overline{\mathbb{R}}\times\mathbb{B}^d), \tag{207}$$

so $u_n\iota_*f\in\mathcal{A}^{(0,0),\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)$. From these observations, we conclude that $(\iota_*f)u\in\mathcal{A}^{((0,0),\alpha),\infty}(\overline{\mathbb{R}}\times\mathbb{B}^d)$. □

Let $P_{\mathrm{e,sc}}$ denote a differential operator on $\overline{\mathbb{R}}\backslash\{0\}\times\mathbb{B}^d$ of the form

$$P_{\mathrm{e,sc}}=\partial_\tau^2+d\tau^{-1}\partial_\tau+\tau^{-2}(\iota_*\Upsilon)R+\mathsf{m}^2 \tag{208}$$

for $R\in\mathfrak{N}_0^2$ and $\Upsilon\in C^\infty_{\mathrm{c}}(\mathbb{X})$, which the Klein–Gordon-Schrödinger operators studied in the rest of the paper have, up to pre-multiplication of the error term by an element of $C^\infty_{\mathrm{c}}(\mathbb{X})$. The function $\iota_*\Upsilon$ appearing here can be fairly badly behaved from the hyperbolic perspective, but $(\iota_*\Upsilon)\varphi\in\mathcal{S}(\overline{\mathbb{R}}\times\mathbb{B}^d)$ for all $\varphi\in\mathcal{S}(\overline{\mathbb{R}}\times\mathbb{B}^d)$ supported away from $\tau=0$. This (and the analogous statement for the partial Taylor series of Υ at the timelike caps) is all is really needed when all of the functions appearing are Schwartz at null infinity.

Proposition 3.16. *Suppose that $u=e^{\pm i\mathsf{m}\tau}\tau^{-d/2}u_0$ for*

$$u_0\in\mathcal{A}^{((0,0),\alpha),\infty}_{\mathrm{loc}}(\overline{\mathbb{R}}\backslash\{0\}\times\mathbb{B}^d) \tag{209}$$

for $\alpha\in\mathbb{R}$ with $\alpha>-1$, and suppose that $Pu=f$ for f of the form $f=e^{\pm i\mathsf{m}\tau}\tau^{-d/2}f_0$, where

$$f_0\in\mathcal{A}^{((0,0),\alpha+2),\infty}_{\mathrm{loc}}(\overline{\mathbb{R}}\backslash\{0\}\times\mathbb{B}^d). \tag{210}$$

Then $u_0\in\mathcal{A}^{((0,0),\alpha+1),\infty}_{\mathrm{loc}}(\overline{\mathbb{R}}\backslash\{0\}\times\mathbb{B}^d)$. ■

Proof. Let $\tilde{P}_{\mathrm{e,sc}} = \tau^{d/2} e^{\mp i\mathsf{m}\tau} P e^{\pm i\mathsf{m}\tau}\tau^{-d/2}$ denote the result of conjugating P by the multiplication operator $\exp(\pm i\mathsf{m}\tau)\tau^{-d/2}$. Since $\mathfrak{N}_0^2$ is closed under conjugations of this form,

$$\tilde{P}_{\mathrm{e,sc}} = \partial_\tau^2 \pm 2i\mathsf{m}\partial_\tau + \tau^{-2}(\iota_*\Upsilon)\tilde{R} \tag{211}$$

for some differential operator $\tilde{R} \in \mathfrak{N}_0^2$. Since $P_{\mathrm{e,sc}}u = f$, we have $\tilde{P}u_0 = f_0$. Thus,

$$\pm 2i\mathsf{m}\partial_\tau u_0 = f_1, \tag{212}$$

where $f_1 = f_0 - \partial_\tau^2 u_0 - \tau^{-2}(\iota_*\Upsilon)\tilde{R}u_0$. Under the hypotheses above, $f_1 \in \mathcal{A}_{\mathrm{loc}}^{((0,0),\alpha+2),\infty}(\overline{\mathbb{R}}\backslash\{0\} \times \mathbb{B}^d)$, like f_0. Integrating eq. (212),

$$u_0(\tau, \mathbf{y}) = u_0(1, \mathbf{y}) \pm \frac{1}{2i}\int_1^\tau f_1(\tau, \mathbf{y})\,\mathrm{d}\tau \tag{213}$$

for $\tau > 1$. The term $u_0(1, \mathbf{y})$ is Schwartz at $y = 1$. Using that $\alpha > -1$, we deduce $u_0 \in \mathcal{A}_{\mathrm{loc}}^{((0,0),\alpha+1),\infty}(\overline{\mathbb{R}}\backslash\{0\} \times \mathbb{B}^d)$. □

Proposition 3.17. *Let $\epsilon > 0$. Suppose that $u_0 \in \mathcal{A}_{\mathrm{loc}}^{-1+\epsilon,\infty}(\overline{\mathbb{R}}\backslash\{0\} \times \mathbb{B}^d)$ satisfies*

$$P_{\mathrm{e,sc}}(e^{\pm i\mathsf{m}\tau}\tau^{-d/2}u_0) = f \tag{214}$$

for some $f \in \mathcal{A}_{\mathrm{loc}}^{\infty,\infty}(\overline{\mathbb{R}}\backslash\{0\} \times \mathbb{B}^d)$. Then,

$$u_0 \in \mathcal{A}_{\mathrm{loc}}^{(0,0),\infty}(\overline{\mathbb{R}}\backslash\{0\} \times \mathbb{B}^d). \tag{215}$$

■

Proof. Since f is Schwartz, $f_0 = e^{\mp i\mathsf{m}\tau}\tau^{d/2}f$ is in $\mathcal{A}_{\mathrm{loc}}^{\infty,(\mathcal{F},\beta)}(\overline{\mathbb{R}}\backslash\{0\} \times \mathbb{B}^d)$ as well. This result therefore follows, via an inductive argument on α, from the previous, in which we take $\mathcal{E} = \{(n, 0) : n \in \mathbb{N}\}$. Note that this index set satisfies $\mathcal{E}_+ = \mathcal{E}$, so the effect of each step of the induction is just to reduce the order of the conormal error α, starting with $\alpha = -1 + \epsilon$. □

By the Sobolev embedding theorem:

Proposition 3.18. *For any $m \in \mathbb{N}$ and $s, \varsigma \in \mathbb{R}$,*

$$H_{\mathrm{e,sc}}^{m,s,\infty;\infty,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d) \subseteq \mathcal{A}^{s+(d+1)/2,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d) \subseteq H_{\mathrm{e,sc}}^{m,s-,\infty;\infty,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d), \tag{216}$$

where $H_{\mathrm{e,sc}}^{m,s,\infty;\infty,\infty} = \bigcap_{k,\kappa\in\mathbb{N}}\bigcap_{\varsigma\in\mathbb{R}} H_{\mathrm{e,sc}}^{m,s,\varsigma;\kappa,k}$ and $H_{\mathrm{e,sc}}^{m,s-,\infty;\infty,\infty} = \bigcap_{k,\kappa\in\mathbb{N}}\bigcap_{\varsigma_0\in\mathbb{R}}\bigcap_{s_0<s} H_{\mathrm{e,sc}}^{m,s_0,\varsigma_0;\infty,\infty}$. ■

Proof. It suffices to consider the $s = 0$ case.

- Let $u \in H_{\mathrm{e,sc}}^{m,0,\infty;\infty,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d)$. Since

$$L \in \mathrm{Diff}_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d) \subseteq \bigcup_{\varsigma\in\mathbb{R},\ell\in\mathbb{N}} (1 - y^2)^\varsigma \mathfrak{M}_0^\ell, \tag{217}$$

  $Lu \in H_{\mathrm{e,sc}}^{m,0,\infty;\infty,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d) \subseteq (1 - y^2)^\varsigma L_{\mathrm{e,sc}}^2(\overline{\mathbb{R}} \times \mathbb{B}^d)$ for each $L \in \mathrm{Diff}_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$ and $\varsigma \in \mathbb{R}$. Since

$$L_{\mathrm{e,sc}}^2(\overline{\mathbb{R}} \times \mathbb{B}^d) = \langle\tau\rangle^{-(d+1)/2}(1 - y^2)^{(d-1)/2} L_{\mathrm{b}}^2(\overline{\mathbb{R}} \times \mathbb{B}^d), \tag{218}$$

  we deduce that u lies in the L_{b}^2-based conormal space

$$\mathcal{I}^{(d+1)/2,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d) = \{u \in L_{\mathrm{b}}^2(\overline{\mathbb{R}} \times \mathbb{B}^d) : Lu \in \langle\tau\rangle^{-(d+1)/2}(1 - y^2)^\varsigma L_{\mathrm{b}}^2(\overline{\mathbb{R}} \times \mathbb{B}^d) \text{ for all } L \in \mathrm{Diff}_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d) \text{ and } \varsigma \in \mathbb{R}\}. \tag{219}$$

  The Sobolev embedding theorem implies that $\mathcal{I}^{(d+1)/2,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d) \subseteq \mathcal{A}^{(d+1)/2,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d)$.

- Conversely, suppose that $u \in \mathcal{A}^{(d+1)/2,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d)$. Then, because $L^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d) \subseteq \langle \tau \rangle^\epsilon (1 - y^2)^{-\epsilon} L^2_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$ for any $\epsilon > 0$,
$$Lu \in \langle \tau \rangle^{\epsilon-(d-1)/2}(1-y^2)^\varsigma L^2_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d) = \langle \tau \rangle^\epsilon (1-y^2)^{\varsigma-(d-1)/2} L^2_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d) \tag{220}$$
for any $\varsigma \in \mathbb{R}$ and $L \in (1-y^2)^{\varsigma_0} \operatorname{Diff}_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$, for any $\varsigma_0 \in \mathbb{R}$. This holds, in particular, for $L = 1$. Since $\partial_\tau \in (1-y^2)^{-1} \operatorname{Diff}_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$ and $\langle \tau \rangle^{-1}(1-y^2)\partial_{y_j} \in \operatorname{Diff}_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$,
$$\operatorname{Diff}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d) \subseteq \bigcup_{\varsigma \in \mathbb{R}} (1-y^2)^\varsigma \operatorname{Diff}_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d). \tag{221}$$
Combining these observations, $L_0 L u \in \langle \tau \rangle^\epsilon (1-y^2)^\varsigma L^2_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$ for any $L_0 \in \operatorname{Diff}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$ and $L \in \operatorname{Diff}_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$. That is,
$$Lu \in H^{m,-\epsilon,\varsigma}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d), \tag{222}$$
for any $m \in \mathbb{N}$. Since $\mathfrak{M}_0 \subset \operatorname{Diff}_{\mathrm{b}}(\overline{\mathbb{R}} \times \mathbb{B}^d)$, we can apply this for all $L \in \bigcap_{\ell \in \mathbb{N}} \mathfrak{M}_0^\ell$ to conclude that $u \in H^{m,-\epsilon,\varsigma;\infty,\infty}_{\mathrm{e,sc}}$. Taking $\epsilon \to 0^+$ and $\varsigma \to \infty$, we conclude the result.

□

Now let $P = \triangle + \mathsf{m}^2 + R_0$ for some $R \in \operatorname{Diff}^{2,-2}_{\mathrm{de,sc}}(\mathbb{O})$.

As in the introduction, let $\chi \in C^\infty(\mathbb{O})$ denote a function supported in $\mathrm{cl}_{\mathbb{O}}\{t^2 \geq r^2\}^\circ = \mathrm{cl}_{\mathbb{O}}\{t^2 \geq r^2\} \setminus \mathrm{cl}_{\mathbb{O}}\{t^2 = r^2\}$ and identically equal to 1 in some neighborhood of $\mathrm{Pf} \cup \mathrm{Ff}$.

Proposition 3.19. *Let $m \geq 0$ and $s > -3/2$. Suppose that $u \in \mathcal{S}'(\mathbb{R}^{1,d})$ satisfies $Pu = f$ for some $f \in \mathcal{S}(\mathbb{R}^{1,d})$. Then:*

- *If $u \in H^{m,(s,\infty,\infty,\infty,\infty);\infty,\infty}_{\mathrm{de,sc};\pm,-}(\mathbb{O})$, then u has the form $u = w + \chi e^{\pm i \mathsf{m} \tau} \langle \tau \rangle^{-d/2} v$ for some $w \in \mathcal{S}(\mathbb{R}^{1,d})$ and $v \in \varrho^\infty_{\mathrm{nPf}} \varrho^\infty_{\mathrm{nFf}} \varrho^\infty_{\mathrm{Ff}} C^\infty(\mathbb{O})$.*
- *If $u \in H^{m,(\infty,\infty,\infty,\infty,s);\infty,\infty}_{\mathrm{de,sc};\pm,+}(\mathbb{O})$, then u has the form $u = w + \chi e^{\pm i \mathsf{m} \tau} \langle \tau \rangle^{-d/2} v$ for some $w \in \mathcal{S}(\mathbb{R}^{1,d})$ and $v \in \varrho^\infty_{\mathrm{Pf}} \varrho^\infty_{\mathrm{nPf}} \varrho^\infty_{\mathrm{nFf}} C^\infty(\mathbb{O})$.*

■

Proof. First of all, observe that P can be written as the pullback $P = \iota^* P_{\mathrm{e,sc}}$ for an e,sc-operator $P_{\mathrm{e,sc}}$ on the Poincaré cylinder, with $P_{\mathrm{e,sc}}$ satisfying the conditions above. We consider the case when
$$u \in H^{m,(\infty,\infty,\infty,\infty,s_{\mathrm{Tf},+});\infty,\infty}_{\mathrm{de,sc};+,+}(\mathbb{O}), \tag{223}$$
and the otheres are similar. Let $g = \chi f + [P,\chi]u$, so that $P(\chi u) = g$. Since $[P,\chi]$ is supported away from timelike infinity, where the de,sc-wavefront set of u is, $g \in \mathcal{S}(\mathbb{R}^{1,d})$. Pushing forward via the diffeomorphism $\iota : \mathbb{X}^\circ \to \mathbb{R} \backslash \{0\} \times \mathbb{B}^{d\circ}$,
$$P_{\mathrm{e,sc}}(\iota_* \chi u) = (\iota_* P)(\iota_* \chi u) = \iota_*(P(\chi u)) = \iota_* g. \tag{224}$$
By the hypothesis and Proposition 3.14, $\iota_* \chi u = e^{\pm i \mathsf{m} \tau} \tau^{-d/2} u_0$ for
$$u_0 \in H^{m,s-d/2,\infty;\infty,\infty}_{\mathrm{e,sc}}(\overline{\mathbb{R}} \times \mathbb{B}^d). \tag{225}$$
Likewise, $\iota_* g \in \dot{C}^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d)$, i.e. g is Schwartz on the Poincaré cylinder.

By Proposition 3.18, $u_0 \in \mathcal{A}^{s+1/2,\infty}(\overline{\mathbb{R}} \times \mathbb{B}^d)$. Since $s > -3/2$, we can appeal to Proposition 3.17 to deduce that
$$u_0 \in \mathcal{A}^{(0,0),\infty}_{\mathrm{loc}}(\overline{\mathbb{R}} \times \mathbb{B}^d). \tag{226}$$
Letting $v = \iota^* \langle \tau \rangle^{d/2} \tau^{-d/2} u_0$, we have $\chi u = e^{+i \mathsf{m} \tau} \langle \tau \rangle^{-d/2} v$, so setting $w = (1-\chi)u \in \mathcal{S}(\mathbb{R}^{1,d})$, we have $u = w + \chi e^{+i \mathsf{m} \tau} \langle \tau \rangle^{-d/2} v$. □

The map $\iota : \mathbb{X} \to \overline{\mathbb{R}} \times \mathbb{B}^d$ discussed in §3 identifies $\{-\infty\} \times \mathbb{B}^d_{\mathbf{y}}$ with the past timelike cap on $\mathbb{M}$. We use this to state:

Proposition 3.20. *Suppose that $v_\pm$ are Schwartz functions on either the the past or the future timelike cap of $\mathbb{M}$, not necessarily the same cap. Then, there exist some*

$$u_\pm \in \varrho_{\mathrm{nPf}}^\infty \varrho_{\mathrm{Sf}}^\infty \varrho_{\mathrm{nFf}}^\infty C^\infty(\mathbb{O}) \tag{227}$$

such that

- *$u_\pm$ has support disjoint from all of the faces of $\mathbb{O}$ except the cap on which $v_\pm$ is given and the adjacent component of* nf*,*
- *$u_\pm$, when restricted to that cap, is $v_\pm$, and*
- *$P(\chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{\pm i\mathsf{m}\sqrt{t^2-r^2}} u_\pm) \in \mathcal{S}(\mathbb{R}^{1,d})$,*

for each choice of sign. ■

Proof. We consider the case where the timelike cap is the future one, with the past case being analogous, and we consider only the plus case of the theorem, the minus case being analogous. We work on $\overline{\mathbb{R}} \times \mathbb{B}^d$, considering $v_+ \in \mathcal{S}(\mathbb{B}^d_{\mathbf{y}})$. It suffices to construct

$$w_+ \in (1-y^2)^\infty C^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d) \tag{228}$$

supported in $[1,\infty]_\tau \times \mathbb{B}^d$ such that $w_+|_{\{\infty\}\times\mathbb{B}^d} = v_+$ and $P_{\mathrm{e,sc}}(\tau^{-d/2} e^{+i\mathsf{m}\tau} w_+) \in \mathcal{S}(\overline{\mathbb{R}} \times \mathbb{B}^d)$. Indeed, given this, set

$$u_+ = \varrho_{\mathrm{Pf}}^{-d/2} (\varrho_{\mathrm{Ff}} \iota^* \tau)^{-d/2} \iota^* w_+ \in C^\infty(\mathbb{O}) \tag{229}$$

(which is supported away from $\mathrm{Pf} \cup \mathrm{nPf} \cup \mathrm{Sf}$). Then, since $P = \iota^* P_{\mathrm{e,sc}}$,

$$P(\chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{+i\mathsf{m}\sqrt{t^2-r^2}} u_+) = [P,\chi](\iota^* \tau^{-d/2} e^{+i\mathsf{m}\tau} w_+) + \chi \iota^* P_{\mathrm{e,sc}}(\tau^{-d/2} e^{+i\mathsf{m}\tau} w_+) \in \mathcal{S}(\mathbb{R}^{1,d}). \tag{230}$$

The construction of w_+ is a straightforward term-by-term construction using the structure of $P_{\mathrm{e,sc}}$ described in eq. (208). Consider a formal series

$$w_{+,\Sigma}(\tau, \mathbf{y}) = \sum_{k=0}^\infty w_{+,k}(\mathbf{y}) \tau^{-k} \in \mathcal{S}(\mathbb{B}^d_y)[[1/\tau]]. \tag{231}$$

Formally applying $P_{\mathrm{e,sc}}$ to $\tau^{-d/2} e^{+i\mathsf{m}\tau} w_{+,\Sigma}$ yields $\tau^{-d/2} e^{+i\mathsf{m}\tau} \tilde{P}_{\mathrm{e,sc}} w_{+,\Sigma}$, where $\tilde{P}_{\mathrm{e,sc}} = \partial_\tau^2 + 2i\mathsf{m}\partial_\tau + \tau^{-2}(\iota_*\Upsilon)\tilde{R}$ for some $\tilde{R} \in \mathfrak{N}_0^2$, as in eq. (211). In order to make sense of $\tilde{R} w_{+,\Sigma}$, we consider the Taylor expansion

$$(\iota_*\Upsilon)\tilde{R} \sim \sum_{k=0}^\infty \tau^{-k} \Big(c_k \partial_\tau^2 + \sum_{j=1}^d c_{k,j}(1-y^2)\partial_\tau \partial_{y_j} + \sum_{j,\ell=1}^d (1-y^2)^2 c_{k,j,\ell} \partial_{y_j}\partial_{y_k} + d_k \partial_\tau + \sum_{j=1}^d d_{k,j}(1-y^2)\partial_{y_j} + e_k \Big), \tag{232}$$

where $c_k, c_{k,j}, c_{k,j,\ell}, d_k, d_{k,j}, e_k \in C^\infty(\mathbb{B}^{d\circ})$, with a polynomial rate of growth at the boundary. Applying $\tilde{P}_{\mathrm{e,sc}}$ to $w_{+,\Sigma}$, the result is the formal series in $1/\tau$, the kth term of which is a linear combination of the $w_{+,0}, \ldots, w_{+,k-1}$ with coefficients in $C^\infty(\mathbb{B}^{d\circ})$ having polynomial growth at the boundary, and with the coefficient of $w_{+,k-1}$ being $-2i\mathsf{m}(k-1)w_{+,k-1}$. Thus, we can recursively define a sequence

$$\{w_{+,k}\}_{k=0}^\infty \subset \mathcal{S}(\mathbb{B}^d) \tag{233}$$

such that $w_{+,0} = v_+$ and such that $\tilde{P}_{\mathrm{e,sc}} w_{+,\Sigma} = 0$, formally. Via the Borel summation lemma, there exists a $w_+ \in (1-y^2)^\infty C^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d)$ whose Taylor series at $\tau = \infty$ is given by eq. (231). Multiplying by a cutoff, we can assume that w_+ is supported in $[1,\infty]_\tau \times \mathbb{B}^d$. The formal manipulations above make sense at the level of computing the Taylor series of $f = P_{\mathrm{e,sc}}(\tau^{-d/2} e^{+i\mathsf{m}\tau} w_{+,\Sigma})$, which is a priori in $\tau^{-d/2} e^{+i\mathsf{m}\tau}(1-y^2)^\infty C^\infty(\overline{\mathbb{R}} \times \mathbb{B}^d)$. The formal manipulations show that the Taylor series of f at $\tau = \infty$ vanishes, which suffices to conclude that f is actually Schwartz. □

4. Classical dynamics on the de,sc- phase space

We now study the (appropriately scaled) Hamiltonian flow of the d'Alembertian – i.e. the null geodesic flow – of an admissible metric on the de,sc-phase space, near null infinity. Attention is restricted to the characteristic set of the Klein–Gordon operator, which depends on m. As seen above, the symbol

$$p = -\tau^2 + \Xi^2 + \eta^2 + \mathsf{m}^2 \in C^\infty(T^*\mathbb{R}^{1,d}) \tag{234}$$

of the Minkowski d'Alembertian is a classical symbol on the de,sc-cotangent bundle of order zero at each face except df, where it is second order (that is, growing quadratically). Let $p[g]$ denote a representative of the principal symbol of $\Box_g$. If g is admissible, then

$$p[g] = p + \varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}p_1[g] \tag{235}$$

for some $p_1 \in S^{2,\mathbf{0}}_{\mathrm{de,sc}}(\mathbb{O})$. Let

$$\tilde{p}[g] = \varrho_{\mathrm{df}}^2 p[g] \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}). \tag{236}$$

Then, the characteristic set $\Sigma_{\mathsf{m}}[g]$ is given by

$$\Sigma_{\mathsf{m}} = \tilde{p}[g]^{-1}(0) \cap \partial({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}). \tag{237}$$

Over the boundary of $\mathbb{O}$, $\Sigma_{\mathsf{m}}[g]$ does not depend on g. In each fiber over the boundary $\Sigma_{\mathsf{m}}[g]$ consists of a two-sheeted hyperboloid (note that this notion does not depend on the choice of coordinates in the base). By the admissibility criteria, which imply time orientability, $\Sigma_{\mathsf{m}}[g]$ has two connected components,

$$\Sigma_{\mathsf{m},\pm}[g] = \Sigma_{\mathsf{m}}[g] \cap \mathrm{cl}_{{}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}}\{\tau \gtreqless 0\}. \tag{238}$$

Recapping the proof of Proposition 2.3, and adding back in the η dependence:

- in $\Omega_{\mathrm{nfTf},\pm,T}$, where we can use the coordinate system $(\varrho_{\mathrm{nf}},\varrho_{\mathrm{Tf}},\theta,\xi,\zeta,\eta) \mapsto \varrho_{\mathrm{nf}}^{-2}\varrho_{\mathrm{Tf}}^{-1}\xi\,\mathrm{d}\varrho_{\mathrm{nf}} + \varrho_{\mathrm{nf}}^{-1}\varrho_{\mathrm{Tf}}^{-2}\zeta\,\mathrm{d}\varrho_{\mathrm{Tf}} + \varrho_{\mathrm{nf}}^{-2}\varrho_{\mathrm{Tf}}^{-1}\eta\,\mathrm{d}\theta$,
$$p = \xi^2 - 2\xi\zeta + \eta^2 + \mathsf{m}^2, \tag{239}$$
and
- in $\Omega_{\mathrm{nfSf},\pm,R}$, using the coordinates $(\varrho_{\mathrm{nf}},\varrho_{\mathrm{Sf}},\theta,\xi,\zeta,\eta) \mapsto \varrho_{\mathrm{nf}}^{-2}\varrho_{\mathrm{Sf}}^{-1}\xi\,\mathrm{d}\varrho_{\mathrm{nf}} + \varrho_{\mathrm{nf}}^{-1}\varrho_{\mathrm{Sf}}^{-2}\zeta\,\mathrm{d}\varrho_{\mathrm{Sf}} + \varrho_{\mathrm{nf}}^{-2}\varrho_{\mathrm{Sf}}^{-1}\eta\,\mathrm{d}\theta$, p is given by $-\xi^2 + 2\xi\zeta + \eta^2 + \mathsf{m}^2$.

Thus, over $\mathrm{nf} \in \{\mathrm{nPf},\mathrm{nFf}\}$, and letting $\sigma \in \{-1,+1\}$ be defined by $\sigma = +1$ if $\mathrm{nf} = \mathrm{nFf}$ and $\sigma = -1$ if $\mathrm{nf} = \mathrm{nPf}$, the set $\Sigma_{\mathsf{m},\pm}[g]$ is given by

$$\Sigma_{\mathsf{m},\pm}[g] \cap \Omega_{\mathrm{nfTf},\pm,T} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}) = \mathrm{cl}_{{}^{\mathrm{de,sc}}\overline{T}^*_{\mathrm{nf}}\mathbb{O}}\{\zeta = (2\xi)^{-1}(\xi^2+\eta^2+\mathsf{m}^2), \mp\sigma\xi > 0\} \tag{240}$$

with respect to the first coordinate system and

$$\Sigma_{\mathsf{m},\pm}[g] \cap \Omega_{\mathrm{nfSf},\pm,R} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}) = \mathrm{cl}_{{}^{\mathrm{de,sc}}\overline{T}^*_{\mathrm{nf}}\mathbb{O}}\{\zeta = (2\xi)^{-1}(\xi^2-\eta^2-\mathsf{m}^2), \mp\sigma\xi > 0\} \tag{241}$$

with respect to the second. These hyperboloids are depicted in Figure 4 in the $d = 2$ case.

In the Cartesian coordinate system $(t,\mathbf{x},\tau,\boldsymbol{\xi}) \mapsto \tau\,\mathrm{d}t + \sum_{i=1}^d \xi_i\,\mathrm{d}x_i \in T^*\mathbb{R}^{1,d}$, the Hamiltonian vector field is

$$H_p = 2\tau\frac{\partial}{\partial t} - 2\sum_{i=1}^d \xi_i \frac{\partial}{\partial x_i} \tag{242}$$

using our sign convention. With respect to the coordinate system $(\varrho_{\mathrm{nf}},\varrho_{\mathrm{Tf}},\theta,\xi,\zeta,\eta) \mapsto \varrho_{\mathrm{nf}}^{-2}\varrho_{\mathrm{Tf}}^{-1}\xi\,\mathrm{d}\varrho_{\mathrm{nf}} + \varrho_{\mathrm{nf}}^{-1}\varrho_{\mathrm{Tf}}^{-2}\zeta\,\mathrm{d}\varrho_{\mathrm{Tf}} + \varrho_{\mathrm{nf}}^{-2}\varrho_{\mathrm{Tf}}^{-1}\eta\,\mathrm{d}\theta$, the rescaled Hamiltonian flow H_p, defined by eq. (37), is given by

$$2^{-1}\varrho_{\mathrm{df}}^{-1}\mathsf{H}_p = (\zeta-\xi)\varrho_{\mathrm{nf}}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} + \xi\varrho_{\mathrm{Tf}}\frac{\partial}{\partial\varrho_{\mathrm{Tf}}} + (2\eta^2+\xi^2-\xi\zeta)\frac{\partial}{\partial\xi} + (\eta^2+(\xi-\zeta)^2)\frac{\partial}{\partial\zeta} + (2\zeta-\xi)V_{\mathbb{S}^{d-1}}, \tag{243}$$

where $V_{\mathbb{S}^{d-1}}$ is the generator of dilations on $T^*\mathbb{S}^{d-1}$. (I.e. $V_{\mathbb{S}^{d-1}} = \sum_{i=1}^{d-1}\eta_i\partial_{\eta_i}$ with respect to any local coordinate system $\theta_1,\ldots,\theta_{d-1}$ on the sphere at infinity.) On the other hand, with respect to the coordinate system $(\varrho_{\mathrm{nf}},\varrho_{\mathrm{Sf}},\theta,\xi,\zeta,\eta)\mapsto \varrho_{\mathrm{nf}}^{-2}\varrho_{\mathrm{Sf}}^{-1}\xi\mathrm{d}\varrho_{\mathrm{nf}}+\varrho_{\mathrm{nf}}^{-1}\varrho_{\mathrm{Sf}}^{-2}\zeta\mathrm{d}\varrho_{\mathrm{Sf}}+\varrho_{\mathrm{nf}}^{-2}\varrho_{\mathrm{Sf}}^{-1}\eta\,\mathrm{d}\theta$, the rescaled Hamiltonian flow H_p is given by

$$2^{-1}\varrho_{\mathrm{df}}^{-1}\mathsf{H}_p=(\xi-\zeta)\varrho_{\mathrm{nf}}\frac{\partial}{\partial\varrho_{\mathrm{nf}}}-\xi\varrho_{\mathrm{Sf}}\frac{\partial}{\partial\varrho_{\mathrm{Sf}}}+(2\eta^2-\xi^2+\xi\zeta)\frac{\partial}{\partial\xi}+(\eta^2-(\xi-\zeta)^2)\frac{\partial}{\partial\zeta}+(\xi-2\zeta)V_{\mathbb{S}^{d-1}}. \tag{244}$$

Defining $\mathsf{H}_{p[g]}=\varrho_{\mathrm{df}}(\varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}})^{-1}H_{p[g]}$,

$$\mathsf{H}_{p[g]}=\mathsf{H}_p \bmod \varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}\mathcal{V}_{\mathrm{b}}({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}). \tag{245}$$

The radial set $\mathcal{R}_+^-\cup\mathcal{R}_+^+\subseteq{}^{\mathrm{de,sc}}T^*_{\mathrm{Ff}}\mathbb{O}$ defined by eq. (42) is given over $\mathrm{nFf}\cap\mathrm{Ff}$ by $\{\xi=\zeta=\pm\mathsf{m},\eta=0\}$, and likewise $\mathcal{R}_-^-\cup\mathcal{R}_-^+\subseteq{}^{\mathrm{de,sc}}T^*_{\mathrm{Pf}}\mathbb{O}$ is given over $\mathrm{nPf}\cap\mathrm{Pf}$ by $\{\xi=\zeta=\pm\mathsf{m},\eta=0\}$.

Likewise, for $\varsigma\in\{-,+\}$, the radial sets $\mathcal{N}_\varsigma^\pm$ are the subsets of $\Sigma_{\mathsf{m},\varsigma}$ defined by

$$\mathcal{N}_\varsigma^\pm\cap{}^{\mathrm{de,sc}}\overline{T}^*_\alpha\mathbb{O}=\begin{cases}\Sigma_{\mathsf{m},\varsigma}\cap{}^{\mathrm{de,sc}}\mathbb{S}^*_\alpha\mathbb{O}\cap\mathrm{cl}_{{}^{\mathrm{de,sc}}\overline{T}^*_\alpha\mathbb{O}}\{\xi=0\} & (\alpha\in\Omega_{\mathrm{nfTf},\pm,T}),\\ \Sigma_{\mathsf{m},\varsigma}\cap{}^{\mathrm{de,sc}}\mathbb{S}^*_\alpha\mathbb{O}\cap\mathrm{cl}_{{}^{\mathrm{de,sc}}\overline{T}^*_\alpha\mathbb{O}}\{\xi=0\} & (\alpha\in\Omega_{\mathrm{nfSf},\pm,R}),\end{cases} \tag{246}$$

these two definitions agreeing on their overlap. The radial sets $\mathcal{C}_\varsigma^\pm,\mathcal{K}_\varsigma^\pm\subset\Sigma_{\mathsf{m},\varsigma}$ are

$$\mathcal{C}_\varsigma^\pm=(\Sigma_{\mathsf{m},\varsigma}\cap{}^{\mathrm{de,sc}}\mathbb{S}^*\mathbb{O}\cap{}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}\cap\mathrm{Tf})\cap\mathrm{cl}_{{}^{\mathrm{de,sc}}\overline{T}^*_\alpha\mathbb{O}}\{\eta=0\})\backslash\mathcal{N}_\varsigma^\pm, \tag{247}$$

$$\mathcal{K}_\varsigma^\pm=(\Sigma_{\mathsf{m},\varsigma}\cap{}^{\mathrm{de,sc}}\mathbb{S}^*\mathbb{O}\cap{}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}\cap\mathrm{Sf})\cap\mathrm{cl}_{{}^{\mathrm{de,sc}}\overline{T}^*_\alpha\mathbb{O}}\{\eta=0\})\backslash\mathcal{N}_\varsigma^\pm, \tag{248}$$

for $\mathrm{Tf}=\mathrm{Ff}$ and $\mathrm{nf}=\mathrm{nFf}$ if $\varsigma=+$ and $\mathrm{Tf}=\mathrm{Pf}$ and $\mathrm{nf}=\mathrm{nPf}$ if $\varsigma=-$. We can now define the final radial sets $\mathcal{A}_\varsigma^\pm$ to be the components of the remaining vanishing set of H_p, which can be seen to lie at fiber infinity.

Consider fiber infinity over $\mathrm{nFf}\cap\mathrm{Ff}$, using the coordinate system in the half-space $\{\zeta<0\}$ over $\Omega_{\mathrm{nfTf},+,T}$ given by

$$\rho=-\frac{1}{\zeta},\qquad s=\frac{\xi}{\zeta},\qquad \hat{\eta}=-\frac{\eta}{\zeta}. \tag{249}$$

Rewriting $\tilde{p}$ in these coordinates, $\tilde{p}=\rho^{-2}\varrho_{\mathrm{df}}^2(s^2-2s+\hat{\eta}^2+\rho^2\mathsf{m}^2)$, which is $s^2-2s+\hat{\eta}^2+\rho^2\mathsf{m}^2=(s-1)^2+\hat{\eta}^2-1+\rho^2\mathsf{m}^2$ up to a smooth, nonvanishing factor in a neighborhood of the part of the characteristic set under consideration. We therefore have, restricting attention to the $\varsigma=+$ case,

$$\Sigma_{\mathsf{m},+}=\{(s-1)^2+\hat{\eta}^2=1-\rho^2\mathsf{m}^2\} \tag{250}$$

locally, in these coordinates. Rewriting eq. (243) in terms of these coordinates, we get

$$2^{-1}\rho\varrho_{\mathrm{df}}^{-1}\mathsf{H}_p=-(1-s)\varrho_{\mathrm{nf}}\frac{\partial}{\partial\varrho_{\mathrm{nf}}}-s\varrho_{\mathrm{Tf}}\frac{\partial}{\partial\varrho_{\mathrm{Tf}}}+\Big[\hat{\eta}^2+(s-1)^2\Big]\rho\frac{\partial}{\partial\rho}-(2-s)\Big[\hat{\eta}^2+s(s-1)\Big]\frac{\partial}{\partial s}+\Big(\hat{\eta}^2+s^2-s-1\Big)V_{\mathbb{S}^{d-1}}. \tag{251}$$

(In local coordinates for the sphere at spatial infinity, $V_{\mathbb{S}^{d-1}}=\sum_{j=1}^{d-1}\hat{\eta}_j\partial_{\hat{\eta}_j}$.) This only vanishes over the boundary of $\mathbb{O}$. We are examining the situation over null infinity, where $\varrho_{\mathrm{nf}}=0$. If $\varrho_{\mathrm{nf}}=0$, then:

- if $\varrho_{\mathrm{Tf}}\neq 0$, then H_p only vanishes on $\Sigma_{\mathsf{m},+}$ at $\Sigma_{\mathsf{m},+}\cap\{s=0\}=\Sigma_{\mathsf{m},+}\cap\{s=0=\rho,\hat{\eta}\}$, which is just the set $\mathcal{N}_+^+$,
- over the corner $\mathrm{nFf}\cap\mathrm{Ff}$, if $\rho\neq 0$ then H_p can only vanish on $\Sigma_{\mathsf{m},+}$ if $\hat{\eta}=0$ and $s=1$, which corresponds to $\mathcal{R}_+^+$,
- at $\rho=0$, H_p vanishes on $\Sigma_{\mathsf{m},+}$ only if $s=2$ or $s=0$ (the latter as already noted), in which case $\hat{\eta}=0$. The former possibility corresponds to $\mathcal{C}_+^\pm$.

The situation on $\Sigma_{\mathsf{m},-}$, and over past null infinity, is similar. In the case $d=2$, H_p restricted to $\Sigma_{\mathsf{m},+}\cap{}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nTf}\cap\mathrm{Ff})$ is depicted in Figure 5.

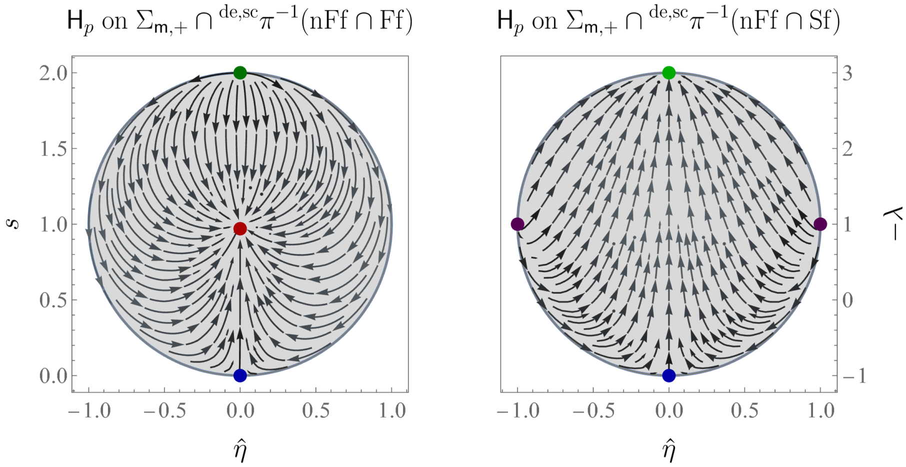

FIGURE 5. The vector field H_p plotted (in the case $d = 2$) on the hyperboloid $\Sigma_{\mathsf{m},+}$ over $\mathrm{nFf} \cap \mathrm{Ff}$ (*left*) and $\mathrm{nFf} \cap \mathrm{Sf}$ (*right*), versus the coordinates $\hat{\eta}$ and s or $\hat{\eta}$ and $-\lambda$. Increasing ρ corresponds to decreasing radii from the plot origin, and the boundary of the disk lies at fiber infinity. In the left plot, we can see the portions over $\mathrm{nFf} \cap \mathrm{Ff}$ of the radial sets $\mathcal{N}^+_+, \mathcal{R}^+_+, \mathcal{C}^+_+$, located at $\hat{\eta} = 0$ and $s = 0, 1, 2$, respectively. In the right plot, we can see the portions of $\mathrm{nFf} \cap \mathrm{Sf}$ of the radial sets $\mathcal{N}^+_+, \mathcal{A}^+_+, \mathcal{K}^+_+$, located at $\hat{\eta} = 0, \pm 1, 0$ and $-\lambda = -1, 1, 3$, respectively.

The coordinate system eq. (249) does not suffice to study the situation over $\Omega_{\mathrm{nfSf},\varsigma,R}$, as each of the sheets $\Sigma_{\mathsf{m},\pm}$ crosses $\{\zeta = 0\}$, as depicted in Figure 4. Instead, consider the coordinate system in the half-space $\{\zeta - \xi > 0\}$ over $\Omega_{\mathrm{nfSf},+,R}$ given by

$$\rho = \frac{1}{\zeta - \xi}, \qquad \lambda = \frac{\zeta + \xi}{\zeta - \xi}, \qquad \hat{\eta} = \frac{\eta}{\zeta - \xi}. \tag{252}$$

In terms of these coordinates, $\tilde{p} = \rho^2 \varrho_{\mathrm{df}}^2(-4^{-1}(1-\lambda)(3+\lambda) + \hat{\eta}^2 + \rho^2 \mathsf{m}^2)$, which is $-4^{-1}(1-\lambda)(3+\lambda) + \hat{\eta}^2 + \rho^2\mathsf{m}^2 = 4^{-1}(\lambda+1)^2 + \hat{\eta}^2 - 1 + \rho^2\mathsf{m}^2$ up to a smooth, nonvanishing factor. Therefore,

$$\Sigma_{\mathsf{m},+} = \{4^{-1}(\lambda+1)^2 + \hat{\eta}^2 = 1 - \rho^2\mathsf{m}^2\} \tag{253}$$

locally. Rewriting eq. (244) in terms of these coordinates,

$$2^{-1}\rho\varrho_{\mathrm{df}}^{-1}\mathsf{H}_p = -\varrho_{\mathrm{nf}}\frac{\partial}{\partial \varrho_{\mathrm{nf}}} + \frac{1}{2}(1-\lambda)\varrho_{\mathrm{Sf}}\frac{\partial}{\partial \varrho_{\mathrm{Sf}}} + \frac{1}{2}\Big[2\hat{\eta}^2 + \lambda + 1\Big]\rho\frac{\partial}{\partial \rho} + \frac{1}{2}(\lambda+3)\Big[2\hat{\eta}^2 + \lambda - 1\Big]\frac{\partial}{\partial \lambda} + (\hat{\eta}^2 - 1)V_{\mathbb{S}^{d-1}}. \tag{254}$$

Then:

- if $\varrho_{\mathrm{nf}} \neq 0$, then H_p is nonvanishing, so we have no radial set over the interior of Sf.
- Over the corner $\mathrm{nFf} \cap \mathrm{Sf}$, if $\rho \neq 0$ then H_p is also nonvanishing (because $\hat{\eta}^2, \lambda + 1 \geq 0$ on $\Sigma_{\mathsf{m},+}$, with equality only at fiber infinity, so the coefficient of the $\rho\partial_\rho$ term in eq. (254) is nonvanishing), so the radial set must be at fiber infinity.
- At fiber infinity, H_p vanishes only if $\lambda \in \{1, -1, -3\}$. If $\lambda = 1, -3$, then $\hat{\eta} = 0$, and, if $\lambda = -1$, then $\|\hat{\eta}\|^2 = 1$. These possibilities correspond to $\mathcal{N}^+_+$, $\mathcal{K}^+_+$, and $\mathcal{A}^+_+$, respectively, where H_p does, in fact, vanish.

The situation on $\Sigma_{\mathsf{m},-}$, and over past null infinity, is similar. The $d = 2$ case is depicted in Figure 5.

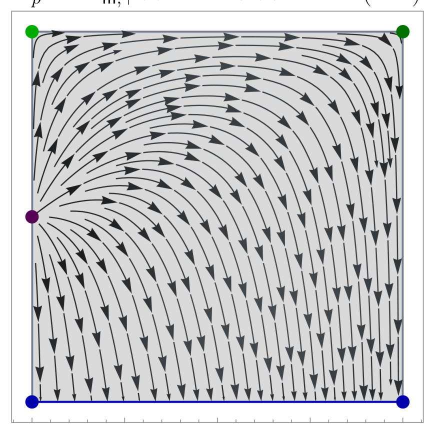

FIGURE 6. The vector field H_p plotted (in the case $d = 2$) at the part of fiber infinity in $\Sigma_{\mathsf{m},+}$ over nFf. Only the $\hat{\eta} > 0$ half is shown. The horizontal axis is parametrized by $\arctan(t-r)$, so the left end $\{\arctan(t-r) = -\pi/2\}$ is over $\mathrm{nFf} \cap \mathrm{Sf}$ and the right end $\{\arctan(t-r) = +\pi/2\}$ is over $\mathrm{nFf} \cap \mathrm{Ff}$, and the vertical axis is parametrized by an appropriate coordinate interpolating between the coordinates s and $-\lambda$ used in Figure 5. The radial sets are colored as in Figure 5: $\mathcal{N}_+^+$ along the bottom, $\mathcal{K}_+^+$ in the top left, $\mathcal{C}_+^+$ in the top right, and $\mathcal{A}_+^+$ in the center left.

Remark. One feature of the dynamics that can be seen from Figure 5 and Figure 6 is the flow from $\mathcal{N}_+^+ \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Sf} \cap \mathrm{nFf})$ to $\mathcal{K}_+^+$ through finite de,sc-frequencies, across nFf along fiber infinity to $\mathcal{C}_+^+$, and then around to $\mathcal{N}_+^+ \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Ff} \cap \mathrm{nFf})$ along fiber infinity (in the $\hat{\eta}$ direction).

Consequently, as forewarned in the introduction, in order to control our solution at $\mathcal{N}_+^+ \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Ff} \cap \mathrm{nFf})$, we need to already have control at $\mathcal{N}_+^+ \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Sf} \cap \mathrm{nFf})$.

We now proceed with a few elementary computations in preparation for the propagation and radial point estimates in the next section. The most basic of these, which captures the fact that the Hamiltonian flow moves us along nf (except at $\mathcal{N}$), is:

Proposition 4.1. *On $\Sigma_{\mathsf{m},\pm} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}^\circ) \backslash \mathcal{N}_\varsigma^\pm$, $\alpha = \varsigma t - r$ satisfies $\pm \mathsf{H}_{p[g]}\alpha > 0$.* ■

Proof. We cover $\Sigma_{\mathsf{m},\pm} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}^\circ)$ by ${}^{\mathrm{de,sc}}\pi^{-1}(\Omega_{\mathrm{nfTf},\varsigma,T}) \cup {}^{\mathrm{de,sc}}\pi^{-1}(\Omega_{\mathrm{nfSf},\varsigma,R})$. In the former, we can write $\alpha = (\varrho_{\mathrm{Tf}} - T)^{-1}$, so, by eq. (243), we have

$$\mathsf{H}_p\alpha = -2\varrho_{\mathrm{df}}\xi \frac{\varrho_{\mathrm{Tf}}}{(\varrho_{\mathrm{Tf}} - T)^2} \tag{255}$$

over nf°, and $\pm\varrho_{\mathrm{df}}\xi < 0$ on $\Sigma_{\mathsf{m},\pm} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\Omega_{\mathrm{nfTf},\varsigma,T}) \backslash \mathcal{N}_\varsigma^\pm$. On the other hand, over $\Omega_{\mathrm{nfSf},\varsigma,R}$, we write $\alpha = -(\varrho_{\mathrm{Sf}} - R)^{-1}$, so by eq. (244) we have

$$\mathsf{H}_p\alpha = -2\xi\varrho_{\mathrm{df}} \frac{\varrho_{\mathrm{Sf}}}{(\varrho_{\mathrm{Sf}} - R)^2} \tag{256}$$

over nf°, so the same holds. So, $\mathsf{H}_p\alpha > 0$ on $\Sigma_{\mathsf{m},\pm} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}^\circ) \backslash \mathcal{N}_\varsigma^\pm$, $\alpha = \varsigma t - r$. The same therefore holds for $\mathsf{H}_{p[g]}$. □

We now discuss the radial sets. To simplify the discussions at the radial sets over the corners of $\mathbb{O}$, we can assume that ϱ_{df} is given by the coordinates labeled ρ in eq. (249) and eq. (252) near the corners of $\mathbb{O}$. It is straightforward to modify the discussion to handle arbitrary choices of bdf at df, and indeed we need to do this anyways to discuss the radial sets $\mathcal{N}$. The arguments are all similar, but this way we do not need to repeat it.

Proposition 4.2. *Fix signs $\varsigma, \sigma \in \{-,+\}$. Letting $\aleph \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ satisfy $\aleph = \varrho_{\mathrm{Sf}}^2 + \rho^2 + (\lambda+1)^2$ near $\mathcal{A}_\sigma^\varsigma$, the symbol*

$$F_1 = F_1[g] = \mathsf{H}_{p[g]}\aleph - 4\varsigma\sigma\aleph \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}) \tag{257}$$

vanishes cubically at $\mathcal{A}^{\varsigma}_{\sigma}$ within $\Sigma_{\mathsf{m}}[g]$, in the sense that

$$F_1 \in \aleph^{3/2}L^\infty + \varrho_{\mathrm{nf}}\aleph L^\infty + \tilde{p}[g]L^\infty \tag{258}$$

locally. ■

Proof. We only consider the case of $\mathcal{A}^{+}_{+}$, the other three being analogous. Before doing so, it is useful to reduce to the case where g is the Minkowski metric. Working in some local coordinate chart $\theta_1,\dots,\theta_{d-1}$ on $\mathbb{S}^{d-1}_{\boldsymbol{\theta}}$, we can write

$$\frac{\mathsf{H}_{p[g]}-\mathsf{H}_p}{\varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}} = V_{\mathrm{nf}}\varrho_{\mathrm{nf}}\frac{\partial}{\partial\varrho_{\mathrm{nf}}} + V_{\mathrm{Sf}}\varrho_{\mathrm{Sf}}\frac{\partial}{\partial\varrho_{\mathrm{Sf}}} + V_\rho\rho\frac{\partial}{\partial\rho} + V_\lambda\frac{\partial}{\partial\lambda} + \sum_{i=1}^{d-1}\Big(V_{\theta_i}\frac{\partial}{\partial\theta_i} + V_{\hat{\eta}_i}\frac{\partial}{\partial\hat{\eta}_i}\Big) \tag{259}$$

for some $V_{\mathrm{nf}}, V_{\varrho_{\mathrm{Sf}}}, V_\rho, V_\lambda, V_{\theta_i}, V_{\hat{\eta}_i} \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$. Applying this to $\aleph$, the result is

$$\frac{\mathsf{H}_{p[g]}-\mathsf{H}_p}{\varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}}\aleph = 2V_{\mathrm{Sf}}\varrho_{\mathrm{Sf}}^2 + 2V_\rho\rho^2 + 2V_\lambda(\lambda+1). \tag{260}$$

The ratios $\varrho_{\mathrm{Sf}}^2/\aleph$, $\rho^2/\aleph$, and $\varrho_{\mathrm{nf}}\varrho_{\mathrm{Sf}}(\lambda+1)/\aleph$ all lie in L^∞ (locally). Thus, we can absorb $(\mathsf{H}_{p[g]}-\mathsf{H}_p)\aleph$ into the $\varrho_{\mathrm{nf}}\aleph L^\infty$ term in eq. (258). Consequently, it suffices to consider only the case where g is the Minkowski metric.

By eq. (254),

$$2^{-1}\mathsf{H}_p\aleph = (1-\lambda)\varrho_{\mathrm{Sf}}^2 + (2\hat{\eta}^2+\lambda+1)\rho^2 + (\lambda+3)(2\hat{\eta}^2+\lambda-1)(\lambda+1) \tag{261}$$

near $\mathcal{A}^{+}_{+}$. We write the right-hand side as $2\aleph + F_{1,0}$ for

$$F_{1,0} = -(1+\lambda)\varrho_{\mathrm{Sf}}^2 + (2\hat{\eta}^2+\lambda-1)(\rho^2+(\lambda+1)^2) + 4(\hat{\eta}^2-1)(\lambda+1), \tag{262}$$

which vanishes cubically at $\mathcal{A}^{+}_{+}$. In order to see this, we write $F_{1,0} = F_{1,1} + \tilde{p}F_{1,2}$ for

$$\begin{aligned} F_{1,1} = -(1+\lambda)\varrho_{\mathrm{Sf}}^2 + 2^{-1}(1-\lambda)(\lambda+1)(\rho^2+(\lambda+1)^2) - 2\rho^2\mathsf{m}^2(\rho^2+(\lambda+1)^2) \\ -(\lambda+1)^3 - 4\rho^2\mathsf{m}^2(\lambda+1) \end{aligned} \tag{263}$$

and $F_{1,2} = 2\rho^2 + 2(\lambda+1)^2 + 4(\lambda+1) \in S^{0,0}_{\mathrm{de,sc}}$. Term-by-term, we see that $F_{1,1} \in \aleph^{3/2}L^\infty$. □

Proposition 4.3. *Fix signs $\varsigma,\sigma \in \{-,+\}$. Setting $a(m,\ell) = \varrho_{\mathrm{df}}^m\varrho_{\mathrm{nf}}^\ell$, the symbol $\alpha = \alpha[g](m,\ell) \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ defined by $\mathsf{H}_{p[g]}a = \alpha a$ satisfies*

(1) $\varsigma\sigma\alpha > 0$ on $\mathcal{N}^{\varsigma}_{\sigma}$ if $m > \ell$, and
(2) $\varsigma\sigma\alpha < 0$ on $\mathcal{N}^{\varsigma}_{\sigma}$ if $m < \ell$.

■

Proof. We check the case of $\mathcal{N}^{+}_{+}$, with the other three being analogous. Before doing so, it is useful to reduce to the simplest case:

- We have $\alpha[g] = \alpha[0] + a^{-1}(\mathsf{H}_{p[g]}-\mathsf{H}_p)a$. Because $\mathsf{H}_{p[g]}-\mathsf{H}_p \in \varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}\mathcal{V}_{\mathrm{b}}({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$,
$$(\mathsf{H}_{p[g]}-\mathsf{H}_p)a = \alpha_1[g]a \tag{264}$$
for some $\alpha_1[g] \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ vanishing over the boundary of $\mathbb{O}$. So, $\alpha[g] = \alpha[0]+\alpha_1[g]$, and this satisfies the conditions in the proposition if and only if $\alpha[0]$ does. It therefore suffices to prove the result in case when g is the Minkowski metric.
- If $\varrho_{\mathrm{df},0}$ is another choice of bdf of df, then
$$\mathsf{H}_{p[g]}\varrho_{\mathrm{df},0}^m\varrho_{\mathrm{nf}}^\ell = \Big(\alpha + \Big(\frac{\varrho_{\mathrm{df}}}{\varrho_{\mathrm{df},0}}\Big)^m\mathsf{H}_{p[g]}\Big(\frac{\varrho_{\mathrm{df},0}}{\varrho_{\mathrm{df}}}\Big)^m\Big)\varrho_{\mathrm{df},0}^m\varrho_{\mathrm{nf}}^\ell, \tag{265}$$
assuming that α satisfies the conclusion of the proposition with the original choice, ϱ_{df}, of bdf of df. As $\mathsf{H}_{p[g]}$ vanishes on the radial sets, the second term in the parentheses vanishes at the radial set in question, so the proposition applies regarding the modified bdf with a

slightly modified α. We can now calculate α over $\Omega_{\mathrm{nfTf},+,T}$ and over $\Omega_{\mathrm{nfSf},+,R}$, using over each whichever choice of ϱ_{df} makes the computation simplest. (And these do not need to be the same choice!)

With these simplifications in mind, we compute:

- Over $\Omega_{\mathrm{nfTf},+,T}$, we use the coordinates eq. (249), and we can take $\varrho_{\mathrm{df}} = \rho$ locally. In this case,
$$\mathsf{H}_p a = 2(m(\hat{\eta}^2 + (s-1)^2) - \ell(1-s))a. \tag{266}$$
Thus, $\alpha[0] = 2m(\hat{\eta}^2 + (s-1)^2) - 2\ell(1-s)$ locally. At $\mathcal{N}^+_+$, $s=0$ and $\hat{\eta}=0$, so $\alpha = 2(m-\ell)$.
- Over $\Omega_{\mathrm{nfSf},+,R}$, we use the coordinates eq. (252), and we can take $\varrho_{\mathrm{df}} = \rho$ locally. In this case,
$$\mathsf{H}_p a = (m(2\hat{\eta}^2 + \lambda + 1) - 2\ell)a, \tag{267}$$
so $\alpha = m(2\hat{\eta}^2 + \lambda + 1) - 2\ell$ locally. At $\mathcal{N}^+_+$, $\lambda = 1$ and $\hat{\eta} = 0$, so $\alpha = 2(m-\ell)$ there as well.

□

Lemma 4.4. *Fix signs $\varsigma, \sigma \in \{-,+\}$.*

- *Given any compact subset $K \subseteq \mathcal{N}^\varsigma_\sigma \cap {}^{\mathrm{de,sc}}\pi^{-1}(\Omega_{\mathrm{nfTf},\sigma,T})$, there exist symbols $s_0, s_1, s_2 \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ such that $s = s_0$ near K, using the coordinates eq. (249), and*
$$s_0 = s_1\tilde{p} + s_2(\hat{\eta}^2 + \mathsf{m}^2\varrho_{\mathrm{df}}^2) \tag{268}$$
globally, with $\varsigma\sigma s_2 > 0$ on $\mathcal{N}^\varsigma_\sigma$.
- *Given any compact subset $K \subseteq \mathcal{N}^\varsigma_\sigma \cap {}^{\mathrm{de,sc}}\pi^{-1}(\Omega_{\mathrm{nfSf},\sigma,R})$, there exist symbols $\lambda_0, \lambda_1, \lambda_2 \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ such that $\lambda = \lambda_0$ near K, using the coordinates eq. (252), and*
$$\lambda_0 = 1 + \lambda_1\tilde{p} + \lambda_2(\hat{\eta}^2 + \mathsf{m}^2\varrho_{\mathrm{df}}^2) \tag{269}$$
globally, with $\varsigma\sigma\lambda_2 > 0$ on $\mathcal{N}^\varsigma_\sigma$.

■

Proof. The proofs of the two parts are similar, so we only write up the first, and we only consider $\mathcal{N}^+_+$, the other three cases being similar. In the coordinates eq. (249), we have
$$s = 1 - (1 + \tilde{p} - \hat{\eta}^2 - \mathsf{m}^2\rho^2)^{1/2} \tag{270}$$
near K, assuming without loss of generality that $\varrho_{\mathrm{df}} = \rho$ locally. It is key that this holds with a single choice of sign (near the other radial set $\mathcal{K}^+_+$, we instead have $s = 1 + (1 + \tilde{p} - \hat{\eta}^2 - \mathsf{m}^2\rho^2)^{1/2}$, and the transition between the two formulas happens away from these two radial sets). We can write
$$(1 - y + z)^{1/2} = (1-y)^{1/2} + zR(y,z) \tag{271}$$
for $R(y,z)$ smooth near $\{y=0, z=0\}$. Applying this with $z = \tilde{p}$ and $y = \hat{\eta}^2 + \mathsf{m}^2\rho^2$,
$$\begin{aligned} s &= 1 - (1 - \hat{\eta}^2 - \mathsf{m}^2\rho^2)^{1/2} - \tilde{p}R(\hat{\eta}^2 + \mathsf{m}^2\rho^2, \tilde{p}) \\ &= R(0, -\hat{\eta}^2 - \mathsf{m}^2\rho^2)(\hat{\eta}^2 + \mathsf{m}^2\rho^2) - \tilde{p}R(\hat{\eta}^2 + \mathsf{m}^2\rho^2, \tilde{p}). \end{aligned} \tag{272}$$
Note that the functions $R(\hat{\eta}^2 + \mathsf{m}^2\rho^2, \tilde{p})$ and $R(0, -\hat{\eta}^2 - \mathsf{m}^2\rho^2)(\hat{\eta}^2 + \mathsf{m}^2\rho^2)$ are or can be extended to smooth functions on some neighborhood of K on ${}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$. Thus, we can find s_0, s_1, s_2 such that eq. (268) holds, with
$$s_1 = -R(\hat{\eta}^2 + \mathsf{m}^2\rho^2, \tilde{p}) \tag{273}$$
and $s_2 = R(0, -\hat{\eta}^2 - \mathsf{m}^2\rho^2)$ locally. Away from K, s_0 is not constrained, so all we need to do is extend s_1, s_2 to smooth functions on the whole radially compactified de,sc-cotangent bundle to satisfy eq. (268) globally, taking eq. (268) as a global definition of s_0. Since $R(0,0) = 1/2$, we have $s_2 > 0$ near K, so we can arrange $s_2 > 0$ globally. □

Note that, away from $\mathrm{cl}_{\mathbb{O}}\{r=0\}$, we can define $\hat{\eta}^2=\eta^2\varrho_{\mathrm{df}}^2$, where $\eta^2=g_{\mathbb{S}^{d-1}}^{-1}(\eta,\eta)$ is defined using the standard spherical metric $g_{\mathbb{S}^{d-1}}$. This semi-global definition depends on the choice of bdf of df, so some care is needed.

Proposition 4.5. *Fix $\varsigma,\sigma\in\{-,+\}$. Letting $\beth\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ be defined by $\beth=\varrho_{\mathrm{nf}}^2+\hat{\eta}^2$ near null infinity, the symbol*

$$F_2=\mathsf{H}_{p[g]}\beth+4\varsigma\sigma\beth\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}) \tag{274}$$

vanishes cubically at $\mathcal{N}^\varsigma_\sigma$ in the sense that $F_2\in\beth^{3/2}L^\infty+\varrho_{\mathrm{df}}\beth L^\infty+\tilde{p}[g]L^\infty$ locally. ■

Proof. In fact, we prove the slightly stronger statement that each of ϱ_{nf}^2 and $\hat{\eta}^2$ have the same property:

$$\mathsf{H}_{p[g]}\varrho_{\mathrm{nf}}^2+4\varsigma\sigma\varrho_{\mathrm{nf}}^2,\mathsf{H}_{p[g]}\hat{\eta}^2+4\varsigma\sigma\hat{\eta}^2\in\beth^{3/2}L^\infty+\varrho_{\mathrm{df}}\beth L^\infty+\tilde{p}L^\infty \tag{275}$$

locally. This version of the proposition has the advantage that it manifestly does not depend on the choice of ϱ_{df}, which affects $\beth$ through $\hat{\eta}$. Indeed, if $\varrho_{\mathrm{df},0}$ is some other boundary-defining function of df, then

$$\mathsf{H}_{p[g]}\Big(\hat{\eta}^2\frac{\varrho_{\mathrm{df},0}^2}{\varrho_{\mathrm{df}}^2}\Big)+4\varsigma\sigma\hat{\eta}^2\frac{\varrho_{\mathrm{df},0}^2}{\varrho_{\mathrm{df}}^2}=\frac{\varrho_{\mathrm{df},0}^2}{\varrho_{\mathrm{df}}^2}F_2+\hat{\eta}^2\mathsf{H}_{p[g]}\Big(\frac{\varrho_{\mathrm{df},0}^2}{\varrho_{\mathrm{df}}^2}\Big). \tag{276}$$

Since $\mathsf{H}_{p[g]}$ vanishes at the radial set $\mathcal{N}^\varsigma_\sigma$, which has $\varrho_{\mathrm{nf}}^2+\hat{\eta}^2+\varrho_{\mathrm{df}}$ as a quadratic defining function within some neighborhood of itself within the characteristic set,

$$\hat{\eta}^2\mathsf{H}_{p[g]}\Big(\frac{\varrho_{\mathrm{df},0}^2}{\varrho_{\mathrm{df}}^2}\Big)\in\beth^{3/2}L^\infty+\varrho_{\mathrm{df}}\beth L^\infty+\tilde{p}[g]L^\infty. \tag{277}$$

Thus, this new term vanishes cubically at $\mathcal{N}^\varsigma_\sigma$, as desired.

In addition, by similar reasoning used in the proof of Proposition 4.2 (although in this case we absorb the contributions into the $\varrho_{\mathrm{df}}\beth L^\infty$ error term), it suffices to consider the case when g is the Minkowski metric.

We consider the case of $\mathcal{N}^+_+$, the other three being analogous.

- We first consider the situation over $\Omega_{\mathrm{nfTf},+,T}$, using the coordinates eq. (249), taking $\varrho_{\mathrm{df}}=\rho$ locally. Then, $\mathsf{H}_p\varrho_{\mathrm{nf}}^2=-4(1-s)\varrho_{\mathrm{nf}}^2$ and $\mathsf{H}_p\hat{\eta}^2=4(\hat{\eta}^2+s^2-s-1)\hat{\eta}^2$ locally. Thus, the claim is that $s\varrho_{\mathrm{nf}}^2$ and $4(\hat{\eta}^2+s^2-s)\hat{\eta}^2$ vanish cubically at $\mathcal{N}^+_+\cap{}^{\mathrm{de,sc}}\pi^{-1}(\Omega_{\mathrm{nfTf},+,T})$, which is true.
- Over $\Omega_{\mathrm{nfSf},+,R}$, we use the coordinates eq. (252), taking $\varrho_{\mathrm{df}}=\rho$. Then, $\mathsf{H}_p\varrho_{\mathrm{nf}}^2=-4\varrho_{\mathrm{nf}}^2$ and $\mathsf{H}_p\hat{\eta}^2=4(\hat{\eta}^2-1)\hat{\eta}^2 \mod \varrho_{\mathrm{nf}}$ locally. Thus, the claim is that $F_2=4\hat{\eta}^4$ vanishes cubically at $\mathcal{N}^+_+\cap{}^{\mathrm{de,sc}}\pi^{-1}(\Omega_{\mathrm{nfSf},+,R})$, and this is true.

□

Proposition 4.6. *Fix signs $\varsigma,\sigma\in\{-,+\}$. Letting $\daleth\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ satisfy $\daleth=\varrho_{\mathrm{nf}}^2+\rho^2+(\lambda+3)^2$ near $\mathcal{K}^\varsigma_\sigma$ in the coordinates eq. (252), there exist $F_3,E_3\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ such that*

$$\mathsf{H}_{p[g]}\daleth=-4\varsigma\sigma\daleth-E_3+F_3, \tag{278}$$

$E_3\geq 0$ everywhere, and F_3 vanishes cubically at $\mathcal{K}^\varsigma_\sigma$ in the sense that $F_3\in\daleth^{3/2}L^\infty+\varrho_{\mathrm{Sf}}\daleth L^\infty+\tilde{p}[g]L^\infty$ locally. ■

Proof. By similar reasoning to that in the proof of Proposition 4.5, it suffices to consider the case when g is the Minkowski metric. We consider the case of $\varsigma,\sigma=+$, that is of $\mathcal{K}^+_+$, the other three being analogous. Then,

$$\begin{aligned}\mathsf{H}_p\daleth&=-4\varrho_{\mathrm{nf}}^2+2(2\hat{\eta}^2+\lambda+1)\rho^2+2(2\hat{\eta}^2+\lambda-1)(\lambda+3)^2\\&=-4\daleth+2(2\hat{\eta}^2+\lambda+3)\rho^2+2(2\hat{\eta}^2+\lambda+1)(\lambda+3)^2.\end{aligned} \tag{279}$$

Choose a symbol $E_3\geq 0$ such that $E_3=4(\lambda+3)^2$ near $\mathcal{K}^+_+$. Thus, we set $F_3=2(2\hat{\eta}^2+\lambda+3)(\rho^2+(\lambda+3)^2)$, and this vanishes cubically at $\mathcal{K}^+_+$. □

Proposition 4.7. *Fix signs $\varsigma,\sigma\in\{-,+\}$. Letting $\daleth\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ satisfy $\daleth=\varrho_{\mathrm{nf}}^2+\rho^2+(s-2)^2$ near $\mathcal{C}^\varsigma_\sigma$ in the coordinates eq. (249), there exist $F_4,E_4\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ such that*

$$\mathsf{H}_p\daleth=4\varsigma\sigma\daleth+E_4+F_4, \tag{280}$$

$E_4\geq 0$ everywhere, and F_4 vanishes cubically at $\mathcal{C}^\varsigma_\sigma$ in the sense that $F_4\in\daleth^{3/2}L^\infty+\varrho_{\mathrm{Tf}}\daleth L^\infty+\tilde{p}[g]L^\infty$ locally. ■

Proof. By similar reasoning to that in the proof of Proposition 4.5, it suffices to consider the case when g is the Minkowski metric. We consider the case of $\varsigma,\sigma=+$, that is of $\mathcal{C}^+_+$, the other three being analogous. Then,

$$\mathsf{H}_p\daleth=4(s-1)\varrho_{\mathrm{nf}}^2+4(\hat{\eta}^2+(s-1)^2)\rho^2+4(s-2)^2(\hat{\eta}^2+s(s-1)). \tag{281}$$

Choose a symbol $E_4\geq 0$ such that $E_4=4(s-2)^2$ near $\mathcal{C}^+_+$. Thus, we set $F_4=-4(2-s)\varrho_{\mathrm{nf}}^2+4(\hat{\eta}^2+s(s-2))\rho^2+4(s-2)^2(\hat{\eta}^2+(s-2)(s+1))$, and this vanishes cubically at $\mathcal{C}^+_+$. □

5. Propagation through null infinity

In this section, let $P\in\mathrm{Diff}^{2,0}_{\mathrm{de,sc}}(\mathbb{O})$ denote a de,sc-differential operator such that

$$P=\square_g+\mathsf{m}^2+\mathrm{Diff}^{1,-2}_{\mathrm{de,sc}}(\mathbb{O}) \tag{282}$$

for an admissible metric g. Thus, the symbol $p[g]:\sum_{i=0}^d\xi_i\,\mathrm{d}x_i\mapsto g^{-1}(\boldsymbol{\xi},\boldsymbol{\xi})+\mathsf{m}^2$ of $\square_g+\mathsf{m}^2$ is a representative of $\sigma^{2,0}_{\mathrm{de,sc}}(P)$.

We state in §5.1 the microlocal version of the proposition that a lack of decay of solutions to $Pu=f$, for nice f, as measured by de-wavefront set on the boundary Penrose diagram, propagates along null infinity, assuming control on $\mathcal{N}$. We then prove a series of radial point type estimates, two at each of the radial sets $\mathcal{A},\mathcal{N},\mathcal{C},\mathcal{K}$ lying over null infinity: $\mathcal{A}$ in §5.2, $\mathcal{N}$ in §5.3, $\mathcal{K}$ in §5.4, and finally $\mathcal{C}$ in §5.5. Each of these is a saddle point of the de,sc-Hamiltonian flow. The radial set $\mathcal{R}$, which lies instead over the timelike caps, is postponed until the next section. The two main results of this section, gotten by concatenating the various estimates, are:

Theorem 2. *Suppose that $m\in\mathbb{R}$ and $\mathsf{s}=(s_{\mathrm{Pf}},s_{\mathrm{nPf}},s_{\mathrm{Sf}},s_{\mathrm{nFf}},s_{\mathrm{Ff}})\in\mathbb{R}^5$ satisfy*

- $m>s_{\mathrm{nf}}+1$,
- $2s_{\mathrm{Sf}}>\max\{-2m+2s_{\mathrm{nf}}+1,m+s_{\mathrm{nf}}-1\}$,
- $2s_{\mathrm{Tf}}<m+s_{\mathrm{nf}}-1$

when $(s_{\mathrm{nf}},s_{\mathrm{Tf}})=(s_{\mathrm{nFf}},s_{\mathrm{Ff}})$ and when $(s_{\mathrm{nf}},s_{\mathrm{Tf}})=(s_{\mathrm{nPf}},s_{\mathrm{Pf}})$. Suppose that $u\in\mathcal{S}'$ is a solution to $Pu=f$ such that, for some $T\in\mathbb{R}$,

$$\mathrm{WF}^{m,s_{\mathrm{Sf}}}_{\mathrm{sc}}(u)\cap{}^{\mathrm{sc}}\pi^{-1}\,\mathrm{cl}_{\mathbb{M}}\{t=T\}=\varnothing \tag{283}$$

and such that $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(f)\subseteq\mathcal{R}$. Then, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\subseteq\mathcal{R}$ as well. ■

Remark. Using the ordinary sc-calculus, it is can be shown that, given the setup of the theorem, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\subseteq\mathcal{R}\cup{}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf})$. Of course, the refinement is only over null infinity.

Theorem 3. *Suppose that $m\in\mathbb{R}$ and $\mathsf{s}=(s_{\mathrm{Pf}},s_{\mathrm{nPf}},s_{\mathrm{Sf}},s_{\mathrm{nFf}},s_{\mathrm{Ff}})\in\mathbb{R}^5$ satisfy*

- $m<s_{\mathrm{nf}}+1$,
- $2s_{\mathrm{Sf}}<\min\{-2m+2s_{\mathrm{nf}}+1,m+s_{\mathrm{nf}}-1\}$,
- $2s_{\mathrm{Tf}}>m+s_{\mathrm{nf}}-1$

when $(s_{\mathrm{nf}},s_{\mathrm{Tf}})=(s_{\mathrm{nFf}},s_{\mathrm{Ff}})$ and when $(s_{\mathrm{nf}},s_{\mathrm{Tf}})=(s_{\mathrm{nPf}},s_{\mathrm{Pf}})$. Suppose that $u\in\mathcal{S}'$ is a solution to $Pu=f$ such that, for some neighborhood $U\subseteq{}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$ of $\mathcal{R}$,

$$\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap U\subseteq\mathcal{R}, \tag{284}$$

and likewise suppose that $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(f)\subseteq\mathcal{R}$. Then, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\subseteq\mathcal{R}$ as well. ■

The e,b-analogues of the results in this section can be found in [HV23, §4]. As the arguments below are very similar to those there (and de,sc-analogues of the standard sc-results described in [Vas18] anyways), we will only sketch the key points. To handle the situation away from null infinity, we can simply cite the propagation results established using the sc-calculus, and so this will be described in even less detail.

Note that, by the definition of admissibility, $\Box_g$ differs from the Minkowski d'Alembertian $\Box$ by an element of $\mathrm{Diff}^{2,-1}(\mathbb{O})$ with real principal symbol. A consequence is that

$$P - P^* \in \mathrm{Diff}^{1,-2}(\mathbb{O}), \tag{285}$$

where P^* is the formal adjoint with respect to the $L^2(\mathbb{R}^{1+d})$ inner product. This restriction on $P - P^*$ simplifies the radial point estimates, which, as in in [Vas18], would otherwise depend on the values of

$$\varrho_{\mathrm{df}}^{1}\varrho_{\mathrm{Pf}}^{-1}\varrho_{\mathrm{nPf}}^{-1}\varrho_{\mathrm{Sf}}^{-1}\varrho_{\mathrm{nFf}}^{-1}\varrho_{\mathrm{Ff}}^{-1}\cdot\sigma_{\mathrm{de,sc}}^{1,-1}(P-P^*)\in S_{\mathrm{de,sc}}^{[0,0]}(\mathbb{O}) \tag{286}$$

along the various radial sets. Let $p_1 \in S^{1,-2}_{\mathrm{de,sc}}(\mathbb{O})$ denote a representative of $i\sigma^{1,-2}_{\mathrm{de,sc}}(P-P^*)$, which we can choose to be real-valued.

5.1. **Propagation Between the Radial Sets.** Using the nonzero component of H_p along the punctured fibers ${}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf})\backslash\mathcal{R}$, we get the following:

Proposition 5.1. *Suppose that $u \in \mathcal{S}'$ satisfies $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap (\mathcal{N} \cup {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{cl}_{\mathbb{O}}\{|t|-r=v\})) = \varnothing$ for some $v \in \mathbb{R}$. Suppose further that, for some $v_1, v_2 \in \mathbb{R}$ satisfying $v_1 < v < v_2$,*

$$\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{cl}_{\mathbb{O}}\{v_1 \leq |t|-r \leq v_2\}) \subseteq \mathcal{N}. \tag{287}$$

Then, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{cl}_{\mathbb{O}}\{t_1 \leq |t|-r \leq v_2\}) = \varnothing$. ■

Proof sketch. As seen by rewriting it in terms of ϱ_{Sf} in $\Omega_{\mathrm{nfSf},\sigma,R}$ and in terms of ϱ_{Tf} in $\Omega_{\mathrm{nfTf},\sigma,T}$, the function $|t|-r$ is monotone under $\mathsf{H}_{p[g]}$ on each component of $(\Sigma_{\mathsf{m},\pm} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{cl}_{\mathbb{O}}\{v_1 < |t|-r < v_2\}))\backslash\mathcal{N}$ (see Proposition 4.1). The proposition therefore follows via the usual proof of Duistermaat–Hörmander type estimates, using elliptic regularity off of the characteristic set. The point is that any integral curve of $\mathsf{H}_{p[g]}$ in the relevant region of the characteristic set has to have one end at one of the sets $\mathcal{N}$ or ${}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{cl}_{\mathbb{O}}\{|t|-r=v\})$, where we have control. □

Proposition 5.2. *Let $m \in \mathbb{R}$ and $\mathsf{s} \in \mathbb{R}^5$, and suppose that $u \in \mathcal{S}'$ satisfies $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{A} = \varnothing$. Then, if $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap ({}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf})\backslash{}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}\cap\mathrm{Tf})) \subseteq \{\hat{\eta}=0\}$,*

$$\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap ({}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf})\backslash{}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}\cap\mathrm{Tf})) \subseteq \{\hat{\eta}=0\}. \tag{288}$$

Moreover, if $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap(\mathcal{A}\cup\mathcal{N})\cap{}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}\cap\mathrm{Sf}) = \varnothing$ and $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu)\cap{}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}\cap\mathrm{Sf}) \subseteq \mathcal{K}$, then $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}\cap\mathrm{Sf}) \subseteq \mathcal{K}$. ■

Proof sketch. By eq. (254), ρ_{Sf} is monotone with respect to $\mathsf{H}_{p[g]}$ along the invariant set $\{|\hat{\eta}| = 1\} \cap \Sigma_{\mathsf{m},\pm}$ (defined using this coordinate system). So the assumption $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{A} = \varnothing$ allows us to conclude, using a Duistermaat–Hörmander estimate, that

$$\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \{|\hat{\eta}|=1\} = \varnothing, \tag{289}$$

using an elliptic estimate off $\Sigma_{\mathsf{m},\pm}$.

By eq. (254), the function $\hat{\eta}^2$ is monotone under $\mathsf{H}_{p[g]}$ on $\Sigma_{\mathsf{m},\pm} \cap ({}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf})\backslash{}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}\cap\mathrm{Tf}))\backslash\{|\hat{\eta}|=0,1\}$ (see Figure 5). We can therefore propagate the control on $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \{|\hat{\eta}|=1\}$ to conclude eq. (288).

To get the second part of the proposition, we use another propagation estimate, this time based on the monotonicity of λ under $\mathsf{H}_{p[g]}$ on $\{\hat{\eta}=0\} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf}\cap\mathrm{Sf})\backslash\mathcal{N}$, which we also read off eq. (254) (again see see Figure 5). □

Propagating in the reverse direction:

Proposition 5.3. *Let $m \in \mathbb{R}$ and $\mathsf{s} \in \mathbb{R}^5$, and suppose that $u \in \mathcal{S}'$ satisfies $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{K} = \varnothing$. Then, if $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Sf}) \subseteq \{\rho = 0, \lambda \in [-1,+1]\}$*

$$\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Sf}) \subseteq \{\rho = 0, \lambda \in [-1,+1]\}. \tag{290}$$

If, in addition, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{N} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Sf}) = \varnothing$ and $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Sf}) \subseteq \mathcal{A}$, then $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Sf}) \subseteq \mathcal{A}$. ■□

Over the other corner:

Proposition 5.4. *Let $m \in \mathbb{R}$ and $\mathsf{s} \in \mathbb{R}^5$, and suppose that $u \in \mathcal{S}'$ satisfies $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{C} = \varnothing$. Then, if*

$$\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) \subseteq \{\hat{\eta} = 0, s \leq 1\}, \tag{291}$$

then $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) \subseteq \{\hat{\eta} = 0, s \leq 1\}$ as well. If, in addition, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{N} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) = \varnothing$ and

$$\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) \subseteq \mathcal{R}, \tag{292}$$

then $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) \subseteq \mathcal{R}$. ■

Proof sketch. The argument is slightly different than the previous in that each part involves two propagation steps. For the first step, we propagate control the rest of $\Sigma_{\mathsf{m},+} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nFf} \cap \mathrm{Ff}) \cap {}^{\mathrm{de,sc}}\mathbb{S}^*\mathbb{O}$ using s as monotone function, which, according to eq. (251), is monotone under $\mathsf{H}_{p[g]}$ on

$$(\Sigma_{\mathsf{m},+} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nFf} \cap \mathrm{Ff}) \cap {}^{\mathrm{de,sc}}\mathbb{S}^*\mathbb{O}) \backslash (\mathcal{C} \cup \mathcal{N}). \tag{293}$$

Having done this, we conclude an absence of wavefront set *at fiber infinity* except possibly at $\mathcal{N}$. Next, this control can be propagated to the rest of $\Sigma_{\mathsf{m},+} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nFf} \cap \mathrm{Ff}) \backslash \{\hat{\eta} = 0, s \leq 1\}$ using ρ as a monotone function, which is monotone in the interior of the fibers according to eq. (251). (See Figure 5.) For the second part of the proposition, the argument is the same, except after the first step we conclude an absence of wavefront set at all of fiber infinity, including at $\mathcal{N}$, and then the second step propagates control to everywhere except $\mathcal{R}$. □

Propagating in the reverse direction:

Proposition 5.5. *Let $m \in \mathbb{R}$ and $\mathsf{s} \in \mathbb{R}^5$, and suppose that $u \in \mathcal{S}'$ satisfies $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{R} = \varnothing$. Then, if*

$$\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) \cap {}^{\mathrm{de,sc}}T^*\mathbb{O} = \varnothing \tag{294}$$

(that is, the de,sc-wavefront over $\mathrm{nf} \cap \mathrm{Tf}$ set is at fiber infinity), then $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) \cap {}^{\mathrm{de,sc}}T^*\mathbb{O} = \varnothing$ as well. If, in addition, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{N} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) = \varnothing$ and

$$\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) \subseteq \mathcal{C}, \tag{295}$$

then $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nf} \cap \mathrm{Tf}) \subseteq \mathcal{C}$ as well. ■□

5.2. **Propagation Through $\mathcal{A}$.** As seen above, $\mathcal{A}^{\pm}_{\pm}$ (with the signs the same) is a source in the fiberwise directions and with respect to the direction along the null face, but it is a sink in the $\partial_{\varrho_{\mathrm{nf}}}$ direction. The same holds for $\mathcal{A}^{\pm}_{\mp}$ (with the signs opposite), with "source" and "sink" switched. Thus, we can prove two estimates:

1. propagation from a band $\{\epsilon_1 < \varrho_{\mathrm{nf}} < \epsilon_2\}$ hitting spacelike infinity into $\mathcal{A}$, and
2. propagation from an appropriate annular set defined using the other coordinates around $\mathcal{A}$ into $\mathcal{A}$.

Proposition 5.6. *Fix signs* $\varsigma,\sigma\in\{-,+\}$. *Suppose that* $m\in\mathbb{R}$ *and* $\mathsf{s}=(s_{\mathrm{Pf}},s_{\mathrm{nPf}},s_{\mathrm{Sf}},s_{\mathrm{nFf}},s_{\mathrm{Ff}})\in\mathbb{R}^5$ *satisfy* $m-s_{\mathrm{nf}}+s_{\mathrm{Sf}}>1/2$, *where* $s_{\mathrm{nf}}\in\{s_{\mathrm{nPf}},s_{\mathrm{nFf}}\}$, *depending on* σ *in the usual way. For any* $\epsilon_1>0$, *there exists some* $\epsilon_0\in(0,\epsilon_1)$ *such that, if* $u\in\mathcal{S}'$ *satisfies*

- $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu)\cap\mathcal{A}^\varsigma_\sigma=\varnothing$,
- $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\{\rho^2+(\|\hat\eta\|-1)^2+(\lambda+1)^2+\varrho_{\mathrm{Sf}}^2<\epsilon_1,\epsilon_2<\varrho_{\mathrm{nf}}<\epsilon_1\}=\varnothing$ *for some* $\epsilon_2\in(0,\epsilon_0)$,

it is the case that $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\mathcal{A}^\varsigma_\sigma=\varnothing$. ■

Remark. Here, $\{\rho^2+(\|\hat\eta\|-1)^2+(\lambda+1)^2+\varrho_{\mathrm{Sf}}^2<\epsilon_1,\epsilon_2<\varrho_{\mathrm{nf}}<\epsilon_1\}$ denotes a subset of ${}^{\mathrm{de,sc}}\pi^{-1}(\Omega_{\mathrm{nfSf},\sigma,R})$. Similar notational conventions will be used below. Note that the condition $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\{\rho^2+(\|\hat\eta\|-1)^2+(\lambda+1)^2+\varrho_{\mathrm{Sf}}^2<\epsilon_1,\epsilon_2<\varrho_{\mathrm{nf}}<\epsilon_1\}=\varnothing$ can be rewritten in terms of $\mathrm{WF}^{m,s_{\mathrm{Sf}}}_{\mathrm{sc}}(u)$, as the second set is disjoint from null infinity.

Proof. We handle the case $\varsigma,\sigma=+$, the other three being analogous. Consider the symbol

$$a_0=\varrho_{\mathrm{df}}^{m_0}\varrho_{\mathrm{nFf}}^{s_0}\varrho_{\mathrm{Sf}}^{\ell_0}\tag{296}$$

for $m_0,s_0,\ell_0\in\mathbb{R}$ given by $m_0=1-2m$, $s_0=-1-2s_{\mathrm{nFf}}$, and $\ell_0=-1-2s_{\mathrm{Sf}}$, where, as above, we arrange for convenience that near $\mathcal{A}^+_+$, $\varrho_{\mathrm{df}}=\rho$, $\varrho_{\mathrm{nFf}}=\varrho_{\mathrm{nf}}$. Then, by eq. (254),

$$\mathsf{H}_pa_0=(m_0(2\hat\eta^2+\lambda+1)-2s_0+\ell_0(1-\lambda))a_0.\tag{297}$$

Thus, we can write $\mathsf{H}_{p[g]}a_0=-\alpha a_0$ for a symbol α given by $\alpha=-m_0(2\hat\eta^2+\lambda+1)+2s_0-\ell_0(1-\lambda)$ over null infinity. Exactly at $\mathcal{A}^+_+$, $\lambda=-1$ and $\hat\eta^2=1$, so

$$\alpha|_{\mathcal{A}^+_+}=2(-m_0+s_0-\ell_0)=2(-1+2m-2s_{\mathrm{nFf}}+2s_{\mathrm{Sf}})>0,\tag{298}$$

with the last inequality coming from our assumption that $m+s_{\mathrm{nf}}+s_{\mathrm{Sf}}>1/2$.

Let $\chi\in C^\infty_{\mathrm{c}}$ be such that $-\operatorname{sgn}(t)\chi'(t)\chi(t)=\chi_0^2(t)$ for some $\chi_0\in C^\infty_{\mathrm{c}}(\mathbb{R})$ and such that $\chi=1$ identically in some neighborhood of the origin (the construction by modifying $e^{-1/t}$ is standard — see e.g. [Vas18]). For $F\in\mathbb{R}^+$, let $\chi_F(t)=\chi(Ft)$, and correspondingly let

$$\chi_{0,F}(t)=F^{1/2}\chi_0^2(Ft),\tag{299}$$

so that $-\operatorname{sgn}(t)\chi_F'(t)\chi_F(t)=\chi_{0,F}(t)^2$. Modify a_0 by using the $\chi_F(t)$ to localize near $\mathcal{A}^+_+$: define $a\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ by

$$a=\chi_F(\tilde p[g])^2\chi_F(\varrho_{\mathrm{nf}})^2\chi_F(\aleph)^2a_0\tag{300}$$

near $\mathcal{A}^+_+$, where $\aleph$ is as in Proposition 4.2. (For convenience, taking F sufficiently large, we arrange that a is identically zero outside of the region for which the definition eq. (300) is taken.) This does indeed localize near $\mathcal{A}^+_+$, in the sense that given any open neighborhood $U\supset\mathcal{A}^+_+$, we can choose $F>0$ sufficiently large so that $\operatorname{supp}a\subseteq U$.

Letting F_1 be as in Proposition 4.2,

$$\begin{aligned}\mathsf{H}_{p[g]}a=-\alpha a+2(1+\varrho_{\mathrm{nf}}g)\chi_F(\tilde p[g])^2\chi_{0,F}(\varrho_{\mathrm{nf}})^2\chi(\aleph)^2\varrho_{\mathrm{nf}}a_0-2\chi_F(\tilde p)^2\chi_F(\varrho_{\mathrm{nf}})^2\chi_{0,F}(\aleph)^2(4\aleph+F_1)a_0\\+2\chi_F'(\tilde p[g])\chi_F(\tilde p[g])\chi_F(\varrho_{\mathrm{nf}})^2\chi_F(\aleph)^2\tilde q\tilde p[g]a_0,\end{aligned}\tag{301}$$

where $\tilde q=\varrho_{\mathrm{df}}^{-2}\mathsf{H}_{p[g]}\varrho_{\mathrm{df}}^2\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$, so that $\mathsf{H}_{p[g]}\tilde p[g]=\tilde q\tilde p[g]$, and

$$g=-\varrho_{\mathrm{nf}}^{-2}(\mathsf{H}_{p[g]}-\mathsf{H}_p)\varrho_{\mathrm{nf}}\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}).\tag{302}$$

Let $w=\varrho_{\mathrm{df}}^{-1}\varrho_{\mathrm{Pf}}\varrho_{\mathrm{nPf}}\varrho_{\mathrm{Sf}}\varrho_{\mathrm{nFf}}\varrho_{\mathrm{Ff}}$, so that $\mathsf{H}_{p[g]}=w^{-1}H_{p[g]}$.

For all $F>0$ sufficiently large, for δ sufficiently small we can define symbols $b,e,f,h\in S^{0,0}_{\mathrm{de,sc}}$ such that

$$\begin{aligned}\mathsf{H}_{p[g]}a+w^{-1}p_1a&=(-\delta a_0^{-2}a^2-b^2+e^2\varrho_{\mathrm{nf}}-f^2+h)a_0,\\H_{p[g]}a+p_1a&=(-\delta a_0^{-2}a^2-b^2+e^2\varrho_{\mathrm{nf}}-f^2+h)wa_0\end{aligned}\tag{303}$$

everywhere, with $b = \chi_{F}(\tilde{p}[g])\chi_{F}(\varrho_{\mathrm{nf}})\chi_{F}(\aleph)(\alpha - \delta a_0^{-1}a - w^{-1}p_1)^{1/2}$,

$$\begin{aligned} e &= \sqrt{2(1+\varrho_{\mathrm{nf}}g)}\chi_{F}(\tilde{p}[g])\chi_{0,F}(\varrho_{\mathrm{nf}})\chi_{F}(\aleph), \\ f &= \sqrt{2}\chi_{F}(\tilde{p}[g])\chi_{F}(\varrho_{\mathrm{nf}})\chi_{0,F}(\aleph)(4\aleph + F_1)^{1/2}, \end{aligned} \tag{304}$$

and $h = 2\chi_F'(\tilde{p}[g])\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{nf}})^2\chi_F(\aleph)^2\tilde{q}\tilde{p}[g]$ near $\mathcal{A}^+_+$. It is because $w^{-1}p_1$ vanishes at $\mathcal{A}^+_+$ (and in fact, over all of the faces of $\mathbb{O}$, because we assumed that p_1 was decaying quadratically there) that we can take F sufficiently large so that the $-w^{-1}p_1$ term under the square root in the definition of b is guaranteed to not spoil the sign.

Quantizing, we get $A = (1/2)(\mathrm{Op}(a) + \mathrm{Op}(a)^*) \in \Psi_{\mathrm{de,sc}}^{-m_0,(-\infty,-\infty,-\ell_0,-s_0,-\infty)}$, this being self-adjoint (here, we are just using the $L^2(\mathbb{R}^{1,d})$ inner product), and

$$\begin{aligned} B &= \mathrm{Op}(w^{1/2}a_0^{1/2}b) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,s_{\mathrm{Sf}},s_{\mathrm{nFf}},-\infty)}, \\ E &= \mathrm{Op}(w^{1/2}a_0^{1/2}\varrho_{\mathrm{nf}}^{1/2}e) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,s_{\mathrm{Sf}},-\infty,-\infty)}, \\ F &= \mathrm{Op}(w^{1/2}a_0^{1/2}f) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,s_{\mathrm{Sf}},s_{\mathrm{nFf}},-\infty)}, \end{aligned} \tag{305}$$

$H = \mathrm{Op}(wa_0h) \in \Psi_{\mathrm{de,sc}}^{2m,(-\infty,-\infty,2s_{\mathrm{Sf}},2s_{\mathrm{nFf}},-\infty)}$, such that

$$-i[P,A] + i(P-P^*)A = -\delta A\Lambda^2 A - B^*B + E^*E - F^*F + H + R \tag{306}$$

for some $R \in \Psi_{\mathrm{de,sc}}^{-m_0,(-\infty,-\infty,-2-\ell_0,-2-s_0,-\infty)}$. Above,

$$\Lambda = (1/2)(\mathrm{Op}(w^{1/2}a_0^{-1/2}) + \mathrm{Op}(w^{1/2}a_0^{-1/2})^*) \in \Psi_{\mathrm{de,sc}}^{1-m,(-1/2,-1/2,-1-s_{\mathrm{Sf}},-1-s_{\mathrm{nFf}},-1/2)}. \tag{307}$$

The quantization procedure can be arranged so as to preserve essential supports, so that

$$\mathrm{WF}'_{\mathrm{de,sc}}(B), \mathrm{WF}'_{\mathrm{de,sc}}(E), \mathrm{WF}'_{\mathrm{de,sc}}(F), \mathrm{WF}'_{\mathrm{de,sc}}(H) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(A) \subseteq \mathrm{supp}(a) \tag{308}$$

which, via the definition eq. (306)) of R, also forces $\mathrm{WF}'_{\mathrm{de,sc}}(R) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(A)$. We have $\mathrm{WF}'_{\mathrm{de,sc}}(E) \subseteq \mathrm{supp}(\chi_F(\tilde{p}[g])\chi_{0,F}(\varrho_{\mathrm{nf}})\chi_F(\aleph))$. For each $\epsilon_2 > 0$, by taking F sufficiently large,

$$\mathrm{WF}'_{\mathrm{de,sc}}(E) \subseteq \{\rho^2 + (\|\hat{\eta}\| - 1)^2 + (\lambda+1)^2 + \varrho_{\mathrm{Sf}}^2 < \epsilon_1, \epsilon_2 < \varrho_{\mathrm{nf}} < \epsilon_1\} \tag{309}$$

as long as ϵ_1 is sufficiently small relative to F.

Computing $\langle -i[P,A]u,u\rangle_{L^2} = i\langle PAu,u\rangle_{L^2} - i(\langle P^*Au,u\rangle_{L^2})^*$, assuming temporarily that u is Schwartz, we get

$$2i\Im\langle Au,Pu\rangle_{L^2} = -\delta\|\Lambda Au\|_{L^2}^2 - \|Bu\|_{L^2}^2 + \|Eu\|_{L^2}^2 - \|Fu\|_{L^2}^2 + \langle Hu,u\rangle_{L^2} + \langle Ru,u\rangle_{L^2}. \tag{310}$$

Thus,

$$\|Bu\|_{L^2}^2 + \delta\|\Lambda Au\|_{L^2}^2 \leq \|Eu\|_{L^2}^2 + |\langle Hu,u\rangle_{L^2}| + |\langle Ru,u\rangle_{L^2}| + 2|\langle Au,Pu\rangle_{L^2}| \tag{311}$$

From this, it can be deduced that, for each $N \in \mathbb{N}$, for some $\tilde{B} \in \Psi_{\mathrm{de,sc}}^{0,0}$ elliptic at $\mathcal{A}^+_+$ and $\tilde{E} \in \Psi_{\mathrm{de,sc}}^{0,0}$ satisfying

$$\mathrm{WF}'_{\mathrm{de,sc}}(\tilde{E}) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(E), \tag{312}$$

we get, the estimate

$$\begin{aligned} \|\tilde{B}u\|^2_{H_{\mathrm{de,sc}}^{m,(N,N,s_{\mathrm{Sf}},s_{\mathrm{nFf}},N)}} \preceq \|\tilde{E}u\|^2_{H_{\mathrm{de,sc}}^{m,(-N,-N,s_{\mathrm{Sf}},-N,-N)}} + \|GPu\|^2_{H_{\mathrm{de,sc}}^{m-1,(-N,-N,s_{\mathrm{Sf}}+1,s_{\mathrm{nFf}}+1,-N)}} \\ + \|Gu\|^2_{H_{\mathrm{de,sc}}^{m-1/2,(-N,-N,s_{\mathrm{Sf}}-1/2,s_{\mathrm{nFf}}-1/2,-N)}} + \|u\|^2_{H_{\mathrm{de,sc}}^{-N,-N}}, \end{aligned} \tag{313}$$

for some $G \in \Psi_{\mathrm{de,sc}}^{0,0}$ having essential support in a small neighborhood of $\mathcal{A}^+_+$ (that can be taken arbitrarily small by making F arbitrarily large), chosen so that

$$\mathrm{WF}'_{\mathrm{de,sc}}(1-G) \cap \mathrm{WF}'_{\mathrm{de,sc}}(A) = \varnothing. \tag{314}$$

Indeed, we can choose such $\tilde{B}, \tilde{E}$ as specified such that

$$\|\tilde{B}u\|_{H_{\mathrm{de,sc}}^{m,(N,N,s_{\mathrm{Sf}},s_{\mathrm{nFf}},N)}} \preceq \|Bu\|_{L^2} + \|u\|_{H_{\mathrm{de,sc}}^{-N,-N}}, \tag{315}$$

$$\|Eu\|_{L^2} \preceq \|\tilde{E}u\|_{H_{\mathrm{de,sc}}^{m,(-N,-N,s_{\mathrm{Sf}},-N,-N)}} + \|u\|_{H_{\mathrm{de,sc}}^{-N,-N}}. \tag{316}$$

The microlocal elliptic parametrix construction in the de,sc-calculus is used to control the $\langle Hu, u\rangle_{L^2}$ term in eq. (311), resulting in the estimate

$$\begin{aligned} |\langle Hu, u\rangle_{L^2}| &\preceq \|GPu\|^2_{H_{\mathrm{de,sc}}^{m-2,(-N,-N,s_{\mathrm{Sf}},s_{\mathrm{nFf}},-N)}} + \|u\|^2_{H_{\mathrm{de,sc}}^{-N,-N}} \\ &\preceq \|GPu\|^2_{H_{\mathrm{de,sc}}^{m-1,(-N,-N,s_{\mathrm{Sf}}+1,s_{\mathrm{nFf}}+1,-N)}} + \|u\|^2_{H_{\mathrm{de,sc}}^{-N,-N}} \end{aligned} \tag{317}$$

which can be done because the essential support of H is located away from the characteristic set of P. The $\langle Ru, u\rangle_{L^2}$ term is just estimated with Cauchy–Schwarz:

$$|\langle Ru, u\rangle_{L^2}| \preceq \|Gu\|^2_{H_{\mathrm{de,sc}}^{m-1/2,(-N,-N,s_{\mathrm{Sf}}-1/2,s_{\mathrm{nFf}}-1/2,-N)}} + \|u\|^2_{H_{\mathrm{de,sc}}^{-N,-N}}. \tag{318}$$

Finally, we can choose $\tilde{G} \in \Psi^{0,0}_{\mathrm{de,sc}}$, dependent on N, such that

$$\|\Lambda_{-1}\tilde{G}Pu\|_{L^2} \preceq \|GPu\|_{H_{\mathrm{de,sc}}^{m-1,(-N,-N,s_{\mathrm{Sf}}+1,s_{\mathrm{nFf}}+1,-N)}}, \tag{319}$$

where Λ_{-1} is a parametrix for Λ, and then we bound the $\langle Au, Pu\rangle_{L^2}$ term in eq. (311) as follows:

$$|\langle Au, Pu\rangle_{L^2}| \leq |\langle Au, \tilde{G}Pu\rangle_{L^2}| + |\langle Au, (1-\tilde{G})Pu\rangle_{L^2}|, \tag{320}$$

and

$$|\langle Au, (1-\tilde{G})Pu\rangle_{L^2}| \preceq \|u\|^2_{H_{\mathrm{de,sc}}^{-N,-N}}, \tag{321}$$

$$\begin{aligned} |\langle Au, \tilde{G}Pu\rangle_{L^2}| &\preceq \|u\|^2_{H_{\mathrm{de,sc}}^{-N,-N}} + \|\Lambda Au\|_{L^2}\|\Lambda_{-1}\tilde{G}Pu\|_{L^2} \\ \|\Lambda Au\|_{L^2}\|\Lambda_{-1}\tilde{G}Pu\|_{L^2}, &\leq 2^{-1}\varepsilon\|\Lambda Au\|^2_{L^2} + 2^{-1}\varepsilon^{-1}\|\Lambda_{-1}\tilde{G}Pu\|^2_{L^2} \\ &\preceq 2^{-1}\varepsilon\|\Lambda Au\|^2_{L^2} + 2^{-1}\varepsilon^{-1}\|GPu\|^2_{H_{\mathrm{de,sc}}^{m-1,(-N,-N,s_{\mathrm{Sf}}+1,s_{\mathrm{nFf}}+1,-N)}} \end{aligned} \tag{322}$$

for any $\varepsilon > 0$, where the bound is independent of ε. If ε is sufficiently small, then we can absorb the $2^{-1}\varepsilon\|\Lambda Au\|_{L^2}$ term into the $\delta\|\Lambda Au\|_{L^2}$ term in eq. (311), yielding eq. (313), as claimed. The constant implicit in eq. (313) depends on all of the operators involved, and on δ and N, but does not depend on u. Thus, assuming that u is Schwartz, we have quantitatively controlled u microlocally near $\mathcal{A}^+_+$ in terms of the quantities on the right-hand side of the estimate.

The standard regularization argument [Vas18][HV23] allows us to make sense of the estimate for general $u \in \mathcal{S}'$, with the conclusion being that if the right-hand side of eq. (313) is finite, then the left-hand side is too, with the stated inequality holding. One key point is that we can regularize in both the differential sense and the decay sense:

- first regularize only in s_{nf} (which we can do by an arbitrarily large number of orders), and assume that
$$Gu \in H_{\mathrm{de,sc}}^{m,(\infty,\infty,s_{\mathrm{Sf}},-N_0,\infty)}. \tag{323}$$
- Apply the same basic argument, but regularize in m and s_{Sf} instead to control
$$\|Gu\|_{H_{\mathrm{de,sc}}^{m,(0,0,s_{\mathrm{Sf}},-N_0,0)}}. \tag{324}$$

  For each value of N_0, we can only regularize by finitely many orders: in order to not spoil the signs involved in the construction of b, we must assume that
$$Gu \in H_{\mathrm{de,sc}}^{-N_1,(0,0,-N_1,-N_0,0)} \tag{325}$$

for N_1 satisfying $-2N_1+N_0>1/2$, which is the threshold condition for the regularized orders.

Combining the two steps, we end up with the estimate

$$\|\tilde{B}u\|_{L^2}^2 \preceq \|\tilde{E}u\|_{L^2}^2 + \|GPu\|^2_{H_{\mathrm{de,sc}}^{m-1,(-N,-N,s_{\mathrm{Sf}}+1,s_{\mathrm{nFf}}+1,-N)}} + \|Gu\|^2_{H_{\mathrm{de,sc}}^{m-1/2,(-N,-N,s_{\mathrm{Sf}}-1/2,s_{\mathrm{nFf}}-1/2,-N)}} + \|u\|^2_{H_{\mathrm{de,sc}}^{-N_1,(-N,-N,-N_1,-N_0,-N)}}, \tag{326}$$

for any $N,N_0\in\mathbb{R}$ and N_1 satisfying $-2N_1+N_0>1/2$, this holding in the strong sense that, if $u\in\mathcal{S}'$ is such that the right-hand side is finite, then the left-hand side is as well. By taking N_0 sufficiently large, we can choose N_1 such that $\max\{N_0,N_1\}>N$. Hence, eq. (313) holds for all $u\in\mathcal{S}'$.

Alternatively, we can regularize in both senses simultaneously with a careful choice of regularizer: for each $\varepsilon,K>0$, consider the locally-defined symbol

$$\varphi_{\varepsilon,K} = \Big(1+\frac{\varepsilon}{\rho^{m_1}\varrho_{\mathrm{nf}}^{s_1}\varrho_{\mathrm{Sf}}^{\ell_1}}\Big)^{-K}, \tag{327}$$

for to-be-decided $m_1,s_1,\ell_1>0$. We can then define a symbol $a_{\varepsilon,K}=\varphi_{\varepsilon,K}^2 a$. The Lie bracket $\mathsf{H}_{p[g]}a_{\varepsilon,K}$ is the same as eq. (301), with an extra factor of $\varphi^2_{\varepsilon,K}$ on the right-hand side, except we have to add the term $2\varphi_{\varepsilon,K}a\mathsf{H}_{p[g]}\varphi_{\varepsilon,K}$, which is equal to

$$\frac{4K\varepsilon}{\varepsilon+\rho^{m_1}\varrho_{\mathrm{nf}}^{s_1}\varrho_{\mathrm{Sf}}^{\ell_1}}\varphi^2_{\varepsilon,K}a\Big(\frac{\ell_1}{2}(1-\lambda)+\frac{m_1}{2}(2\hat{\eta}^2+\lambda+1)-s_1\Big) \tag{328}$$

at nFf. Note that, at $\mathcal{A}^+_+$, the bracketed term is given by $\ell_1+m_1-s_1$. Choose $s_1=2$ and $m_1,\ell_1=1/2$. Then,

$$\varphi^{-1}_{\varepsilon,K}\mathsf{H}_{p[g]}\varphi_{\varepsilon,K}|_{\mathcal{A}^+_+}<0. \tag{329}$$

Then, for all $\digamma>0$ sufficiently large, for δ sufficiently small,

$$b_\varepsilon = \varphi^2_{\varepsilon,K}\chi_\digamma(\tilde{p}[g])\chi_\digamma(\varrho_{\mathrm{nf}})\chi_\digamma(\aleph)\sqrt{\alpha-\delta a_0^{-1}a_\varepsilon - w^{-1}p_1 - \frac{2}{\varphi_{\varepsilon,K}}\mathsf{H}_{p[g]}\varphi_{\varepsilon,K}} \tag{330}$$

is a well-defined symbol near $\mathcal{A}^+_+$. Defining $e_\varepsilon=\varphi^2_{\varepsilon,K}e$, $f_\varepsilon=\varphi^2_{\varepsilon,K}f$, and so on,

$$\mathsf{H}_{p[g]}a_\varepsilon + w^{-1}p_1a_\varepsilon = (-\delta a_0^{-2}a_\varepsilon^2 - b_\varepsilon^2 + e_\varepsilon^2\varrho_{\mathrm{nf}} - f_\varepsilon^2 + h_\varepsilon)a_0 \tag{331}$$

Quantizing, with the extra weights thrown in as in eq. (305), we get operators $A_\varepsilon,B_\varepsilon,F_\varepsilon,H_\varepsilon,R_\varepsilon$, with similar essential support properties to their non-regularized counterparts, such that

$$-i[P,A_\varepsilon]+i(P-P^*)A_\varepsilon = -\delta A_\varepsilon\Lambda^2A_\varepsilon - B_\varepsilon^*B_\varepsilon + E_\varepsilon^*E_\varepsilon - F_\varepsilon^*F_\varepsilon + H_\varepsilon + R_\varepsilon. \tag{332}$$

Each of these is a uniform family of de,sc-operators with the same orders as their non-regularized counterparts. But, for each individual $\varepsilon>0$, they are regularizing operators. For each $N\in\mathbb{R}$ and tempered distribution $u\in H^{-N,-N}_{\mathrm{de,sc}}$, we can choose K sufficiently large such that the algebraic manipulations above are all justified, and via the usual strong convergence argument the estimate eq. (313) follows, now contingent only on the weak hypothesis that

$$u\in H^{-N,-N}_{\mathrm{de,sc}}. \tag{333}$$

Since N was arbitrary, we can conclude that eq. (313) holds in the strong sense, for any $u\in\mathcal{S}'$.

We now finish the conclusion of the proposition from the strong estimate eq. (313). Suppose that $u\in\mathcal{S}'$ satisfies the hypotheses:

- $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu)\cap\mathcal{A}^\varsigma_\sigma=\varnothing$,
- $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\{\rho^2+(\|\hat{\eta}\|-1)^2+(\lambda+1)^2+\varrho_{\mathrm{Sf}}^2<\epsilon_1,\epsilon_2<\varrho_{\mathrm{nf}}<\epsilon_1\}$.

Then, the first two terms on the right-hand side of eq. (313) are finite, for any N, as long as $\digamma$ is sufficiently large (and correspondingly $\mathrm{WF}'_{\mathrm{de,sc}}(G)$ is taken sufficiently small). If N is sufficiently large then, $u \in H_{\mathrm{de,sc}}^{-N,-N}$, so the final term is finite as well. If

$$\mathrm{WF}_{\mathrm{de,sc}}^{m-1/2,(-N,-N,s_{\mathrm{Sf}}-1/2,s_{\mathrm{nFf}}-1/2,-N)}(u) \cap \mathcal{A}_+^+ = \varnothing, \tag{334}$$

then (for $\digamma \gg 0$, G with small essential support) such that the third term on the right-hand side of is finite. Having checked that each term on the right-hand side is finite, we conclude that the left-hand side is finite as well. Since $\tilde{B}$ is elliptic at the radial set, we conclude that

$$\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \cap \mathcal{A}_+^+ = \varnothing. \tag{335}$$

The condition

$$\mathrm{WF}_{\mathrm{de,sc}}^{m-1/4,(-N,-N,s_{\mathrm{Sf}}-1/4,s_{\mathrm{nFf}}-1/2,-N)}(u) \cap \mathcal{A}_+^+ = \varnothing \tag{336}$$

implies eq. (334), but has the advantage that the orders in eq. (336) satisfy the threshold condition if and only if the originals do (as we are assuming as a hypothesis of the proposition). The orders in eq. (336) are (for N sufficiently large) a quarter-order smaller than those in eq. (335), so the proposition follows via an inductive argument (taking the case when all of the orders are $\leq -N$ as the base case). □

Similarly:

Proposition 5.7. *Fix signs $\varsigma, \sigma \in \{-,+\}$. Suppose that $m \in \mathbb{R}$ and $\mathsf{s} = (s_{\mathrm{Pf}}, s_{\mathrm{nPf}}, s_{\mathrm{Sf}}, s_{\mathrm{nFf}}, s_{\mathrm{Ff}}) \in \mathbb{R}^5$ satisfying $m - s_{\mathrm{nf}} + s_{\mathrm{Sf}} < 1/2$, where $s_{\mathrm{nf}} \in \{s_{\mathrm{nPf}}, s_{\mathrm{nFf}}\}$, depending on σ. For any $\epsilon_1 > 0$, there exists some $\epsilon_0 \in (0, \epsilon_1)$ such that, if $u \in \mathcal{S}'$ satisfies $\epsilon_2 \in (0, \epsilon_1)$, then, if $u \in \mathcal{S}'$ satisfies*

- $\mathrm{WF}_{\mathrm{de,sc}}^{m-1,\mathsf{s}+1}(Pu) \cap \mathcal{A}_\sigma^\varsigma = \varnothing$,
- $\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \cap \{\epsilon_2 < \rho^2 + (\|\hat{\eta}\| - 1)^2 + (\lambda + 1)^2 + \varrho_{\mathrm{Sf}}^2 < \epsilon_1, \varrho_{\mathrm{nf}} < \epsilon_1\} = \varnothing$ *for some* $\epsilon_1 > 0$ *and* $\epsilon_2 \in (0, \epsilon_1)$,

it is the case that $\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \cap \mathcal{A}_\sigma^\varsigma = \varnothing$. ■□

The argument is the same as that in the previous proposition, with a few sign switches, which result in switching the signs of the b, B-terms terms. (The signs of the terms proportional to δ then have to be switched as well.) Thus, instead of the $\|Eu\|_{L^2}$ term in eq. (313), we need to keep the $\|Fu\|_{L^2}$ term, resulting, in the $\mathcal{A}_+^+$ case, in an estimate (holding in the strong sense, for all $u \in \mathcal{S}'$) of the form

$$\begin{aligned}\|\tilde{B}u\|^2_{H_{\mathrm{de,sc}}^{m,(N,N,s_{\mathrm{Sf}},s_{\mathrm{nFf}},N)}} \preceq \|\tilde{F}u\|^2_{H_{\mathrm{de,sc}}^{m,(-N,-N,s_{\mathrm{Sf}},s_{\mathrm{nFf}},-N)}} + \|GPu\|^2_{H_{\mathrm{de,sc}}^{m-1,(-N,-N,s_{\mathrm{Sf}}+1,s_{\mathrm{nFf}}+1,-N)}} \\ + \|Gu\|^2_{H_{\mathrm{de,sc}}^{m-1/2,(-N,-N,s_{\mathrm{Sf}}-1/2,s_{\mathrm{nFf}}-1/2,-N)}} + \|u\|^2_{H_{\mathrm{de,sc}}^{-N,-N}},\end{aligned} \tag{337}$$

for $\tilde{F} \in \Psi_{\mathrm{de,sc}}^{0,0}$ with

$$\mathrm{WF}'_{\mathrm{de,sc}}(\tilde{F}) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(F). \tag{338}$$

The argument is analogous, with a few minor modifications. For instance, instead of taking $s_1 = 2$ and $m_1, \ell_1 = 1/2$ in the regularizer eq. (327), we can take $m_1, s_1, \ell_1 = 1$, so that eq. (328) has the opposite sign, which matches the switched signs of the b, B-terms. From eq. (337) (with the parameters $\digamma, G$ chosen appropriately, as above), the statement of the proposition follows.

5.3. **Propagation Through $\mathcal{N}$.** We now prove two different radial point estimates at $\mathcal{N}$. By a ray, we mean a subset of $\mathcal{N}_\sigma^\varsigma$ of the form

$$\mathcal{N}_I = \mathcal{N}_\sigma^\varsigma \cap \mathrm{cl}_{{}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}}\{|t| - r \in I\} \tag{339}$$

for some closed interval $I \subseteq [-\infty, +\infty]$ such that at least one of $\pm\infty$ in I. So, for instance, $\mathcal{N}_{\{-\infty\}} = \mathcal{N} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{sf})$ is a ray. If $-\infty \in I$, we will call $\mathcal{N}_I$ *spacelike-adjacent*, and if $+\infty \in I$, we will call $\mathcal{N}_I$ *timelike-adjacent.* As we will see below, we can only propagate in one direction on each type of ray for each pair of admissible Sobolev orders $(m,\mathsf{s})\in\mathbb{R}\times\mathbb{R}^5$. The one exception is $\mathcal{N}^\varsigma_\sigma=\mathcal{N}_{[-\infty,+\infty]}$ itself, which is both spacelike-adjacent and timelike-adjacent. We say that $\mathcal{N}_I$ is strictly spacelike-adjacent or strictly timelike-adjacent if $\mathcal{N}_I\neq\mathcal{N}$.

The timelike-adjacent case is:

Proposition 5.8. *Fix signs $\varsigma,\sigma\in\{-,+\}$, and let $\mathcal{N}_I$ denote a strictly timelike-adjacent ray of $\mathcal{N}^\varsigma_\sigma$, which we can write using the coordinates eq. (249) (over $\Omega_{\mathrm{nfTf},\sigma,T}$, for some large $T>0$) as*

$$\mathcal{N}_I=\{\varrho_{\mathrm{Tf}}\leq\bar{\varrho}_{\mathrm{Tf}},\rho=0,s=0,\beth=0\} \tag{340}$$

for some $\bar{\varrho}_{\mathrm{Tf}}>0$, where $\beth$ is as in Proposition 4.5. Let $m\in\mathbb{R}$ and $\mathsf{s}\in\mathbb{R}^5$ satisfy $m<s_{\mathrm{nFf}}+1$, where $s\in\{s_{\mathrm{nPf}},s_{\mathrm{nFf}}\}$, depending on the sign σ. Suppose that $u\in\mathcal{S}'$ satisfies

- $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu)\cap\mathcal{N}_I=\varnothing$ *and*
- $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\{\beth,s^2<\epsilon_1,\varrho_{\mathrm{Tf}}\leq\bar{\varrho}_{\mathrm{Tf}}+\epsilon_1,\epsilon_2<\rho<\epsilon_1\}=\varnothing$

for some $\epsilon_1>0$ and sufficiently small $\epsilon_2\in(0,\epsilon_1)$. Then, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\mathcal{N}_I=\varnothing$. ■

Proof. We handle the case $\varsigma,\sigma=+$, the others being analogous. Let $a_0=\varrho_{\mathrm{df}}^{m_0}\varrho_{\mathrm{nFf}}^{s_0}\varrho_{\mathrm{Tf}}^{\ell_0}$, where $m_0=1-2m$, $s_0=-1-2s_{\mathrm{nFf}}$, $\ell_0=-1-2s_{\mathrm{Tf}}$. Then, by Proposition 4.3, we have

$$\mathsf{H}_{p[g]}a_0=\tilde{\alpha}a_0 \tag{341}$$

for a symbol $\tilde{\alpha}$ such that $\tilde{\alpha}=\alpha-s\ell_0$ at $\mathcal{N}_I$, where $\alpha=\alpha(m_0,s_0)$ is as defined in that proposition. At $\mathcal{N}^+_+$, $s=0$ and so $(\alpha+s\ell_0)>0$ (by Proposition 4.3 and the observation that $m_0>s_0 \iff m<s_{\mathrm{nFf}}+1$).

Let $\Upsilon\in\mathbb{R}$, $\digamma,\digamma'>0$, and $\tilde{\varrho}_{\mathrm{Tf}}=\Upsilon\varrho_{\mathrm{nf}}+\varrho_{\mathrm{Tf}}$. Define $a\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ by

$$a=\chi_\digamma(\tilde{p}[g])^2\chi_{\digamma'}(\rho_{\mathrm{df}})^2\chi_\digamma(\beth)^2\chi_\digamma(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})^2a_0 \tag{342}$$

near $\mathcal{N}_I$, and we can take a to be supported nearby. Then, near $\mathcal{N}_I$,

$$\begin{aligned}\mathsf{H}_{p[g]}a=\tilde{\alpha}a&-2\alpha_{\mathrm{df}}\chi_\digamma(\tilde{p}[g])^2\chi_{0,\digamma}(\varrho_{\mathrm{df}})^2\chi_{\digamma'}(\beth)^2\chi_\digamma(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})^2\varrho_{\mathrm{df}}a_0\\&+2\chi_\digamma(\tilde{p}[g])^2\chi_\digamma(\varrho_{\mathrm{df}})^2\chi_{0,\digamma'}(\beth)^2\chi_\digamma(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})^2(4\beth-F_2)a_0\\&+4(s\varrho_{\mathrm{Tf}}+(1-s)\Upsilon\varrho_{\mathrm{nf}}+\varrho_{\mathrm{nf}}\varrho_{\mathrm{Tf}}c)\chi_\digamma(\tilde{p}[g])^2\chi_\digamma(\varrho_{\mathrm{df}})^2\chi_{\digamma'}(\beth)^2\chi_{0,\digamma}(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})^2a_0\\&+2\chi_\digamma'(\tilde{p}[g])\chi_\digamma(\tilde{p}[g])\chi_\digamma(\varrho_{\mathrm{df}})^2\chi_{\digamma'}(\beth)^2\chi_\digamma(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})^2\tilde{q}\tilde{p}[g]a_0,\end{aligned} \tag{343}$$

where $\alpha_{\mathrm{df}}=\alpha(1,0)$, $\alpha_{\mathrm{nf}}=\alpha(0,1)$, and $c\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ comes from applying $\varrho_{\mathrm{nf}}^{-1}\varrho_{\mathrm{Tf}}^{-1}(\mathsf{H}_{p[g]}-\mathsf{H}_p)$ to $\tilde{\varrho}_{\mathrm{Tf}}$. (Here, we are assuming without loss of generality that ϱ_{df} agrees with ρ near $\mathcal{N}_I$.)

By Proposition 4.3, $\alpha_{\mathrm{df}}>0$ near $\mathcal{N}^+_+$. By choosing $\digamma$ sufficiently large, by Lemma 4.4, we can write $s=s_1\tilde{p}+s_2(\hat{\eta}^2+\mathsf{m}^2\rho^2)$ nearby, where s_1,s_2 are as in the lemma. We want to work with $\tilde{p}[g]$, not $\tilde{p}$, so we will write this as

$$s=s_1\tilde{p}[g]+\varrho_{\mathrm{nf}}\varrho_{\mathrm{Tf}}c_2+s_2(\hat{\eta}^2+\mathsf{m}^2\rho^2) \tag{344}$$

for $c_2\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ defined by $c_2=s_1(\tilde{p}-\tilde{p}[g])\varrho_{\mathrm{nf}}^{-1}\varrho_{\mathrm{Tf}}^{-1}$. The key feature of eq. (344) is that each term on the right-hand side is amenable to the positive commutator argument. The term proportional to $\tilde{p}[g]$, when quantized and applied to u, will yield a term involving the forcing. The term involving c_2 is suppressed by a factor of ϱ_{nf}, which we will be able to dominate by a term of semidefinite sign by choosing Υ large. Finally, the terms $s_2\hat{\eta}^2,s_2\mathsf{m}^2\rho^2$ have a semidefinite sign, because s_2 does.

For all $\Upsilon>0$ sufficiently large and $\digamma,\digamma'>0$ sufficiently large, for δ sufficiently small (depending on $\digamma,\digamma'$), we can define symbols

$$b,e,f,g,h,z\in S^{0,0}_{\mathrm{de,sc}} \tag{345}$$

such that

$$\mathsf{H}_{p[g]}a + w^{-1}p_1 a = \Big(\delta a_0^{-2}a^2 + b^2 - \varrho_{\mathrm{df}}e^2 + f^2 + \rho_{\mathrm{nf}}z^2 + \varrho_{\mathrm{Tf}}\sum_{i=1}^{d} g_i^2 + h\tilde{p}[g]\Big)a_0 \tag{346}$$

everywhere, with the following definitions:

$$\begin{aligned} b &= \chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{df}})\chi_{F'}(\sqsupset)\chi_F(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})(\tilde{\alpha}-\delta a_0^{-1}a + w^{-1}p_1)^{1/2}, \\ e &= \sqrt{2}\alpha_{\mathrm{df}}^{1/2}\chi_F(\tilde{p}[g])\chi_{0,F}(\varrho_{\mathrm{df}})\chi_{F'}(\sqsupset)\chi_F(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\}), \\ f &= \sqrt{2}\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{df}})\chi_{0,F'}(\sqsupset)\chi(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})(4\sqsupset - F_2)^{1/2}, \end{aligned} \tag{347}$$

and, for $i = 1, \ldots, d-1$,

$$g_i = 2\sqrt{s_2}\chi(\tilde{p}[g])\chi_F(\varrho_{\mathrm{df}})\chi_{F'}(\sqsupset)\chi_{0,F}(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})\hat{\eta}_i \tag{348}$$

$$g_d = 2\sqrt{s_2}\chi(\tilde{p}[g])\chi_F(\varrho_{\mathrm{df}})\chi_{F'}(\sqsupset)\chi_{0,F}(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})\mathsf{m}\rho, \tag{349}$$

(recall that $s_2 > 0$ on $\mathcal{N}_+^+$), and, finally,

$$z = 2\sqrt{\varrho_{\mathrm{Tf}}^2 c_2 + (1-s)\Upsilon + \varrho_{\mathrm{Tf}}c}\cdot\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{df}})\chi_{F'}(\sqsupset)\chi_{0,F}(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\}) \tag{350}$$

(recall that $1 - s = 1$ on $\mathcal{N}_+^+$) and

$$\begin{aligned} h = {}& 2\chi_F'(\tilde{p}[g])\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{df}})^2\chi_{F'}(\sqsupset)^2\chi_F(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})^2\tilde{q} \\ &+ 4s_1\varrho_{\mathrm{Tf}}\chi_F(\tilde{p}[g])^2\chi_F(\varrho_{\mathrm{df}})^2\chi_{F'}(\sqsupset)^2\chi_{0,F}(\max\{0,\tilde{\varrho}_{\mathrm{Tf}}-\bar{\varrho}_{\mathrm{Tf}}\})^2 \end{aligned} \tag{351}$$

near $\mathcal{N}_+^+$.

If F, F' are sufficiently large, then $\mathrm{WF}'_{\mathrm{de,sc}}(e) \subseteq \{\sqsupset, s^2 < \epsilon_1, \varrho_{\mathrm{Tf}} \leq \bar{\varrho}_{\mathrm{Tf}} + \epsilon_1, \epsilon_2 < \rho < \epsilon_1\}$.

Quantizing, we get $A = (1/2)(\mathrm{Op}(a) + \mathrm{Op}(a)^*) \in \Psi_{\mathrm{de,sc}}^{-m_0,(-\infty,-\infty,-\infty,-s_0,-\ell_0)}$,

$$\begin{aligned} B &= \mathrm{Op}(w^{1/2}a_0^{1/2}b) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}, \\ E &= \mathrm{Op}(w^{1/2}a_0^{1/2}\varrho_{\mathrm{df}}^{1/2}e) \in \Psi_{\mathrm{de,sc}}^{-\infty,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}, \\ F &= \mathrm{Op}(w^{1/2}a_0^{1/2}f) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}, \\ G_i &= \mathrm{Op}(w^{1/2}a_0^{1/2}\varrho_{\mathrm{Tf}}^{1/2}g_i) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},-\infty)}, \\ Z &= \mathrm{Op}(w^{1/2}a_0^{1/2}\varrho_{\mathrm{nf}}^{1/2}z) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}}-1,-\infty)} \end{aligned} \tag{352}$$

$H = \mathrm{Op}(wa_0h) \in \Psi_{\mathrm{de,sc}}^{2m,(-\infty,-\infty,-\infty,2s_{\mathrm{nFf}},2s_{\mathrm{Tf}})}$, and $R \in \Psi_{\mathrm{de,sc}}^{-m_0,(-\infty,-\infty,-\infty,-2-s_0,-2-\ell_0)}$ such that

$$-i[P,A] + i(P^*-P) = \delta A\Lambda^2 A + B^*B - E^*E + F^*F + \sum_{i=1}^{d} G_i^*G_i + Z^*Z + H\tilde{P} + R \tag{353}$$

for

$$\Lambda = (1/2)(\mathrm{Op}(w^{1/2}a_0^{-1/2}) + \mathrm{Op}(w^{1/2}a_0^{-1/2})) \in \Psi_{\mathrm{de,sc}}^{1-m,(-1/2,-1/2,-1/2,-1-s_{\mathrm{nFf}},-1-s_{\mathrm{Tf}})}, \tag{354}$$

with the operators A, B, E, F, G_i, Z, H all having essential supports contained within $\operatorname{supp} a$.

The argument proceeds as usual from here, where the key observation is that the F^*F term, $G_i^*G_i$, Z^*Z terms have the same sign as the B^*B term (and therefore can ultimately be discarded from the estimate) except we estimate the contribution $\langle u, H\tilde{P}u\rangle$ in the following way: for $u \in \mathcal{S}$,

$$|\langle u, H\tilde{P}u\rangle| \preceq \|Ou\|^2_{H_{\mathrm{de,sc}}^{m-1,\mathsf{s}-1}} + \|OPu\|^2_{H_{\mathrm{de,sc}}^{m-1,\mathsf{s}+1}} + \|Pu\|^2_{H_{\mathrm{de,sc}}^{-N,-N}} \tag{355}$$

for some $O \in \Psi^{0,0}_{\mathrm{de,sc}}$ with essential support contained near $\mathcal{N}^+_+$ and for $N \in \mathbb{N}$. Thus, rather than controlling this term with elliptic regularity as before, we use the assumption

$$\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap \mathcal{N}_I = \varnothing, \tag{356}$$

which implies that

$$\|OPu\|_{H^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}} < \infty \tag{357}$$

as long as F is sufficiently large. The end result, after carrying out the regularization argument and the typical inductive argument, is the estimate, holding in the strong sense for all $u \in \mathcal{S}'$,

$$\|\tilde{B}u\|^2_{H^{m,(N,N,N,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}_{\mathrm{de,sc}}} \preceq \|\tilde{E}u\|^2_{H^{m,(-N,-N,-N,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}_{\mathrm{de,sc}}} + \|QPu\|^2_{H^{m-1,(-N,-N,-N,s_{\mathrm{nFf}}+1,s_{\mathrm{Ff}}+1)}_{\mathrm{de,sc}}} + \|u\|^2_{H^{-N,-N}_{\mathrm{de,sc}}}, \tag{358}$$

for some $\tilde{B} \in \Psi^{0,0}_{\mathrm{de,sc}}$ elliptic along $\mathcal{N}_I$, $\tilde{E} \in \Psi^{0,0}_{\mathrm{de,sc}}$ satisfying

$$\mathrm{WF}'_{\mathrm{de,sc}}(\tilde{E}) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(E), \tag{359}$$

and some Q dependent on the other operators whose essential support can be made to be an arbitrarily small neighborhood of $\mathcal{N}_I$ by making F, F' larger. The estimate eq. (358) finishes the proof. □

Proposition 5.9. *Fix signs $\varsigma, \sigma \in \{-,+\}$, and let $\mathcal{N}_I$ denote a strictly spacelike-adjacent ray of $\mathcal{N}^\varsigma_\sigma$, which we can write using the coordinates eq. (252) (over $\Omega_{\mathrm{nfSf},\sigma,R}$, for some large $R > 0$) as*

$$\mathcal{N}_I = \{\varrho_{\mathrm{Sf}} \leq \bar{\varrho}_{\mathrm{Sf}}, \rho = 0, \lambda = 1, \sqsupset = 0\} \tag{360}$$

for some $\bar{\varrho}_{\mathrm{Sf}} > 0$, where $\sqsupset$ is as in Proposition 4.5. Let $m \in \mathbb{R}$ and $\mathsf{s} \in \mathbb{R}^5$ satisfy $m > s_{\mathrm{nFf}} + 1$, where $s \in \{s_{\mathrm{nPf}}, s_{\mathrm{nFf}}\}$, depending on the sign σ. Suppose that $u \in \mathcal{S}'$ satisfies

- $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap \mathcal{N}_I = \varnothing$ *and*
- $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \{\epsilon_2 < \sqsupset < \epsilon_1, \varrho_{\mathrm{Sf}} \leq \bar{\varrho}_{\mathrm{Sf}} + \epsilon_1, \rho^2 + (\lambda-1)^2 < \epsilon_1\} = \varnothing$

for some $\epsilon_1 > 0$ and sufficiently small $\epsilon_2 \in (0, \epsilon_1)$. Then, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{N}_I = \varnothing$. ■□

The proof is analogous argument to that above, with the usual sign switches.

For the special case of the full ray, the conclusions of both propositions hold:

Proposition 5.10. *Fix signs $\varsigma, \sigma \in \{-,+\}$. Suppose that $u \in \mathcal{S}'$ satisfies $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap \mathcal{N}^\varsigma_\sigma = \varnothing$ and at least one of*

- *$m < s_{\mathrm{nFf}} + 1$ and $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)$ is disjoint from a neighborhood of $\mathcal{N}^\varsigma_\sigma$ of the form $\{\sqsupset, s^2 < \epsilon_1, \varrho_{\mathrm{Tf}} \leq \bar{\varrho}_{\mathrm{Tf}} + \epsilon_1, \epsilon_2 < \rho < \epsilon_1\}$,*
- *$m > s_{\mathrm{nFf}} + 1$ and $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)$ is disjoint from a neighborhood $\mathcal{N}^\varsigma_\sigma$ of the form $\{\epsilon_2 < \sqsupset < \epsilon_1, \varrho_{\mathrm{Sf}} \leq \bar{\varrho}_{\mathrm{Sf}} + \epsilon_1, \rho^2 + (\lambda-1)^2 < \epsilon_1\}$,*

hold. Then, $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{N}^\varsigma_\sigma = \varnothing$. ■□

The proof is analogous those above, except we no longer need the cutoff along null infinity, and we must make sure that the symbols are well-defined in both coordinate patches.

5.4. **Propagation Through $\mathcal{K}$.**

Proposition 5.11. *Fix signs $\varsigma, \sigma \in \{-,+\}$. Suppose that $m \in \mathbb{R}$ and $\mathsf{s} = (s_{\mathrm{Pf}}, s_{\mathrm{nPf}}, s_{\mathrm{Sf}}, s_{\mathrm{nFf}}, s_{\mathrm{Ff}}) \in \mathbb{R}^5$ satisfying $m + s_{\mathrm{nf}} - 2s_{\mathrm{Sf}} > 1$, where $s_{\mathrm{nf}} \in \{s_{\mathrm{nPf}}, s_{\mathrm{nFf}}\}$, depending on σ. Then, if $u \in \mathcal{S}'$ satisfies*

- $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu) \cap \mathcal{K}^\varsigma_\sigma = \varnothing$,
- $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \{\rho^2 + \hat{\eta}^2 + (\lambda+3)^2 + \varrho^2_{\mathrm{nf}} < \epsilon_1, \epsilon_2 < \varrho_{\mathrm{Sf}} < \epsilon_1\} = \varnothing$ *for some $\epsilon_1 > 0$ and $\epsilon_2 \in (0, \epsilon_1)$ sufficiently small,*

it is the case that $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \mathcal{K}^\varsigma_\sigma = \varnothing$. ■

Proof. We handle the case $\varsigma,\sigma=+$, the other three being analogous. Let $a_0=\varrho_{\mathrm{df}}^{m_0}\varrho_{\mathrm{nFf}}^{s_0}\varrho_{\mathrm{Sf}}^{\ell_0}$, where $m_0=1-2m$, $s_0=-1-2s_{\mathrm{nFf}}$, $\ell_0=-1-2s_{\mathrm{Sf}}$. Exactly at $\mathcal{K}^+_+$, $\lambda=-3$ and $\hat{\eta}^2=0$, so eq. (254) yields

$$\mathsf{H}_{p[g]}a_0=\alpha a_0 \tag{361}$$

for some $\alpha\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ equal to $2(-m_0-s_0+2\ell_0)=4(m+s_{\mathrm{nFf}}-2s_{\mathrm{Sf}}-1)$ at $\mathcal{K}^+_+$. Note that $\alpha>0$ near $\mathcal{K}^+_+$.

Define $a\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ by

$$a=\chi_F(\tilde{p}[g])^2\chi_F(\varrho_{\mathrm{Sf}})^2\chi_F(\daleth)^2a_0 \tag{362}$$

near $\mathcal{K}^+_+$, where $\daleth$ is as in Proposition 4.6, with a supported near $\mathcal{K}^+_+$. We compute

$$\begin{aligned}\mathsf{H}_{p[g]}a&=\alpha a-2(1-\lambda+\varrho_{\mathrm{nf}}c)\chi_F(\tilde{p}[g])^2\chi_{0,F}(\varrho_{\mathrm{Sf}})^2\chi_F(\daleth)^2\varrho_{\mathrm{Sf}}a_0\\&+2\chi_F(\tilde{p}[g])^2\chi_F(\varrho_{\mathrm{Sf}})^2\chi_{0,F}(\daleth)^2(4\daleth+E_3-F_3)a_0+2\chi_F'(\tilde{p}[g])\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{Sf}})^2\chi_F(\daleth)^2\tilde{q}\tilde{p}[g]a_0\end{aligned} \tag{363}$$

for some $c\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$. For all $F>0$ sufficiently large, for δ sufficiently small, we can define symbols $b,e,f,h\in S^{0,0}_{\mathrm{de,sc}}$ such that

$$\mathsf{H}_{p[g]}a+w^{-1}p_1a=(\delta a_0^{-2}a^2+b^2-e^2\varrho_{\mathrm{Sf}}+f^2+h)a_0, \tag{364}$$

$$H_{p[g]}a+p_1a=(\delta a_0^{-2}a^2+b^2-e^2\varrho_{\mathrm{Sf}}+f^2+h)wa_0 \tag{365}$$

everywhere, with $b=\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{Sf}})\chi_F(\daleth)(\alpha-\delta a_0^{-1}a+w^{-1}p_1)^{1/2}$,

$$e=\sqrt{2(1-\lambda+\varrho_{\mathrm{nf}}c)}\chi_F(\tilde{p}[g])\chi_{0,F}(\varrho_{\mathrm{Sf}})\chi_F(\daleth) \tag{366}$$

(as $\lambda=3$ at $\mathcal{K}^+_+$, the function under the square root is positive near the radial set),

$$f=\sqrt{2}\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{Sf}})\chi_{0,F}(\daleth)(4\daleth+E_3-F_3)^{1/2}, \tag{367}$$

and $h=2\chi_F'(\tilde{p}[g])\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{Sf}})^2\chi_F(\daleth)^2\tilde{q}\tilde{p}[g]$ near $\mathcal{K}^+_+$.

Quantizing, we get $A=(1/2)(\mathrm{Op}(a)+\mathrm{Op}(a)^*)\in\Psi_{\mathrm{de,sc}}^{-m_0,(-\infty,-\infty,-\ell_0,-s_0,-\infty)}$,

$$\begin{aligned}B&=\mathrm{Op}(w^{1/2}a_0^{1/2}b)\in\Psi_{\mathrm{de,sc}}^{(1-m_0)/2,(-\infty,-\infty,s_{\mathrm{Sf}},s_{\mathrm{nFf}},-\infty)},\\E&=\mathrm{Op}(w^{1/2}a_0^{1/2}\varrho_{\mathrm{Sf}}e)\in\Psi_{\mathrm{de,sc}}^{(1-m_0)/2,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},-\infty)},\\F&=\mathrm{Op}(w^{1/2}a_0^{1/2}f)\in\Psi_{\mathrm{de,sc}}^{(1-m_0)/2,(-\infty,-\infty,s_{\mathrm{Sf}},s_{\mathrm{nFf}},-\infty)},\end{aligned} \tag{368}$$

$h=\mathrm{Op}(wa_0h)\in\Psi_{\mathrm{de,sc}}^{2m,(-\infty,-\infty,2s_{\mathrm{Sf}},2s_{\mathrm{nFf}},-\infty)}$, and $R\in\Psi_{\mathrm{de,sc}}^{-m_0,(-\infty,-2-s_0,-2-\ell_0,-\infty,-\infty)}$ such that

$$-i[P,A]+i(P-P^*)A=\delta A\Lambda^2A+B^*B-E^*E+F^*F+H+R, \tag{369}$$

where Λ is as in Proposition 5.6. If F is sufficiently large, then

$$\mathrm{WF}'_{\mathrm{de,sc}}(E)\subseteq\{\rho^2+\hat{\eta}^2+(\lambda+3)^2+\varrho_{\mathrm{nf}}^2<\epsilon_1,\epsilon_2<\varrho_{\mathrm{Sf}}<\epsilon_1\}. \tag{370}$$

The proof now proceeds as usual, where the key observation is that the F^*F term has the same sign as the B^*B term and therefore can be ultimately discarded from the estimates. Thus, for some

$$G,\tilde{B},\tilde{E}\in\Psi^{0,0}_{\mathrm{de,sc}} \tag{371}$$

with $\tilde{B}$ elliptic at $\mathcal{K}^+_+$ and $\mathrm{WF}'_{\mathrm{de,sc}}(\tilde{E})\subseteq\mathrm{WF}'_{\mathrm{de,sc}}(E)$, we have, for all $u\in\mathcal{S}'$, the estimate

$$\begin{aligned}\|\tilde{B}u\|^2_{H_{\mathrm{de,sc}}^{m,(N,N,s_{\mathrm{Sf}},s_{\mathrm{nFf}},N)}}\preceq\|\tilde{E}u\|^2_{H_{\mathrm{de,sc}}^{m,(-N,-N,-N,s_{\mathrm{nFf}},-N)}}+\|GPu\|^2_{H_{\mathrm{de,sc}}^{m-1,(-N,-N,s_{\mathrm{Sf}}+1,s_{\mathrm{nFf}}+1,-N)}}\\+\|u\|^2_{H_{\mathrm{de,sc}}^{-N,-N}},\end{aligned} \tag{372}$$

holding in the strong sense, where the essential support of G can be made to be in an arbitrarily small neighborhood of $\mathcal{K}_+^+$ by making $\digamma$ larger. This estimate completes the proof. □

Similarly:

Proposition 5.12. *Fix signs $\varsigma,\sigma\in\{-,+\}$. Suppose that $m\in\mathbb{R}$ and $\mathsf{s}=(s_{\mathrm{Pf}},s_{\mathrm{nPf}},s_{\mathrm{Sf}},s_{\mathrm{nFf}},s_{\mathrm{Ff}})\in\mathbb{R}^5$ satisfying $m+s_{\mathrm{nf}}-2s_{\mathrm{Sf}}<1$, where $s_{\mathrm{nf}}\in\{s_{\mathrm{nPf}},s_{\mathrm{nFf}}\}$, depending on σ. Then, if $u\in\mathcal{S}'$ satisfies*

- $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu)\cap\mathcal{K}^\varsigma_\sigma=\varnothing$,
- $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\{\epsilon_2<\rho^2+\hat{\eta}^2+(\lambda+3)^2+\varrho_{\mathrm{nf}}^2<\epsilon_1,\varrho_{\mathrm{Sf}}<\epsilon_1\}=\varnothing$ *for some* $\epsilon_1>0$ *and* $\epsilon_2\in(0,\epsilon_1)$ *sufficiently small,*

it is the case that $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\mathcal{K}^\varsigma_\sigma=\varnothing$. ■□

The proof follows the proof of Proposition 5.11, except the sign of the B^*B term (along with the sign of the $\delta A\Lambda^2A$ term) in eq. (369) has to be switched, with results in having to keep the F^*F term in estimates rather than the E^*E term.

5.5. Propagation Through $\mathcal{C}$.

Proposition 5.13. *Fix signs $\varsigma,\sigma\in\{-,+\}$. Suppose that $m\in\mathbb{R}$ and $\mathsf{s}=(s_{\mathrm{Pf}},s_{\mathrm{nPf}},s_{\mathrm{Sf}},s_{\mathrm{nf}},s_{\mathrm{Tf}})\in\mathbb{R}^5$ satisfying $m+s_{\mathrm{nFf}}-2s_{\mathrm{Ff}}<1$, where $s_{\mathrm{Tf}}\in\{s_{\mathrm{Pf}},s_{\mathrm{Ff}}\}$, depending on σ. Then, if $u\in\mathcal{S}'$ satisfies*

- $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu)\cap\mathcal{C}^\varsigma_\sigma=\varnothing$,
- $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\{\epsilon_2<\rho^2+\hat{\eta}^2+(s-2)^2+\varrho_{\mathrm{nf}}^2<\epsilon_1,\varrho_{\mathrm{Tf}}<\epsilon_1\}=\varnothing$ *for some* $\epsilon_1>0$ *and* $\epsilon_2\in(0,\epsilon_1)$ *sufficiently small,*

it is the case that $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)\cap\mathcal{C}^\varsigma_\sigma=\varnothing$. ■

Proof. We handle the case $\varsigma,\sigma=+$, the other three being analogous. Let $a_0=\varrho_{\mathrm{df}}^{m_0}\varrho_{\mathrm{nFf}}^{s_0}\varrho_{\mathrm{Tf}}^{\ell_0}$, where $m_0=1-2m$, $s_0=-1-2s_{\mathrm{nFf}}$, $\ell_0=-1-2s_{\mathrm{Ff}}$. Without loss of generality, we assume that $\varrho_{\mathrm{df}}=\rho$ locally. Then, $\mathsf{H}_{p[g]}a_0=\alpha a_0$ for $\alpha\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ given by

$$\frac{\alpha}{2}=(\hat{\eta}^2+(s-1)^2)m_0+(s-1)s_0-s\ell_0 \tag{373}$$

over null infinity. Exactly at $\mathcal{C}_+^+$, $s=2$ and $\hat{\eta}=0$, so $\alpha=m_0+s_0-2\ell_0=-2(m+s_{\mathrm{nFf}}-2s_{\mathrm{Ff}}-1)>0$ there.

Define $a\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ by $a=\chi_\digamma(\tilde{p}[g])^2\chi_\digamma(\varrho_{\mathrm{Sf}})^2\chi_\digamma(\daleth)^2a_0$ near $\mathcal{C}_+^+$, where $\daleth$ is in Proposition 4.7, with a supported near $\mathcal{C}_+^+$. We calculate

$$\begin{split}\mathsf{H}_{p[g]}a&=\alpha a+4(s+\varrho_{\mathrm{nf}}c)\chi_\digamma(\tilde{p}[g])^2\chi_{0,\digamma}(\varrho_{\mathrm{Tf}})^2\chi_\digamma(\daleth)^2\varrho_{\mathrm{Tf}}a_0\\&\quad-2\chi_\digamma(\tilde{p}[g])^2\chi_\digamma(\varrho_{\mathrm{Tf}})^2\chi_{0,\digamma}(\daleth)^2(4\daleth+E_4+F_4)a_0+2\chi_\digamma'(\tilde{p}[g])\chi_\digamma(\tilde{p}[g])\chi_\digamma(\varrho_{\mathrm{Sf}})^2\chi_\digamma(\daleth)^2\tilde{q}\tilde{p}[g]a_0\end{split} \tag{374}$$

for some $c\in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$.

For all $\digamma>0$ sufficiently large, for δ sufficiently small, we can define symbols $b,e,f,h\in S^{0,\mathbf{0}}_{\mathrm{de,sc}}$ such that

$$\mathsf{H}_{p[g]}ap+w^{-1}p_1a=(\delta a_0^{-2}a^2+b^2+e^2\varrho_{\mathrm{Tf}}-f^2+h)a_0, \tag{375}$$

$$H_{p[g]}a+p_1a=(\delta a_0^{-2}a^2+b^2+e^2\varrho_{\mathrm{Tf}}-f^2+h)wa_0 \tag{376}$$

everywhere, with $b=\chi_\digamma(\tilde{p}[g])\chi_\digamma(\varrho_{\mathrm{Tf}})\chi_\digamma(\daleth)(\alpha-\delta a_0^{-1}a+w^{-1}p_1)^{1/2}$,

$$e=2\sqrt{s+\varrho_{\mathrm{nf}}c}\,\chi_\digamma(\tilde{p}[g])\chi_{0,\digamma}(\varrho_{\mathrm{Tf}})\chi_\digamma(\daleth)\varrho_{\mathrm{Tf}}^{1/2}, \tag{377}$$

$f = \sqrt{2}\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{Tf}})\chi_{0,F}(\daleth)(4\daleth + E_3 + F_4)^{1/2}$, and $h = 2\chi'_F(\tilde{p}[g])\chi_F(\tilde{p}[g])\chi_F(\varrho_{\mathrm{Sf}})^2\chi_F(\daleth)^2\tilde{q}\tilde{p}[g]$ near $\mathcal{C}^+_+$. If F is sufficiently large, then

$$\mathrm{WF}'_{\mathrm{de,sc}}(F) \subseteq \{\epsilon_2 < \rho^2 + \hat{\eta}^2 + (s-2)^2 + \varrho_{\mathrm{nf}}^2 < \epsilon_1, \varrho_{\mathrm{Tf}} < \epsilon_1\}. \tag{378}$$

Quantizing, we get $A = (1/2)(\mathrm{Op}(a) + \mathrm{Op}(a)^*) \in \Psi_{\mathrm{de,sc}}^{-m_0,(-\infty,-\infty,-\infty,-s_0,-\ell_0)}$,

$$\begin{aligned} B &= \mathrm{Op}(w^{1/2}a_0^{1/2}b) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}, \\ E &= \mathrm{Op}(w^{1/2}a_0^{1/2}\varrho_{\mathrm{Tf}}e) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},-\infty)}, \\ F &= \mathrm{Op}(w^{1/2}a_0^{1/2}f) \in \Psi_{\mathrm{de,sc}}^{m,(-\infty,-\infty,-\infty,s_{s_{\mathrm{nFf}}},s_{\mathrm{Ff}})}, \\ h &= \mathrm{Op}(wa_0h) \in \Psi_{\mathrm{de,sc}}^{2m,(-\infty,-\infty,-\infty,2s_{\mathrm{nFf}},2s_{\mathrm{Ff}})}, \end{aligned} \tag{379}$$

and $R \in \Psi_{\mathrm{de,sc}}^{-m_0,(-\infty,-\infty,-\infty,2s_{\mathrm{nFf}}-1,2s_{\mathrm{Ff}}-1)}$ such that

$$-i[P,A] + i(P-P^*)A = \delta A\Lambda^2 A + B^*B + E^*E - F^*F + H + R \tag{380}$$

for $\Lambda = (1/2)(\mathrm{Op}(w^{1/2}a_0^{-1/2}) + \mathrm{Op}(w^{1/2}a_0^{-1/2})^*) \in \Psi_{\mathrm{de,sc}}^{1-m,(-1/2,-1/2,-1/2,-1-s_{\mathrm{nFf}},-1-s_{\mathrm{Ff}})}$.

The proof now proceeds as usual, where the key observation is that the E^*E term has the same sign as the B^*B term and therefore can be ultimately discarded from the estimates. Thus, for some

$$G, \tilde{B}, \tilde{F} \in \Psi_{\mathrm{de,sc}}^{0,\mathbf{0}} \tag{381}$$

with $\tilde{B}$ elliptic at $\mathcal{C}^+_+$ and $\mathrm{WF}'_{\mathrm{de,sc}}(\tilde{F}) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(F)$, we have, for all $u \in \mathcal{S}'$, the estimate

$$\begin{aligned} \|\tilde{B}u\|^2_{H_{\mathrm{de,sc}}^{m,(N,N,N,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}} \preceq \|\tilde{F}u\|^2_{H_{\mathrm{de,sc}}^{m,(-N,-N,-N,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}} + \|GPu\|^2_{H_{\mathrm{de,sc}}^{m-1,(-N,-N,-N,s_{\mathrm{nFf}}+1,s_{\mathrm{Sf}}+1)}} \\ + \|u\|^2_{H_{\mathrm{de,sc}}^{-N,-N}}, \end{aligned} \tag{382}$$

holding in the strong sense, where the essential support of G can be made to be in an arbitrarily small neighborhood of $\mathcal{C}^+_+$ by making F larger. This estimate completes the proof. □

Similarly:

Proposition 5.14. *Fix signs $\varsigma, \sigma \in \{-,+\}$. Suppose that $m \in \mathbb{R}$ and $\mathsf{s} = (s_{\mathrm{Pf}}, s_{\mathrm{nPf}}, s_{\mathrm{Sf}}, s_{\mathrm{nFf}}, s_{\mathrm{Ff}}) \in \mathbb{R}^5$ satisfying $m + s_{\mathrm{nf}} - 2s_{\mathrm{Tf}} < 1$, where $s_{\mathrm{Tf}} \in \{s_{\mathrm{Pf}}, s_{\mathrm{Ff}}\}$, depending on σ. Then, if $u \in \mathcal{S}'$ satisfies*

- $\mathrm{WF}_{\mathrm{de,sc}}^{m-1,\mathsf{s}+1}(Pu) \cap \mathcal{C}^\varsigma_\sigma = \varnothing$,
- $\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \cap \{\rho^2 + \hat{\eta}^2 + (s-2)^2 + \varrho_{\mathrm{nf}}^2 < \epsilon_1, \epsilon_2 < \varrho_{\mathrm{Tf}} < \epsilon_1\} = \varnothing$ *for some $\epsilon_1 > 0$ and $\epsilon_2 \in (0, \epsilon_1)$ sufficiently small,*

it is the case that $\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \cap \mathcal{C}^\varsigma_\sigma = \varnothing$. ■□

6. The radial set $\mathcal{R}$

Let P be as in the previous section. We encode the radial point estimate at $\mathcal{R}$ in the qualitative statements:

Theorem 4 (Propagation out of $\mathcal{R}$)**.** *Suppose that $m \in \mathbb{R}$ and $\mathsf{s} \in (s_{\mathrm{Pf}}, s_{\mathrm{nPf}}, s_{\mathrm{Sf}}, s_{\mathrm{nFf}}, s_{\mathrm{Ff}}) \in \mathbb{R}^5$ and $s_{\mathrm{Pf},0}, s_{\mathrm{Ff},0} \in \mathbb{R}$ satisfy*

$$s_{\mathrm{Pf}} > s_{\mathrm{Pf},0} > -1/2 \quad \textit{and } s_{\mathrm{Ff}} > s_{\mathrm{Ff},0} > -1/2. \tag{383}$$

Let $\mathsf{s}_0 = (s_{\mathrm{Pf},0}, s_{\mathrm{nPf},0}, s_{\mathrm{Sf},0}, s_{\mathrm{nFf},0}, s_{\mathrm{Ff},0}) \in \mathbb{R}^5$. Fix signs $\varsigma, \sigma \in \{-,+\}$ and $k, \kappa \in \mathbb{N}$. Then, if $u \in \mathcal{S}'$ is a solution to $Pu = f$ such that

$$\mathrm{WF}_{\mathrm{de,sc}}^{-N,\mathsf{s}_0}(Au) \cap \mathcal{R}^\varsigma_\sigma = \varnothing \tag{384}$$

and $\mathrm{WF}^{-N,\mathsf{s}+1}_{\mathrm{de,sc}}(Af) \cap \mathcal{R}^{\varsigma}_{\sigma} = \varnothing$ *for all* $A \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$ *for some* $N > 0$*, then* $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(Au) \cap \mathcal{R}^{\varsigma}_{\sigma} = \varnothing$ *for all* $A \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$ *as well.* ■

Theorem 5 (Propagation into $\mathcal{R}$)**.** *Suppose that* $m \in \mathbb{R}$ *and* $\mathsf{s} \in (s_{\mathrm{Pf}}, s_{\mathrm{nPf}}, s_{\mathrm{Sf}}, s_{\mathrm{nFf}}, s_{\mathrm{Ff}}) \in \mathbb{R}^5$ *and* $s_{\mathrm{Pf},0}, s_{\mathrm{Ff},0} \in \mathbb{R}$ *satisfy* $\max\{s_{\mathrm{Pf}}, s_{\mathrm{Ff}}\} < -1/2$*. Fix signs* $\varsigma, \sigma \in \{-,+\}$ *and* $k, \kappa \in \mathbb{N}$*. Let* $u \in \mathcal{S}'$ *denote a solution to* $Pu = f$*. Then, if there exists a neighborhood* $U \subseteq {}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$ *of* $\mathcal{R}$ *such that*

$$\mathrm{WF}^{-N,\mathsf{s}}_{\mathrm{de,sc}}(Au) \cap U \subseteq \mathcal{R}^{\varsigma}_{\sigma}, \tag{385}$$

and if further $\mathrm{WF}^{-N,\mathsf{s}+1}_{\mathrm{de,sc}}(Af) \cap \mathcal{R}^{\varsigma}_{\sigma} = \varnothing$ *for all* $A \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$*, then* $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(Au) \cap U = \varnothing$ *for all* $A \in \mathfrak{M}^{\kappa,k}_{\varsigma,\sigma}$ *as well.* ■

We only consider the case of $\mathcal{R}^+_+$ explicitly. The case of $\mathcal{R}^-_-$ is essentially identical, and the cases of $\mathcal{R}^+_-, \mathcal{R}^-_+$ have overall signs switched in the computations but are otherwise identical.

We will prove the result in three parts: in §6.1, we handle the $k = 0, \kappa = 0$ case (which is the de,sc-analogue of the standard radial point result described in [Vas18; Vas20]), in §6.2 we handle $k > 0$ via induction (this being done via a somewhat involved secondary positive commutator argument), and in §6.3 we handle $\kappa > 0$ via another, more straightforward induction. The argument is a modification of that in [HMV04, §6][HMV08, Appendix A][GR+20, §3].

6.1. **Base case.** Let $\kappa, k = 0$. We now use ρ_0 to denote a quadratic defining function of $\mathcal{R}^+_+$ in $\Sigma_{\mathsf{m},+} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Ff})$, such that

$$\mathcal{R}^+_+ = \rho_0^{-1}(\{0\}) \cap \Sigma_{\mathsf{m},+} \cap {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{Ff}). \tag{386}$$

Over $\Omega_{\mathrm{nfTf},+,0}$, we can take this to be of the form $\rho_0 = \hat{\eta}^2 + (s-1)^2$ with respect to the coordinate system eq. (252).

We first observe that the symbol $F \in S^{0,0}_{\mathrm{de,sc}}$ defined by

$$\mathsf{H}_p \rho_0 = -4\rho_0 + F \tag{387}$$

vanishes cubically at $\mathcal{R}^+_+$. In order to show this, it suffices to check the claim in local coordinate patches. Away from null infinity, this is familiar [Vas18] from the radial point estimate for Klein–Gordon in the sc-calculus, so we only need to check the situation near null infinity. Near null infinity (recall that we are taking $\varrho_{\mathrm{df}} = \rho$ as usual over $\mathrm{nFf} \cap \mathrm{Tf}$), eq. (251) yields

$$\mathsf{H}_p \rho_0 = -4(2-s)(\hat{\eta}^2 + s(s-1))(s-1) + 4(\hat{\eta}^2 + s^2 - s - 1)\hat{\eta}^2, \tag{388}$$

from which it can be seen that F vanishes cubically at $\mathcal{R}^+_+$. Thus,

$$\mathsf{H}_{p[g]} \rho_0 = -4\rho_0 + F + \tilde{F}\varrho_{\mathrm{Ff}} \tag{389}$$

for some $\tilde{F} \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$.

Let $a_0 = \varrho_{\mathrm{nFf}}^{s_0} \varrho_{\mathrm{Ff}}^{\ell_0}$, where $s_0 = -1 - 2s_{\mathrm{nFf}}$, $\ell_0 = -1 - 2s_{\mathrm{Ff}}$. Then, we can write

$$\mathsf{H}_{p[g]} a_0 = \alpha a_0 \tag{390}$$

for $\alpha \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ given by

$$\alpha = -2(s_0(1-s) + \ell_0 s) = 2(1+2s_{\mathrm{nFf}})(1-s) + 2(1+2s_{\mathrm{Ff}})s \tag{391}$$

at ${}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nFf} \cap \mathrm{Ff})$, assuming without loss of generality that $\varrho_{\mathrm{Ff}} = \varrho_{\mathrm{Tf}}$ in local coordinates near $\mathrm{nFf} \cap \mathrm{Ff}$. Exactly at $\mathcal{R}^+_+$, this is $2(1+2s_{\mathrm{Ff}})$, which has a definite sign as long as $s_{\mathrm{Ff}} \neq -1/2$. The sign found here is the same as over the whole of Ff in the standard sc-analysis.

Now consider, as usual, a symbol $a = \chi_F(\tilde{p}[g])^2\chi_{F'}(\varrho_{\mathrm{Ff}})^2\chi_F(\rho_0)^2 a_0$ near $\mathcal{R}^+_+$ and supported away from df, Sf, nPf, Pf. Then

$$\begin{aligned}\mathsf{H}_{p[g]}a = \alpha a + 2\chi_F(\tilde{p}[g])^2\chi_{F'}(\varrho_{\mathrm{Ff}})^2\chi_{0,F}(\rho_0)^2 a_0(4\rho_0 - F - \tilde{F}\varrho_{\mathrm{Ff}}) \\ + 2\chi_F(\tilde{p}[g])^2\chi_{0,F'}(\varrho_{\mathrm{Ff}})^2\chi_F(\rho_0)^2 a_0\varrho_{\mathrm{Ff}}(2-\varrho_{\mathrm{Ff}}c) \\ + 2\chi_F'(\tilde{p}[g])\chi_F(\tilde{p}[g])\chi_{F'}(\varrho_{\mathrm{Ff}})^2\chi_F(\rho_0)^2\tilde{q}\tilde{p}[g]a_0\end{aligned} \tag{392}$$

for some $c \in C^\infty({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$, coming from applying $\varrho_{\mathrm{Tf}}^{-1}(\mathsf{H}_{p[g]} - \mathsf{H}_p) \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{de,sc}}\overline{T}^*\mathbb{O})$ to ϱ_{Ff}.

- First suppose that $s_{\mathrm{Tf}} > -1/2$, so that $\alpha > 0$ near $\mathcal{R}^+_+$. For all F sufficiently large, for all F' sufficiently large relative to F, for all $\delta > 0$ sufficiently small, we can define symbols $b, e, \tilde{g}, h \in S^{0,0}_{\mathrm{de,sc}}$ such that

$$\mathsf{H}_{p[g]}a = (\delta a_0^{-2}a^2 + b^2 + e^2 + \varrho_{\mathrm{Ff}}\tilde{g}^2 + h)a_0 \tag{393}$$

$$H_{p[g]}a = (\delta a_0^{-2}a^2 + b^2 + e^2 + \varrho_{\mathrm{Ff}}\tilde{g}^2 + h)wa_0 \tag{394}$$

everywhere, with $b = \chi_F(\tilde{p}[g])\chi_{F'}(\varrho_{\mathrm{Ff}})\chi_F(\rho_0)(\alpha - \delta a_0^{-1}a)^{1/2}$,

$$e = \sqrt{2}\chi_F(\tilde{p}[g])\chi_{F'}(\varrho_{\mathrm{Ff}})\chi_{0,F}(\rho_0)(4\rho_0 - F - \tilde{F}\varrho_{\mathrm{nf}})^{1/2} \tag{395}$$

(for each fixed value of F, the function $\chi_{0,F}(\rho_0)$ is supported away from $\rho_0 = 0$, so, as long as F' is chosen sufficiently large, the function $4\rho_0 - F - \tilde{F}\varrho_{\mathrm{nf}}$ under the square root will be bounded away from zero on the support of the prefactor),

$$\tilde{g} = \sqrt{4 - 2\varrho_{\mathrm{Ff}}c}\,\chi_F(\tilde{p}[g])\chi_{0,F'}(\varrho_{\mathrm{Ff}})\chi_F(\rho_0), \tag{396}$$

and $h = 2\chi_F'(\tilde{p}[g])\chi_F(\tilde{p}[g])\chi_{F'}(\varrho_{\mathrm{Ff}})\chi_F(\rho_0)\tilde{q}\tilde{p}[g]$ near $\mathcal{R}^+_+$.

Quantizing, we get $A = (1/2)(\mathrm{Op}(a) + \mathrm{Op}(a)^*) \in \Psi_{\mathrm{de,sc}}^{-\infty,(-\infty,-\infty,-\infty,-s_0,-\ell_0)}$,

$$\begin{aligned}B &= \mathrm{Op}(w^{1/2}a_0^{1/2}b) \in \Psi_{\mathrm{de,sc}}^{-\infty,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}, \\ E &= \mathrm{Op}(w^{1/2}a_0^{1/2}e) \in \Psi_{\mathrm{de,sc}}^{-\infty,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}, \\ \tilde{G} &= \mathrm{Op}(w^{1/2}a_0^{1/2}\varrho_{\mathrm{Ff}}^{1/2}\tilde{g}) \in \Psi_{\mathrm{de,sc}}^{-\infty,(-\infty,-\infty,-\infty,s_{\mathrm{nFf}},-\infty)}, \\ H &= \mathrm{Op}(wa_0h) \in \Psi_{\mathrm{de,sc}}^{-\infty,(-\infty,-\infty,-\infty,2s_{\mathrm{nFf}},2s_{\mathrm{Ff}})},\end{aligned} \tag{397}$$

and $R \in \Psi_{\mathrm{de,sc}}^{-\infty,(-\infty,-\infty,-\infty,2s_{\mathrm{nFf}}-1,2s_{\mathrm{Ff}}-1)}$ such that

$$-i[P,A] + i(P-P^*)A = \delta A\Lambda^2 A + B^*B + E^*E + \tilde{G}^*\tilde{G} + H + R \tag{398}$$

for

$$\Lambda = (1/2)(\mathrm{Op}(w^{1/2}a_0^{-1/2}) + \mathrm{Op}(w^{1/2}a_0^{-1/2})^*) \in \Psi_{\mathrm{de,sc}}^{1-m,(-1/2,-1/2,-1/2,-1-s_{\mathrm{nFf}},-1-s_{\mathrm{Ff}})}. \tag{399}$$

(Unlike in the estimates in the previous section, the $i(P-P^*)A$ has the same order as R, so we do not need to take it into account in the principal symbolic construction.) So, given sufficiently nice u,

$$2i\Im\langle Au, Pu\rangle_{L^2} = -\delta\|\Lambda Au\|^2_{L^2} + \|Bu\|^2_{L^2} + \|Eu\|^2_{L^2} + \|\tilde{G}u\|^2_{L^2} + \langle Hu,u\rangle_{L^2} + \langle Ru,u\rangle_{L^2}. \tag{400}$$

The rest of the argument proceeds as in the other propagation estimates, except the regularization argument is more delicate, but in a standard way. Indeed, it is only possible to regularize by a finite amount. Consider, for each $\varepsilon, K, K' > 0$, the regularizer

$$\varphi_{\varepsilon,K,K'} = \Big(1 + \frac{\varepsilon}{\varrho_{\mathrm{Ff}}\varrho_{\mathrm{nFf}}^{K'}}\Big)^{-K}. \tag{401}$$

Assuming, in addition to $\varrho_{\mathrm{Ff}} = \varrho_{\mathrm{Tf}}$ in the local coordinate system near the corner, $\varrho_{\mathrm{nFf}} = \varrho_{\mathrm{nf}}$,

$$\mathsf{H}_{p[g]}\varphi_{\varepsilon,K,K'} = -\frac{2K\varepsilon}{\varepsilon + \varrho_{\mathrm{Ff}}\varrho_{\mathrm{nFf}}^{K'}}\varphi_{\varepsilon,K,K'}(s + K'(1-s)) \tag{402}$$

over $\partial\mathbb{O}$, in some neighborhood of $\mathrm{nFf}\cap\mathrm{Ff}$. Notice that, at $\mathcal{R}^+_+$, over $\Omega_{\mathrm{nfTf},+,0}$, we have $s + K'(1-s) = 1$. Combining this calculation with the one done over $\mathrm{cl}_{\mathbb{O}}\{r=0\}$ as part of the standard sc-analysis, we can conclude that

$$\frac{2}{\varphi_{\varepsilon,K,K'}}\mathsf{H}_{p[g]}\varphi_{\varepsilon,K,K'}\Big|_{\mathcal{R}^+_+} \leq 4K. \tag{403}$$

Define

$$a_\varepsilon = \varphi^2_{\varepsilon,K,K'}a, \tag{404}$$

and likewise for the other symbols above with the exception of b, and define

$$b_\varepsilon = \varphi^2_{\varepsilon,K,K'}\chi_{\digamma}(\tilde{p}[g])\chi_{\digamma'}(\varrho_{\mathrm{Ff}})\chi_{\digamma}(\rho_0)\sqrt{\alpha - \delta a_0^{-1}a_\varepsilon + \frac{2}{\varphi_{\varepsilon,K,K'}}\mathsf{H}_{p[g]}\varphi_{\varepsilon,K,K'}}, \tag{405}$$

assuming that the symbol under the square root is positive on the support of the prefactor, so that this is a well-defined symbol. Exactly at $\mathcal{R}^+_+$, we have $\alpha = 4s_{\mathrm{Ff}} + 2$, so, as long as

$$K < s_{\mathrm{Ff}} + \frac{1}{2}, \tag{406}$$

then eq. (403) guarantees that the symbol b_ε is well-defined *for all* $\varepsilon > 0$, as long as $\digamma$ is sufficiently large and δ is sufficiently small *relative to* K. As long as these conditions are met, instead of eq. (393), we have

$$\mathsf{H}_{p[g]}a_\varepsilon = (\delta a_0^{-2}a_\varepsilon^2 + b_\varepsilon^2 + e_\varepsilon^2 + \varrho_{\mathrm{Ff}}\tilde{g}_\varepsilon^2 + h_\varepsilon)a_0. \tag{407}$$

Thus, quantizing, we get operators, all of which are uniform families of de,sc-ΨDOs of the same orders as their non-regularized counterparts, such that

$$-i[P, A_\varepsilon] + i(P - P^*)A_\varepsilon = \delta A_\varepsilon\Lambda^2A_\varepsilon + B_\varepsilon^*B_\varepsilon + E_\varepsilon^*E_\varepsilon + \tilde{G}_\varepsilon^*\tilde{G}_\varepsilon + H_\varepsilon + R_\varepsilon. \tag{408}$$

Given u with $\mathrm{WF}^{-N,\mathsf{s}_0}_{\mathrm{de,sc}}(u)\cap\mathcal{R}^+_+ = \varnothing$, we can take K, K' large enough such that, as long as $\digamma$ is sufficiently large, then we can deduce from eq. (408) that

$$2i\Im\langle A_\varepsilon u, Pu\rangle_{L^2} = \delta\|\Lambda A_\varepsilon u\|^2_{L^2} + \|B_\varepsilon u\|^2_{L^2} + \|E_\varepsilon u\|^2_{L^2} + \|\tilde{G}_\varepsilon u\|^2_{L^2} + \langle H_\varepsilon u, u\rangle_{L^2} + \langle R_\varepsilon u, u\rangle_{L^2}, \tag{409}$$

from which the estimate

$$\|B_\varepsilon u\|^2_{L^2} + \delta\|\Lambda A_\varepsilon u\|^2_{L^2} \leq |\langle H_\varepsilon u, u\rangle_{L^2}| + |\langle R_\varepsilon u, u\rangle_{L^2}| + 2|\langle A_\varepsilon u, Pu\rangle_{L^2}| \tag{410}$$

follows. Indeed, for $\digamma$ sufficiently large:

(1) We have

$$\Lambda A_\varepsilon u, B_\varepsilon u, E_\varepsilon u, \tilde{G}_\varepsilon u \in H^{N,(N,N,N,KK'+s_{\mathrm{nFf},0}-s_{\mathrm{nFf}},K+s_{\mathrm{Ff},0}-s_{\mathrm{Ff}})}_{\mathrm{de,sc}} \subseteq L^2 \tag{411}$$

as long as $K > s_{\mathrm{Ff}} - s_{\mathrm{Ff},0}$ and KK' is sufficiently large. Since $s_{\mathrm{Ff},0} \in (-1/2, s_{\mathrm{Ff}})$, $s_{\mathrm{Ff}} - s_{\mathrm{Ff},0} < s_{\mathrm{Ff}} + 1/2$, the interval $(s_{\mathrm{Ff}} - s_{\mathrm{Ff},0}, s_{\mathrm{Ff}} + 1/2)$ is nonempty, which means there exists K large enough to satisfy both eq. (406) and eq. (411).

(2) $H_\varepsilon u \in H^{N,(N,N,N,2KK'+s_{\mathrm{nFf},0}-2s_{\mathrm{nFf}},2K+s_{\mathrm{Ff},0}-2s_{\mathrm{Ff}})}_{\mathrm{de,sc}}$, and, for N sufficiently large,

$$H^{N,(N,N,N,KK'+s_{\mathrm{nFf},0}-2s_{\mathrm{nFf}},K+s_{\mathrm{Ff},0}-2s_{\mathrm{Ff}})}_{\mathrm{de,sc}} \subseteq (H^{-N,\mathsf{s}_0}_{\mathrm{de,sc}})^* \tag{412}$$

as long as $K > s_{\mathrm{Ff}} - s_{\mathrm{Ff},0}$ and KK' is sufficiently large, which means that the $\langle H_\varepsilon u, u\rangle_{L^2}$ term is well-defined in the sense of Hörmander, and likewise for $\langle R_\varepsilon u, u\rangle_{L^2}$.

(3) $\mathrm{WF}^{-N-2,\mathsf{s}_0}_{\mathrm{de,sc}}(Pu)\cap\mathcal{R}^+_+=\varnothing$, and, for N sufficiently large,

$$A_\varepsilon u\in H^{N+2,(N,N,N,-1+KK'+s_{\mathrm{nFf},0}-2s_{\mathrm{nFf}},-1+K+s_{\mathrm{Ff},0}-2s_{\mathrm{Ff}})}_{\mathrm{de,sc}}\subseteq(\mathrm{WF}^{-N-2,\mathsf{s}_0}_{\mathrm{de,sc}})^*, \tag{413}$$

assuming the conditions above are satisfied. This implies that $\langle A_\varepsilon u,Pu\rangle_{L^2}$ is well-defined in the sense of Hörmander.

Having eq. (410), we get, after taking $\varepsilon\to0^+$, an estimate of the form

$$\|\tilde{B}u\|^2_{H^{N,(N,N,N,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}_{\mathrm{de,sc}}}\preceq\|GPu\|^2_{H^{-N,(-N,-N,-N,s_{\mathrm{nFf}}+1,s_{\mathrm{Sf}}+1)}_{\mathrm{de,sc}}}+\|Gu\|^2_{H^{-N,(-N,-N,-N,s_{\mathrm{nFf}}-1/2,s_{\mathrm{Ff}}-1/2)}_{\mathrm{de,sc}}}+\|u\|^2_{H^{-N,-N}_{\mathrm{de,sc}}}, \tag{414}$$

where $\tilde{B}\in\Psi^{0,0}_{\mathrm{de,sc}}$ is elliptic on $\mathcal{R}^+_+$ and $G\in\Psi^{0,0}_{\mathrm{de,sc}}$ whose essential support can be taken to be arbitrarily close to $\mathcal{R}^+_+$ by making F larger. In order to make this an estimate in the strong sense, we can simply add a term to the right-hand side:

$$\|\tilde{B}u\|^2_{H^{N,(N,N,N,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}_{\mathrm{de,sc}}}\preceq\|GPu\|^2_{H^{-N,(-N,-N,-N,s_{\mathrm{nFf}}+1,s_{\mathrm{Sf}}+1)}_{\mathrm{de,sc}}}+\|Gu\|^2_{H^{-N,\mathsf{s}_0}_{\mathrm{de,sc}}}+\|Gu\|^2_{H^{-N,(-N,-N,-N,s_{\mathrm{nFf}}-1/2,s_{\mathrm{Ff}}-1/2)}_{\mathrm{de,sc}}}+\|u\|^2_{H^{-N,-N}_{\mathrm{de,sc}}}. \tag{415}$$

The usual inductive argument then allows the removal of the penultimate term, and so we end up with the strong estimate

$$\|\tilde{B}u\|^2_{H^{N,(N,N,N,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}_{\mathrm{de,sc}}}\preceq\|GPu\|^2_{H^{-N,(-N,-N,-N,s_{\mathrm{nFf}}+1,s_{\mathrm{Sf}}+1)}_{\mathrm{de,sc}}}+\|Gu\|^2_{H^{-N,\mathsf{s}_0}_{\mathrm{de,sc}}}+\|u\|^2_{H^{-N,-N}_{\mathrm{de,sc}}}, \tag{416}$$

which completes the proof.

- On the other hand, if $s_{\mathrm{Tf}}<-1/2$, then $\alpha<0$ near $\mathcal{R}^+_+$, then quantization yields operators as above, modulo some sign switches in their definitions, such that

$$-i[P,A]+i(P-P^*)A=\delta A\Lambda^2A-B^*B+E^*E+\tilde{G}^*\tilde{G}+H+R. \tag{417}$$

From this, the strong estimate of the form

$$\|\tilde{B}u\|^2_{H^{N,(N,N,N,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}_{\mathrm{de,sc}}}\preceq\|GPu\|^2_{H^{-N,(-N,-N,-N,s_{\mathrm{nFf}}+1,s_{\mathrm{Sf}}+1)}_{\mathrm{de,sc}}}+\|\tilde{E}u\|^2_{H^{-N,(-N,-N,-N,s_{\mathrm{nFf}},s_{\mathrm{Ff}})}_{\mathrm{de,sc}}}+\|\tilde{\tilde{G}}u\|^2_{H^{-N,(-N,-N,-N,s_{\mathrm{nFf}},-N)}_{\mathrm{de,sc}}}+\|u\|^2_{H^{-N,-N}_{\mathrm{de,sc}}} \tag{418}$$

follows. The necessary regularization argument is simpler than the previous, as an arbitrarily large amount of regularization can be done.

6.2. **First induction.** In order to carry out the construction of the commutant for $k>0$, we recall the following algebraic computation, which is essentially [HMV04, eq. 6.16][GR+20, eq. 3.23]. Let $A_0,\ldots,A_N\in\mathfrak{N}_+$ denote a spanning set over $\Psi^{0,0}_{\mathrm{de,sc}}$, with $A_0=1$. For each multi-index $\alpha\in\mathbb{N}^N$ with $|k|=\alpha$ and tuple $\mathsf{s}\in\mathbb{R}^5$, let

$$A_{\alpha,s}=\varrho^{-\mathsf{s}}A_1^{\alpha_1}\cdots A_N^{\alpha_N}, \tag{419}$$

where

$$\varrho^{\mathsf{s}}=\varrho_{\mathrm{Pf}}^{s_{\mathrm{Pf}}}\varrho_{\mathrm{nPf}}^{s_{\mathrm{nPf}}}\varrho_{\mathrm{Sf}}^{s_{\mathrm{Sf}}}\varrho_{\mathrm{nFf}}^{s_{\mathrm{nFf}}}\varrho_{\mathrm{Ff}}^{s_{\mathrm{Ff}}}. \tag{420}$$

Also, let $Q\in\Psi^{-\infty,0}_{\mathrm{de,sc}}$. For the purposes of this computation, $P\in\Psi^{2,0}_{\mathrm{de,sc}}$ can be arbitrary. Let

$$D_{\mathsf{s}}=i\varrho^{-1}[P,\varrho^{-\mathsf{s}}]\varrho^{\mathsf{s}}\in\mathrm{Diff}^{1,0}_{\mathrm{de,sc}}. \tag{421}$$

Then:

Lemma 6.1. *There exist* $E_{\alpha,\mathsf{s}-1/2} = E_{\alpha,\mathsf{s}-1/2}(Q) \in \varrho^{-\mathsf{s}+1/2}\mathfrak{N}_+^{k-1}$*, with* $\mathrm{WF}'_{\mathrm{de,sc}}(E_{\alpha,\mathsf{s}-1/2}) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(Q)$ *such that*

$$\begin{aligned} i[P, A^*_{\alpha,\mathsf{s}+1/2}Q^*QA_{\alpha,\mathsf{s}+1/2}] = 2A^*_{\alpha,\mathsf{s}+1/2}Q^*\varrho^{1/2}\Re\Big[D_{\mathsf{s}+1/2} + \sum_{j=1}^N \alpha_j C_{j,j}\Big]\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2} \\ + 2\sum_{|\beta|=k,\beta\neq\alpha} A^*_{\alpha,\mathsf{s}+1/2}Q^*\varrho^{1/2}\Re[C_{\alpha,\beta}]\varrho^{1/2}QA_{\beta,\mathsf{s}+1/2} \\ + A^*_{\alpha,\mathsf{s}+1/2}Q^*E_{\alpha,\mathsf{s}-1/2} + E^*_{\alpha,\mathsf{s}-1/2}QA_{\alpha,\mathsf{s}+1/2} + A^*_{\alpha,\mathsf{s}+1/2}i[P,Q^*Q]A_{\alpha,\mathsf{s}+1/2} \end{aligned} \tag{422}$$

holds for some $C_{\alpha,\beta} \in \Psi^{1,0}_{\mathrm{de,sc}}$ *satisfying* $i\varrho^{-1}[P,A_\alpha] = \sum_{|\beta|\leq k} C_{\alpha,\beta}A_\beta$ *and*

$$\sigma^{0,0}_{\mathrm{de,sc}}(C_{\alpha,\beta})|_{\mathcal{R}^+_+} = 0 \tag{423}$$

for all multindices $\alpha,\beta \in \mathbb{N}^N$ *with* $|\alpha| = |\beta| = k$. ■

Here, for $j,k \in \{1,\ldots,N\}$, $C_{j,k}$ is as in Proposition 3.9. When α,β are multi-indices that are zero except in the jth and kth slots respectively, where they are one, then we can take $C_{\alpha,\beta} = C_{j,k}$.

Proof. We show that given any collection of $C_{\alpha,\beta} \in \Psi^{1,0}_{\mathrm{de,sc}}$ satisfying

$$i\varrho^{-1}[P,A_\alpha] = \sum_{|\beta|\leq|\alpha|} C_{\alpha,\beta}A_\beta, \tag{424}$$

there exists a collection of $E^{(0)}_{\alpha,\mathsf{s}-1/2} \in \varrho^{-\mathsf{s}+1/2}\mathfrak{N}^{k-1}$ with $\mathrm{WF}'_{\mathrm{de,sc}}(E^{(0)}_{\alpha,\mathsf{s}-1/2}) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(Q)$ such that

$$\begin{aligned} i[P, A^*_{\alpha,\mathsf{s}+1/2}Q^*QA_{\alpha,\mathsf{s}+1/2}] = 2\sum_{|\beta|\leq k} A^*_{\alpha,\mathsf{s}+1/2}Q^*\varrho^{1/2}\Re\Big[\delta_{\alpha,\beta}D_{\mathsf{s}+1/2} + C_{\alpha,\beta}\Big]\varrho^{1/2}QA_{\beta,\mathsf{s}+1/2} \\ + A^*_{\alpha,\mathsf{s}+1/2}Q^*E^{(0)}_{\alpha,\mathsf{s}-1/2} + E^{(0)*}_{\alpha,\mathsf{s}-1/2}QA_{\alpha,\mathsf{s}+1/2} + A^*_{\alpha,\mathsf{s}+1/2}i[P,Q^*Q]A_{\alpha,\mathsf{s}+1/2} \end{aligned} \tag{425}$$

holds for each $\alpha \in \mathbb{N}^N$. We then show that we can choose $C_{\alpha,\beta}$ such that, if $|\beta| = |\alpha|$,

$$C_{\alpha,\beta} = \begin{cases} \sum_{j=1}^N \alpha_j C_{j,j} & (\alpha=\beta), \\ \alpha_\delta C_{j,\nu} & (|\alpha-\beta|=2), \\ 0 & (\text{otherwise}), \end{cases} \tag{426}$$

where in the second case j,ν are the indices in which α,β differ, with $\alpha_j = \beta_j + 1$ and $\beta_\nu = \alpha_\nu + 1$. Defining

$$E_{\alpha,\mathsf{s}-1/2} = E^{(0)}_{\alpha,\mathsf{s}-1/2} + 2\varrho^{1/2}\sum_{|\beta|<k}\Re[C_{\alpha,\beta}]\varrho^{1/2}QA_{\beta,\mathsf{s}+1/2} \in \varrho^{-\mathsf{s}+1/2}\mathfrak{N}_+^{k-1}, \tag{427}$$

eq. (422) holds, and by Proposition 3.9, the $C_{\alpha,\beta}$ defined by eq. (426) satisfy eq. (423).

- Suppose that we are given $C_{\alpha,\beta}$ satisfying eq. (424). The left-hand side of eq. (425) is given by

$$i[P, A^*_{\alpha,\mathsf{s}+1/2}Q^*QA_{\alpha,\mathsf{s}+1/2}] = A^*_{\alpha,\mathsf{s}+1/2}Q^*i[P,QA_{\alpha,\mathsf{s}+1/2}] + [P,A_{\alpha,\mathsf{s}+1/2}Q^*]QA. \tag{428}$$

Consider $i[P,A_{\alpha,\mathsf{s}+1/2}] = i\varrho^{-\mathsf{s}-1/2}[P,A_\alpha] + i[P,\varrho^{-\mathsf{s}-1/2}]A_\alpha$. Using eq. (424), this becomes

$$\begin{aligned} i[P,A_{\alpha,\mathsf{s}+1/2}] &= \sum_{|\beta|=|\alpha|}\varrho^{-\mathsf{s}+1/2}C_{\alpha,\beta}A_\beta + D_{\mathsf{s}+1/2}A_{\alpha,\mathsf{s}-1/2} \\ &= \sum_{|\beta|=|\alpha|}C_{\alpha,\beta}A_{\beta,\mathsf{s}-1/2} + \sum_{|\beta|=|\alpha|}[\varrho^{-\mathsf{s}+1/2},C_{\alpha,\beta}]A_\beta + D_{\mathsf{s}+1/2}A_{\alpha,\mathsf{s}-1/2}. \end{aligned} \tag{429}$$

Therefore, $i[P,QA_{\alpha,\mathsf{s}+1/2}] = Qi[P,A_{\alpha,\mathsf{s}+1/2}]+i[P,Q]A_{\alpha,\mathsf{s}+1/2}$ can be written, after rearrangement of the terms in $Qi[P,A_{\alpha,\mathsf{s}+1/2}]$, as

$$\begin{aligned} i[P,QA_{\alpha,\mathsf{s}+1/2}] &= \varrho D_{\mathsf{s}+1/2}QA_{\alpha,\mathsf{s}+1/2} + \varrho \sum_{|\beta|=|\alpha|} C_{\alpha,\beta}QA_{\beta,\mathsf{s}+1/2} + [D_{\mathsf{s}+1/2}Q,\varrho]A_{\alpha,\mathsf{s}+1/2} \\ &+ \sum_{|\beta|=|\alpha|}[C_{\alpha,\beta}Q,\varrho]A_{\beta,\mathsf{s}+1/2} + \sum_{|\beta|=|\alpha|}\Big([Q,C_{\alpha,\beta}] + Q[\varrho^{-\mathsf{s}+1/2},C_{\alpha,\beta}]\varrho^{\mathsf{s}-1/2}\Big)A_{\beta,\mathsf{s}-1/2} \\ &+ [Q,D_{\mathsf{s}+1/2}]A_{\alpha,\mathsf{s}-1/2} + i[P,Q]A_{\alpha,\mathsf{s}+1/2}, \end{aligned} \tag{430}$$

and similarly for $[P,A^*_{\alpha,\mathsf{s}+1/2}] = -[P,A_{\alpha,\mathsf{s}+1/2}]^*$. So, the operator

$$i[P,A^*_{\alpha,\mathsf{s}+1/2}Q^*QA_{\alpha,\mathsf{s}+1/2}] = A^*_{\alpha,\mathsf{s}+1/2}Q^*i[P,QA_{\alpha,\mathsf{s}+1/2}] + [P,A_{\alpha,\mathsf{s}+1/2}Q^*]QA \tag{431}$$

is given by

$$\begin{aligned} i[P,A^*_{\alpha,\mathsf{s}+1/2}Q^*QA_{\alpha,\mathsf{s}+1/2}] &= A^*_{\alpha,\mathsf{s}+1/2}Q^*\Big(\varrho D_{\mathsf{s}+1/2} + D^*_{\mathsf{s}+1/2}\varrho + \sum_{|\beta|=|\alpha|}(\varrho C_{\alpha,\beta} + C^*_{\alpha,\beta}\varrho)\Big)QA_{\beta,\mathsf{s}+1/2} \\ &+ A^*_{\alpha,\mathsf{s}+1/2}Q^*E^{(1)}_{\alpha,\mathsf{s}-1/2} + E^{(1)*}_{\alpha,\mathsf{s}-1/2}QA_{\alpha,\mathsf{s}+1/2} + A^*_{\alpha,\mathsf{s}+1/2}i[P,Q^*Q]A_{\alpha,\mathsf{s}+1/2} \end{aligned} \tag{432}$$

for

$$\begin{aligned} E^{(1)}_{\alpha,\mathsf{s}-1/2} &= [D_{\mathsf{s}+1/2}Q,\varrho]A_{\alpha,\mathsf{s}+1/2} + \sum_{|\beta|=|\alpha|}[C_{\alpha,\beta}Q,\varrho]A_{\beta,\mathsf{s}+1/2} \\ &+ \sum_{|\beta|=|\alpha|}\Big([Q,C_{\alpha,\beta}] + Q[\varrho^{-\mathsf{s}+1/2},C_{\alpha,\beta}]\varrho^{\mathsf{s}-1/2}\Big)A_{\beta,\mathsf{s}-1/2} + [Q,D_{\mathsf{s}+1/2}]A_{\alpha,\mathsf{s}-1/2}. \end{aligned} \tag{433}$$

Here, we have recombined

$$A^*_{\alpha,\mathsf{s}+1/2}Q^*i[P,Q]A_{\alpha,\mathsf{s}+1/2} + A^*_{\alpha,\mathsf{s}+1/2}i[P,Q^*]QA_{\alpha,\mathsf{s}+1/2} = A^*_{\alpha,\mathsf{s}+1/2}i[P,Q^*Q]A_{\alpha,\mathsf{s}+1/2}. \tag{434}$$

Term by term, we see that $E^{(1)}_{\alpha,\mathsf{s}-1/2} \in \Psi^{-\infty,\mathsf{s}-1/2}_{\mathrm{de,sc}}\mathfrak{N}^{k-1}_+$. For example, $[D_{\mathsf{s}+1/2}Q,\varrho] \in \Psi^{-\infty,-2}_{\mathrm{de,sc}}$, so

$$[D_{\mathsf{s}+1/2}Q,\varrho]A_{\alpha,\mathsf{s}+1/2} = [D_{\mathsf{s}+1/2}Q,\varrho]\varrho^{-\mathsf{s}-1/2}A_\alpha \tag{435}$$

is the product of an element of $\Psi^{-\infty,-3/2+\mathsf{s}}_{\mathrm{de,sc}}$ and an element of $\mathfrak{N}^k_+$ Since $\Psi^{-\infty,-1}_{\mathrm{de,sc}}\mathfrak{N}^k_+ \subseteq \mathfrak{N}^{k-1}_+$, we get

$$[D_{\mathsf{s}+1/2}Q,\varrho]A_{\alpha,\mathsf{s}-1/2} \in \Psi^{-\infty,\mathsf{s}-1/2}_{\mathrm{de,sc}}\mathfrak{N}^{k-1}_+. \tag{436}$$

The other terms in eq. (433) are analyzed similarly.

Equation (432) looks very similar to the desired eq. (425), except we want to commute a factor of $\varrho^{1/2}$ through each of the $D_{\mathsf{s}+1/2}$'s and $C_{\alpha,\beta}$'s. If we set

$$E^{(0)}_{\alpha,\mathsf{s}-1/2} = E^{(1)}_{\alpha,\mathsf{s}-1/2} + \varrho^{1/2}\Big([\varrho^{1/2},D_{\mathsf{s}+1/2}] + \sum_{|\beta|=|\alpha|}[\varrho^{1/2},C_{\alpha,\beta}]\Big)QA_{\alpha,\mathsf{s}+1/2}. \tag{437}$$

This lies in $\Psi^{-\infty,\mathsf{s}-1/2}_{\mathrm{de,sc}}\mathfrak{N}^{k-1}_+$, and eq. (432) becomes eq. (425). Observe that

$$\mathrm{WF}'_{\mathrm{de,sc}}(E^{(0)}_{\alpha,\mathsf{s}-1/2}) \subseteq \mathrm{WF}'_{\mathrm{de,sc}}(Q), \tag{438}$$

so we have accomplished our first task.

- We compute

$$
\begin{aligned}
i\varrho^{-1}[P,A_\alpha] &= \varrho^{-1}\sum_{j=1}^N \Big[\Big(\prod_{\ell=1}^{j-1} A_\ell^{\alpha_\ell}\Big)\Big(\sum_{\varkappa=1}^{\alpha_j} A_j^{\varkappa-1} i[P,A_j] A_j^{\alpha_j-\varkappa}\Big)\Big(\prod_{\ell=j+1}^N A_\ell^{\alpha_\ell}\Big)\Big] \\
&= \varrho^{-1}\sum_{j=1}^N \Big[\Big(\prod_{\ell=1}^{j-1} A_\ell^{\alpha_\ell}\Big)\Big(\sum_{\varkappa=1}^{\alpha_j} A_j^{\varkappa-1} \varrho \sum_{\nu=0}^N C_{j,\nu}A_\nu A_j^{\alpha_j-\varkappa}\Big)\Big(\prod_{\ell=j+1}^N A_\ell^{\alpha_\ell}\Big)\Big] \\
&= \sum_{j=1}^N \sum_{\varkappa=1}^{\alpha_j}\sum_{\nu=0}^N \Big[\varrho^{-1}\Big(\prod_{\ell=1}^{j-1} A_\ell^{\alpha_\ell}\Big)\Big(A_j^{\varkappa-1}\varrho C_{j,\nu}A_i A_j^{\alpha_j-\varkappa}\Big)\Big(\prod_{\ell=j+1}^N A_\ell^{\alpha_\ell}\Big)\Big].
\end{aligned}
\tag{439}
$$

Consider the summand, $\varrho^{-1}(\prod_{\ell=1}^{j-1} A_\ell^{\alpha_\ell})(A_j^{\varkappa-1}\varrho C_{j,\nu}A_iA_j^{\alpha_j-\varkappa})(\prod_{\ell=j+1}^N A_\ell^{\alpha_\ell})$. In the case $\nu=j$, commuting the term $\varrho C_{j,\nu}=\varrho C_{j,j}$ to the left yields

$$
\varrho^{-1}\Big(\prod_{\ell=1}^{j-1} A_\ell^{\alpha_\ell}\Big)\Big(A_j^{\varkappa-1}\varrho C_{j,j}A_j^{\alpha_j-\varkappa+1}\Big)\Big(\prod_{\ell=j+1}^N A_\ell^{\alpha_\ell}\Big) - C_{j,j}A_\alpha \in \Psi^{1,0}_{\mathrm{de,sc}}\mathfrak{N}_+^{k-1}. \tag{440}
$$

A similar computation applies when $\nu\notin\{j,0\}$, with the result

$$
\varrho^{-1}\Big(\prod_{\ell=1}^{j-1} A_\ell^{\alpha_\ell}\Big)\Big(A_j^{\varkappa-1}\varrho C_{j,\nu}A_\nu A_j^{\alpha_j-\varkappa}\Big)\Big(\prod_{\ell=j+1}^N A_\ell^{\alpha_\ell}\Big) - C_{j,\nu}A_\beta \in \Psi^{1,0}_{\mathrm{de,sc}}\mathfrak{N}_+^{k-1}, \tag{441}
$$

where β differs from α by decrementing the jth entry and incrementing the νth. For the remaining case, $\nu=0$, we simply use that

$$
\varrho^{-1}\Big(\prod_{\ell=1}^{j-1} A_\ell^{\alpha_\ell}\Big)\Big(A_j^{\varkappa-1}\varrho C_{j,0}A_j^{\alpha_j-\varkappa}\Big)\Big(\prod_{\ell=j+1}^N A_\ell^{\alpha_\ell}\Big) \in \Psi^{1,0}_{\mathrm{de,sc}}\mathfrak{N}_+^{k-1}. \tag{442}
$$

So,

$$
i\varrho^{-1}[P,A_\alpha] = \sum_{\beta\in\mathbb{N}^N,|\beta|=k} C_{\alpha,\beta}A_\beta \bmod \Psi^{1,0}_{\mathrm{de,sc}}\mathfrak{N}_\sigma^{k-1}, \tag{443}
$$

where $C_{\alpha,\beta}$ are defined for $|\alpha|=|\beta|=k$ by eq. (426).

Since the A_α's for $|\alpha|<k$ span $\mathfrak{N}_+^{k-1}$ over $\Psi^{0,0}_{\mathrm{de,sc}}$, the $\Psi^{1,0}_{\mathrm{de,sc}}\mathfrak{N}_+^{k-1}$ error term in eq. (443) can be written as

$$
\sum_{\beta\in\mathbb{N}^N,|\beta|<k} C_{\alpha,\beta}A_\beta \tag{444}
$$

for some $\{C_{\alpha,\beta}\}_{|\beta|<k}\subseteq\Psi^{1,0}_{\mathrm{de,sc}}$. So, the $\{C_{\alpha,\beta}\}_{|\alpha|,|\beta|\le k}$ defined here satisfy eq. (424) on the nose, and they satisfy eq. (426) when $|\alpha|=|\beta|=k$. This completes the second part of the proof.

□

We now return to the main line of argument, with P as in the introduction. Let $k\in\mathbb{N}^+$, still taking $\kappa=0$. As in the lemma above, let $D_{\mathsf{s}}\in\mathrm{Diff}^{1,0}_{\mathrm{de,sc}}$ be defined by $i\varrho^{-1}[P,\varrho^{-\mathsf{s}}]=D_{\mathsf{s}}\varrho^{-\mathsf{s}}$. Then,

$$
\begin{cases}
\sigma^{0,0}_{\mathrm{de,sc}}(D_{\mathsf{s}})|_{\mathcal{R}_+^+} > 0 & (s_{\mathrm{Ff}}>0) \\
\sigma^{0,0}_{\mathrm{de,sc}}(D_{\mathsf{s}})|_{\mathcal{R}_+^+} < 0 & (s_{\mathrm{Ff}}<0).
\end{cases} \tag{445}
$$

We consider

$$
\{C'_{\alpha,\beta}\}_{|\alpha|,|\beta|=k} = \{\delta_{\alpha,\beta}(D_{\mathsf{s}+1/2}+D^*_{\mathsf{s}+1/2}) + C_{\alpha,\beta} + C^*_{\alpha,\beta}\}_{|\alpha|,|\beta|=k} \tag{446}
$$

as a matrix-valued de,sc-ΨDO C' whose matrix elements are indexed by multiindices $\alpha,\beta\in\mathbb{N}^N$ with $|\alpha|=k$ and $|\beta|=k$. The matrix-valued principal symbol of C' is the matrix

$$\sigma^{1,0}_{\mathrm{de,sc}}(C') = \{\delta_{\alpha,\beta}2\Re\sigma^{1,0}_{\mathrm{de,sc}}(D_{\mathsf{s}+1/2}) + 2\Re\sigma^{1,0}_{\mathrm{de,sc}}(C_{\alpha,\beta})\}_{|\alpha|,|\beta|=k}. \tag{447}$$

Choosing representatives of $\sigma^{1,0}_{\mathrm{de,sc}}(D_{\mathsf{s}+1/2})$ and $\sigma^{1,0}_{\mathrm{de,sc}}(C_{\alpha,\beta})$, for each α,β, we get a representative c' of $\sigma^{1,0}_{\mathrm{de,sc}}(C')$, which assigns to each point of ${}^{\mathrm{de,sc}}T^*\mathbb{O}$ an ordinary matrix. As long as $s_{\mathrm{Ff}}\neq -1/2$, this matrix is – owing to eq. (423) and eq. (445) – either positive definite or negative definite near $\mathcal{R}^+_+$, so

$$b = |c'|^{1/2} \tag{448}$$

is, near $\mathcal{R}^+_+$, a well-defined symmetric matrix whose entries are elements of $S^{1,0}_{\mathrm{de,sc}}$ defined near the radial set. Let $\{b_{\alpha,\beta}\}_{|\alpha|,|\beta|=k}$ denote the entries of b. Squaring eq. (448), we see that, for each α,β,

$$c'_{\alpha,\beta} = \pm\sum_{|\gamma|=k} b_{\alpha,\gamma}b_{\gamma,\beta}, \tag{449}$$

where the sign is positive if $s_{\mathrm{Ff}} > -1/2$ and negative otherwise.

Quantizing (and remembering that the discussion above is only valid near the radial set), there exist

$$B_{\alpha,\beta} = B_{\beta,\alpha}\in\Psi^{-\infty,0}_{\mathrm{de,sc}}, \tag{450}$$

$R_{\alpha,\beta}\in\Psi^{-\infty,-1}_{\mathrm{de,sc}}$, and $E\in\Psi^{-\infty,-\infty}_{\mathrm{de,sc}}$ such that

$$Q^*\varrho^{1/2}C'_{\alpha,\beta}\varrho^{1/2}Q = Q^*\varrho^{1/2}\Big[\pm\sum_{|\gamma|=k}B^*_{\alpha,\gamma}B_{\gamma,\beta}+R_{\alpha,\beta}\Big]\varrho^{1/2}Q+E, \tag{451}$$

with

$$\sigma^{0,0}_{\mathrm{de,sc}}(B_{\alpha,\beta}) = b_{\alpha,\beta} \tag{452}$$

near $\mathcal{R}^+_+$, at least as Q has essential support in a sufficiently small neighborhood of the radial set (so that the discussion above is valid within it). Moreover, since b is invertible (as it is strictly definite and not just semidefinite) near $\mathcal{R}^+_+$, there exist $\Upsilon_{\alpha,\beta}\in\Psi^{-\infty,0}_{\mathrm{de,sc}}$ such that

$$Q^*\varrho^{1/2}\Big(\sum_{|\gamma|=k}\Upsilon_{\alpha,\gamma}B_{\gamma,\beta}-\delta_{\alpha,\beta}\Big)\varrho^{1/2}Q\in\Psi^{-\infty,-\infty}_{\mathrm{de,sc}} \tag{453}$$

$$Q^*\varrho^{1/2}\Big(\sum_{|\gamma|=k}B_{\alpha,\gamma}\Upsilon_{\gamma,\beta}-\delta_{\alpha,\beta}\Big)\varrho^{1/2}Q\in\Psi^{-\infty,-\infty}_{\mathrm{de,sc}} \tag{454}$$

for each α,β, where $\delta_{\bullet,\bullet}$ denotes the Kronecker δ (once again, as long as the essential support of Q is sufficiently close to $\mathcal{R}^+_+$). (That we can arrange for the errors above to be residual rather than merely one uniform order better is an instance of the iterative parametrix construction.)

Now consider $u\in\mathcal{S}'$ as in the setup of the proposition. Assuming we can justify the algebraic manipulations:

$$\begin{aligned}\sum_{|\alpha|=k}&\langle u, i[P, A^*_{\alpha,\mathsf{s}+1/2}Q^*QA_{\alpha,\mathsf{s}+1/2}]u\rangle\\ &= \Big[\sum_{|\alpha|,|\beta|=k}\Big\langle u, A^*_{\alpha,\mathsf{s}+1/2}Q^*\varrho^{1/2}\Big(\pm\sum_{|\gamma|=k}B^*_{\alpha,\gamma}B_{\gamma,\beta}+R_{\alpha,\beta}\Big)\varrho^{1/2}QA_{\beta,\mathsf{s}+1/2}u\Big\rangle\\ &+\langle\varrho^{1/2}Q^*A^*_{\alpha,\mathsf{s}+1/2}u, E_{\alpha,\mathsf{s}}u\rangle+\langle E_{\alpha,\mathsf{s}}u,\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rangle+\langle u, A^*_{\alpha,\mathsf{s}+1/2}i[P,Q^*Q]A_{\alpha,\mathsf{s}+1/2}u\rangle\Big].\end{aligned} \tag{455}$$

The main term is

$$\pm\sum_{|\alpha|,|\beta|,|\gamma|=k}\langle B_{\gamma,\alpha}\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u, B_{\gamma,\beta}\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rangle = \pm\sum_{|\gamma|=k}\Big\|\sum_{|\alpha|=k}B_{\gamma,\alpha}\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\Big\|^2_{L^2}. \tag{456}$$

Abbreviate this as $\lVert B\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rVert_{L^2}^2$. Thus, eq. (455) yields

$$\lVert B\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rVert_{L^2}^2 \leq \sum_{|\alpha|=k}\Big[|\langle u, A^*_{\alpha,\mathsf{s}+1/2}i[P,Q^*Q]A_{\alpha,\mathsf{s}+1/2}u\rangle| + 2|\langle E_{\alpha,\mathsf{s}}u, \varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rangle| + 2|\langle QA_{\alpha,\mathsf{s}+1/2}Pu, QA_{\alpha,\mathsf{s}+1/2}u\rangle|\Big] + \sum_{|\alpha|,|\beta|=k}|\langle \varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u, R_{\alpha,\beta}\varrho^{1/2}QA_{\beta,\mathsf{s}+1/2}u\rangle|. \tag{457}$$

If the right-hand side of eq. (456) is finite, i.e. if

$$\sum_{|\alpha|=k} B_{\gamma,\alpha}\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u \in L^2, \tag{458}$$

then, applying Υ, we conclude that

$$\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u = \varrho^{1/2}Q\varrho^{-\mathsf{s}-1/2}A_\alpha u \in H^{\infty,\mathbf{0}}_{\mathrm{de,sc}}. \tag{459}$$

As long as Q is elliptic at $\mathcal{R}^+_+$, we conclude that $\mathrm{WF}^{*,\mathsf{s}}_{\mathrm{de,sc}}(A_\alpha u)\cap\mathcal{R}^+_+ = \varnothing$. Quantitatively, this means that the $H^{-N,\mathsf{s}}_{\mathrm{de,sc}}$ norm of $A_\alpha u$ near $\mathcal{R}^+_+$ is controlled by the inequality

$$\lVert Q_1A_\alpha u\rVert_{H^{-N,\mathsf{s}}_{\mathrm{de,sc}}} \preceq \lVert B\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rVert_{L^2} + \lVert u\rVert_{H^{-N,-N}_{\mathrm{de,sc}}} \tag{460}$$

for some $Q_1\in\Psi^{0,\mathbf{0}}_{\mathrm{de,sc}}$ that is elliptic at $\mathcal{R}^+_+$, this holding for each N and $\mathsf{s}\in\mathbb{R}^5$, and for every $u\in\mathcal{S}'$.

Each of the terms on the right-hand side of eq. (457) can be controlled using the hypotheses of the propositions and the inductive hypothesis:

- First of all assuming that $1-Q$ has essential support away from $\mathcal{R}^+_+$, $[P,Q^*Q]\in\Psi^{-\infty,-\mathbf{1}}_{\mathrm{de,sc}}$ has essential support which is disjoint from $\mathcal{R}^+_+$ as well. As long as Q has essential support sufficiently close to $\mathcal{R}^+_+$, the hypotheses of either proposition considered imply that

$$\mathrm{WF}^{*,\mathsf{s}}_{\mathrm{de,sc}}(A_\alpha u)\cap\mathrm{WF}'_{\mathrm{de,sc}}([P,Q^*Q]) = \varnothing \tag{461}$$

  where either s. Thus, the $|\langle u, A^*_{\alpha,\mathsf{s}+1/2}i[P,Q^*Q]A_{\alpha,\mathsf{s}+1/2}u\rangle$ term in eq. (457) is finite and can be quantitatively controlled by the $H^{-N,\mathsf{s}}_{\mathrm{de,sc}}$ norms of $A_\alpha u$ in an annular region around $\mathcal{R}^+_+$.
- Now, letting $\tilde{Q}\in\Psi^{-\infty,\mathbf{0}}_{\mathrm{de,sc}}$ be such that $\mathrm{WF}'_{\mathrm{de,sc}}(1-\tilde{Q})\cap\mathrm{WF}'_{\mathrm{de,sc}}(Q)=\varnothing$,

$$|\langle E_{\alpha,\mathsf{s}}u, \varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rangle| \leq |\langle (1-\tilde{Q})E_{\alpha,\mathsf{s}}u, \varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rangle| + |\langle \tilde{Q}E_{\alpha,\mathsf{s}}u, \varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rangle|. \tag{462}$$

  The first term on the right-hand side is straightforward to estimate, as

$$E_{\alpha,\mathsf{s}}(1-\tilde{Q})^*\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}\in\Psi^{-\infty,-\infty}_{\mathrm{de,sc}}. \tag{463}$$

  On the other hand, by Cauchy-Schwarz and AM-GM,

$$2|\langle \tilde{Q}E_{\alpha,\mathsf{s}}u, \varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u\rangle| \leq \epsilon^{-1}\lVert\tilde{Q}E_{\alpha,\mathsf{s}}u\rVert_{L^2}^2 + \epsilon\lVert\varrho^{1/2}QA^*_{\alpha,\mathsf{s}+1/2}u\rVert^2_{L^2} \tag{464}$$

  for any $\epsilon>0$. The term $\lVert\tilde{Q}E_{\alpha,\mathsf{s}}u\rVert^2_{L^2}$ can be controlled by the inductive hypothesis, since $E_{\alpha,\mathsf{s}}\in\Psi^{-\infty,-\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{k-1}_+$.

  The other term, which is controlled in terms of the $H^{-N,\mathsf{s}}_{\mathrm{de,sc}}$ norms of $A_\alpha u$ near $\mathcal{R}^+_+$, but this is suppressed a factor of ϵ and therefore can be absorbed into the left-hand side of the ultimate estimate.
- Because $R_{\alpha,\beta}\in\Psi^{-\infty,-\mathbf{1}}_{\mathrm{de,sc}}$, we have $R_{\alpha,\beta}\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}\in\Psi^{-\infty,-\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{k-1}_+$. Thus, the

$$|\langle\varrho^{1/2}QA_{\alpha,\mathsf{s}+1/2}u, R_{\alpha,\beta}\varrho^{1/2}QA_{\beta,\mathsf{s}+1/2}u\rangle| \tag{465}$$

  terms in eq. (457) can be estimated like the previous class of terms.

- Finally, consider the $|\langle QA_{\alpha,\mathsf{s}+1/2}Pu, QA_{\alpha,\mathsf{s}+1/2}u\rangle|$ term in eq. (457). We can write
$$Q\varrho^{-1/2} = \varrho^{-1/2}Q + \varrho^{-1/2}F \tag{466}$$
for some $F \in \Psi^{-\infty,-1}_{\mathrm{de,sc}}$. Then,
$$|\langle QA_{\alpha,\mathsf{s}+1/2}Pu, QA_{\alpha,\mathsf{s}+1/2}u\rangle| \leq |\langle QA_{\alpha,\mathsf{s}+1/2}Pu, \varrho^{-1/2}QA_{\alpha,\mathsf{s}}u\rangle| + |\langle \varrho^{-1/2}QA_{\alpha,\mathsf{s}+1/2}Pu, FA_{\alpha,\mathsf{s}}u\rangle|. \tag{467}$$
By Cauchy-Schwarz and AM-GM, the second term on the right-hand side is bounded above as follows:
$$|\langle \varrho^{-1/2}QA_{\alpha,\mathsf{s}}Pu, FA_{\alpha,\mathsf{s}+1/2}u\rangle| \preceq \|\varrho^{-1/2}QA_{\alpha,\mathsf{s}+1/2}Pu\|^2_{L^2} + \|FQA_{\alpha,\mathsf{s}}u\|^2_{L^2}. \tag{468}$$
The first term on the right-hand side of eq. (468) can be controlled using the hypotheses of the propositions to be proven. On the other hand,
$$FQA_{\alpha,\mathsf{s}} \in \Psi^{-\infty,\mathsf{s}}_{\mathrm{de,sc}}\mathfrak{M}^{k-1}_{+}, \tag{469}$$
so the second term on the right-hand side of eq. (468) is controlled using the inductive hypothesis. The first term on the right-hand side of eq. (467) can be bounded above by
$$2| \leq |\langle QA_{\alpha,\mathsf{s}+1/2}Pu, \varrho^{-1/2}QA_{\alpha,\mathsf{s}}u\rangle| \leq \epsilon^{-1}\|\varrho^{-1/2}QA_{\alpha,\mathsf{s}+1/2}Pu\|^2_{L^2} + \epsilon\|QA_{\alpha,\mathsf{s}}u\|^2_{L^2}. \tag{470}$$
As above, the $\epsilon\|QA_{\alpha,\mathsf{s}}u\|^2_{L^2}$ will be able to be absorbed into the left-hand side of the ultimate estimates. The remaining term is controllable, for each $\epsilon > 0$, in terms of the $H^{-N,\mathsf{s}+1}_{\mathrm{de,sc}}$ norms of $A_\alpha Pu$ near the radial set, which are finite by the hypotheses of the propositions to be proven.

The upshot is that, assuming the algebraic manipulations above are justified, then the $H^{-N,\mathsf{s}}_{\mathrm{de,sc}}$ norms of the $A_\alpha u$ near $\mathcal{R}^+_+$ in terms of quantities already under control by the inductive hypothesis or assumptions of the propositions. Regularizing, in a manner completely analogous to that in [HMV04][HMV08][GR+20], suffices to show that this estimate holds in the strong sense that if the terms on the right-hand side are all finite, then the left-hand side is finite as well. This then yields the next step in the first induction.

6.3. **Second induction.**

We now induct on κ. We first prove:

Lemma 6.2 (Cf. [GR+20], eq. 3.31)**.** *Let $m \in \mathbb{R}$, $\mathsf{s} \in \mathbb{R}^5$ be arbitrary, and let $k \in \mathbb{N}^+$ and $\kappa \in \mathbb{N}$. Suppose that $u \in \mathcal{S}'$ satisfies*

- $\mathrm{WF}^{*,\mathsf{s}}_{\mathrm{de,sc}}(Au) \cap \mathcal{R}^+_+ = \varnothing$ *and*
- $\mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}(APu) \cap \mathcal{R}^+_+ = \varnothing$

for all $A \in \mathfrak{M}^{\kappa,k}_{+,+}$. Then,
$$\mathrm{WF}^{*,\mathsf{s}}_{\mathrm{de,sc}}(Au) \cap \mathcal{R}^+_+ = \varnothing \tag{471}$$
for all $A \in \mathfrak{M}^{\kappa+1,k-1}_{+,+}$. ■

Proof. We will prove that, under the hypotheses of the lemma, eq. (471) holds for all $A \in \mathfrak{M}^{\kappa+1,j}_{+,+}$ for $j \in \{0,\ldots,k-1\}$. The proof proceeds inductively on j, with $j = 0$ as the base case.

For each j, that it suffices to check eq. (471) for a set of ΨDOs spanning $\mathfrak{M}^{\kappa+1,j}_{+,+}$ as a left $\Psi^{0,0}_{\mathrm{de,sc}}$-module. Since $\mathfrak{M}^{\kappa+1,j}_{+,+}$ is generated as a left $\Psi^{0,0}_{\mathrm{de,sc}}$-module by products of the form V_+A_0 for $A_0 \in \mathfrak{M}^{\kappa,j}_{+,+}$ together with elements of $\mathfrak{M}^{\kappa,j+1}_{+,+}$, in order to show that eq. (471) holds for all $A \in \mathfrak{M}^{\kappa+1,j}_{+,+}$ it suffices to prove that
$$\mathrm{WF}^{*,\mathsf{s}}_{\mathrm{de,sc}}(V_+A_0u) \cap \mathcal{R}^+_+ = \varnothing \tag{472}$$

for $A_0 \in \mathfrak{M}^{\kappa,j}_{+,+}$. (Indeed, since $j \leq k-1$, if $A \in \mathfrak{M}^{\kappa,j+1}_{+,+}$ then $A \in \mathfrak{M}^{\kappa,k}_{+,+}$, so eq. (471) holds for such A by hypothesis.) In particular, in order to prove the result for $\kappa, j = 0$, we only need to prove that

$$\mathrm{WF}^{*,\mathsf{s}}_{\mathrm{de,sc}}(V_+u) \cap \mathcal{R}^+_+ = \varnothing, \tag{473}$$

that is, we only need to consider $A_0 = 1$.

Applying Proposition 3.10, we write $P = \tau^{-2}V_-V_+ + \tau^{-2}(d-1)V_- + R$ for some $R \in \mathfrak{N}^2$. Consequently, for any $A_0 \in \Psi_{\mathrm{de,sc}}$,

$$\begin{aligned}\tau^{-2}V_-V_+A_0u = A_0Pu - \tau^{-2}(d-1)V_-A_0u - \varrho^2RA_0u \\ + [\tau^{-2}V_-V_+, A_0]u + [\tau^{-2}(d-1)V_-, A_0]u + [\varrho^2R, A_0]u.\end{aligned} \tag{474}$$

Since $\tau^{-2}V_- \in \mathrm{Diff}^{1,(-1,-1,-\infty,-1,-1)}_{\mathrm{de,sc}}$ is elliptic at $\mathcal{R}^+_+$, it suffices to prove that the sets

$$\mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu), \mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}(\tau^{-2}(d-1)V_-u), \mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}(\varrho^2Ru) \tag{475}$$

$$\mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}([\tau^{-2}V_-V_+, A_0]u), \mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}([\tau^{-2}(d-1)V_-, A_0]u), \mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}([\varrho^2R, A_0]u) \tag{476}$$

are all disjoint from $\mathcal{R}^+_+$. That this is true for A_0Pu is a hypothesis. That this is true for $\tau^{-2}(d-1)V_-A_0u$ follows from the other hypothesis, which says that $\mathrm{WF}^{*,\mathsf{s}}_{\mathrm{de,sc}}(A_0u) \cap \mathcal{R}^+_+ = \varnothing$, and from

$$\tau^{-2}(d-1)V_- \in \mathrm{Diff}^{1,-1}_{\mathrm{de,sc}}. \tag{477}$$

Because $\mathfrak{N}^2 \subset \Psi^{1,1}_{\mathrm{de,sc}}\mathfrak{N}$, we have

$$\varrho^2R \in \Psi^{1,-1}_{\mathrm{de,sc}}\mathfrak{N}, \tag{478}$$

so the same logic applies to ϱ^2RA_0u. It only remains to check the terms in the second line of eq. (474). If $\kappa = 0$ and $j = 0$, then, since we are only considering $A_0 = 1$, all of the terms in the second line of eq. (474) are just zero, so we are done. Otherwise:

- Suppose that $A_0 \in \mathfrak{M}^{\kappa,j}_{+,+}$. From $\varrho^2R \in \Psi^{1,-1}_{\mathrm{de,sc}}\mathfrak{N}_+$, we get
$$[\varrho^2R, A_0] \in \Psi^{1,-1}_{\mathrm{de,sc}}(1_{\kappa>0}\mathfrak{M}^{\kappa-1,j+1}_{+,+} + \mathfrak{M}^{\kappa,j}_{+,+}) \tag{479}$$
via eq. (190), so the first hypothesis above implies that the final term in eq. (474) has $\mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}$ disjoint from $\mathcal{R}^+_+$.
- Analogous reasoning also applies to the penultimate term in eq. (474).
- On the other hand,
$$[\tau^{-2}V_-V_+, A_0] = \tau^{-2}V_-[V_+, A_0] + [\tau^{-2}V_-, A_0]V_+; \tag{480}$$
$$\tau^{-2}V_-[V_+, A_0] \in \Psi^{1,-1}_{\mathrm{de,sc}}(\mathfrak{M}^{\kappa,j}_{+,+} + 1_{j>0}\mathfrak{M}^{\kappa+1,j-1}_{+,+}), \tag{481}$$
$$[\tau^{-2}V_-, A_0]V_+ \in \Psi^{1,-1}_{\mathrm{de,sc}}(1_{\kappa>0}\mathfrak{M}^{\kappa,j}_{+,+} + 1_{j>0}\mathfrak{M}^{\kappa+1,j-1}_{+,+} + \mathfrak{M}^{1,0}_{+,+}) \tag{482}$$
via eq. (190) and the second clause of Proposition 3.12. The first hypothesis above implies that
$$\mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}(Bu) \cap \mathcal{R}^+_+ = \varnothing \tag{483}$$
for all $B \in \Psi^{1,-1}_{\mathrm{de,sc}}\mathfrak{M}^{\kappa,j}_{+,+}$. Likewise, since we already know the $\kappa = 0$ and $j = 0$ case of the result,
$$\mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}(Bu) \cap \mathcal{R}^+_+ = \varnothing \tag{484}$$
holds for $B \in \Psi^{1,-1}_{\mathrm{de,sc}}\mathfrak{M}^{1,0}_{+,+}$. If $j = 0$, then we can immediately conclude
$$\begin{aligned}\mathrm{WF}^{m,\mathsf{s}+1}_{\mathrm{de,sc}}(\tau^{-2}V_-[V_+, A]u) \cap \mathcal{R}^+_+ = \varnothing, \\ \mathrm{WF}^{m,\mathsf{s}+1}_{\mathrm{de,sc}}([\tau^{-2}V_-, A]V_+u) \cap \mathcal{R}^+_+ = \varnothing.\end{aligned} \tag{485}$$

This completes the proof in the $j=0$ case. If $j\geq 1$, then the inductive hypothesis says that

$$\mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}(Bu)\cap\mathcal{R}^+_+=\varnothing \tag{486}$$

for all $B\in\Psi^{1,-1}_{\mathrm{de,sc}}\mathfrak{M}^{\kappa+1,j-1}_{+,+}$, so we can still conclude eq. (485).

□

Consequently:

Proposition 6.3. *If*

$$\mathrm{WF}^{*,\mathsf{s}}_{\mathrm{de,sc}}(Au)\cap\mathcal{R}^+_+=\varnothing \tag{487}$$

for all $A\in\mathfrak{N}^{\kappa+k}_+$ *and* $\mathrm{WF}^{*,\mathsf{s}+1}_{\mathrm{de,sc}}(APu)\cap\mathcal{R}^+_+=\varnothing$ *for all* $A\in\mathfrak{M}^{\kappa,k}_{+,+}$*, then eq. (487) holds for all* $A\in\mathfrak{M}^{\kappa,k}_{+,+}$. ■□

Theorem 4 and Theorem 5 follow.

7. Proofs of main theorems

We now spell out the precise hypotheses under which the main theorems are proven. We do not aim to be maximally general here; we call a Lorentzian metric g on $\mathbb{R}^{1,d}$ *admissible* if the following conditions are satisfied:

- g satisfies $g-g_{\mathbb{M}}\in\varrho^2_{\mathrm{Pf}}\varrho^2_{\mathrm{nPf}}\varrho^2_{\mathrm{Sf}}\varrho^2_{\mathrm{nFf}}\varrho^2_{\mathrm{Ff}}C^\infty(\mathbb{O};{}^{\mathrm{de,sc}}\mathrm{Sym}\,T^*\mathbb{O})$, where $g_{\mathbb{M}}$ is the exact Minkowski metric,
- $(\mathbb{R}^{1,d},g)$ is globally hyperbolic and t serves as a time function, so that $\mathrm{d}t$ timelike,
- $\Sigma_T=\{(t,\mathbf{x})\in\mathbb{R}^{1,d}:t=T\}$ is a Cauchy hypersurface for each $T\in\mathbb{R}$,
- any null geodesic, when projected down to $\mathbb{M}^\circ$, tends to null infinity in both directions.

The first condition specifies the precise sense in which g is asymptotically flat. If $g\in g_{\mathbb{M}}+(1+t^2+r^2)^{-1}C^\infty(\mathbb{M};{}^{\mathrm{sc}}\mathrm{Sym}\,T^*\mathbb{M})$, then the first condition is satisfied. Indeed, a frame for ${}^{\mathrm{sc}}\mathrm{Sym}\,T^*\mathbb{M}$ is given by the sections $\mathrm{d}x_i\odot\mathrm{d}x_j$ for $i\in\{0,\dots,d\}$, and the computations in §2 show that

$$\mathrm{d}x_i\odot\mathrm{d}x_j\in\varrho^{-1}_{\mathrm{nPf}}\varrho^{-1}_{\mathrm{nFf}}C^\infty(\mathbb{O};{}^{\mathrm{de,sc}}\mathrm{Sym}\,T^*\mathbb{O}). \tag{488}$$

Since $(1+t^2+r^2)^{-1}\in\varrho^2_{\mathrm{Pf}}\varrho^4_{\mathrm{nPf}}\varrho^2_{\mathrm{Sf}}\varrho^4_{\mathrm{nFf}}\varrho^2_{\mathrm{Ff}}C^\infty(\mathbb{O})$, this implies that

$$g-g_{\mathbb{M}}\in\varrho^2_{\mathrm{Pf}}\varrho^3_{\mathrm{nPf}}\varrho^2_{\mathrm{Sf}}\varrho^3_{\mathrm{nFf}}\varrho^2_{\mathrm{Ff}}C^\infty(\mathbb{O};{}^{\mathrm{de,sc}}\mathrm{Sym}\,T^*\mathbb{O}), \tag{489}$$

so g is asymptotically flat in the sense above. It is not difficult to construct $g\in g_{\mathbb{M}}+(1+t^2+r^2)^{-1}C^\infty(\mathbb{M};{}^{\mathrm{sc}}\mathrm{Sym}\,T^*\mathbb{M})$ besides $g_{\mathbb{M}}$ itself satisfying the other conditions above, so the discussion below applies to more than just exact Minkowski spacetime.

Given the setup above, the d'Alembertian $\Box_g$ satisfies $\Box_g-\Box\in\mathrm{Diff}^{2,-2}_{\mathrm{de,sc}}(\mathbb{O})$. Consider now an operator of the form

$$P=\Box_g+Q+\mathsf{m}^2 \tag{490}$$

for $Q\in\mathrm{Diff}^{1,-2}_{\mathrm{de,sc}}(\mathbb{O})$. Such an operator has all of the properties required in each of the previous sections, so we can cite the various results.

7.1. **Initial value problem.** We now prove Theorem 1. Let χ be as in that theorem. First, for solutions to the IVP that are assumed to be tempered:

Proposition 7.1. *Suppose that* $u\in\mathcal{S}'(\mathbb{R}^{1,d})$ *is a solution to the IVP*

$$\begin{cases}Pu=f\\ u|_{t=0}=u^{(0)},\\ \partial_t u|_{t=0}=u^{(1)}\end{cases} \tag{491}$$

for some $f \in \mathcal{S}(\mathbb{R}^{1,d})$*,* $u^{(0)}, u^{(1)} \in \mathcal{S}(\mathbb{R}^d)$*. Then,* u *has the form*

$$u = u_0 + \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{-i\mathsf{m}\sqrt{t^2-r^2}} u_- + \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{+i\mathsf{m}\sqrt{t^2-r^2}} u_+ \tag{492}$$

for some $u_0 \in \mathcal{S}(\mathbb{R}^{1,d})$ *and some* $u_\pm \in \varrho_{\mathrm{nPf}}^\infty \varrho_{\mathrm{Sf}}^\infty \varrho_{\mathrm{nFf}}^\infty C^\infty(\mathbb{O}) = \bigcap_{k \in \mathbb{N}} \varrho_{\mathrm{nPf}}^k \varrho_{\mathrm{Sf}}^k \varrho_{\mathrm{nFf}}^k C^\infty(\mathbb{O})$. ■

Proof. In order to get started, we need to know that u (which can be deduced to be smooth via the Duistermaat–Hörmander theorem or propagation of singularities in physical space) is Schwartz in a neighborhood of $\mathrm{cl}_{\mathbb{M}}\{t = 0\}$ in $\mathbb{M}$. One way to see this is to consider the advanced and retarded components $u^-(t, \mathbf{x}) = (1 - \Theta(t))u(t, \mathbf{x})$ and $u^+(t, \mathbf{x}) = \Theta(t)u(t, \mathbf{x})$, where Θ denotes a Heaviside function. We have $u^\pm \in \mathcal{S}'(\mathbb{R}^{1,d})$, as can be seen from the energy estimate corollary $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{R}^d))$. These satisfy

$$Pu^\pm(t, \mathbf{x}) = \pm(\delta'(t) f_1(\mathbf{x}) + \delta(t) f_2(\mathbf{x})) \tag{493}$$

for some $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$ depending on $u^{(0)}$ and $u^{(1)}$ and on P. The sc-wavefront sets $\mathrm{WF}_{\mathrm{sc}}(\delta'(t) f_1(\mathbf{x}))$, $\mathrm{WF}_{\mathrm{sc}}(\delta(t) f_2(\mathbf{x}))$ are disjoint from the sc-characteristic set of P. Indeed, it can be checked (either directly, or via an argument presented after the end of this proof) that

$$\mathrm{WF}_{\mathrm{sc}}(\delta'(t) f_1(\mathbf{x})), \mathrm{WF}_{\mathrm{sc}}(\delta(t) f_2(\mathbf{x})) \subseteq {}^{\mathrm{sc}}N^* \mathrm{cl}_{\mathbb{M}}\{t = 0\} \cap {}^{\mathrm{sc}}\mathbb{S}^* \mathbb{M}, \tag{494}$$

and the right-hand side is disjoint from the sc-characteristic set of P which intersects ${}^{\mathrm{sc}}N^* \mathrm{cl}_{\mathbb{M}}\{t = 0\} \cap^{\mathrm{sc}}$ only away from fiber infinity. Since $u^\pm$ vanish identically in one of the two temporal hemispheres $\mathrm{cl}_{\mathbb{M}}\{\mp t > 0\} \backslash \mathrm{cl}_{\mathbb{M}}\{t = 0\}$, $u^\pm$ has no sc-wavefront set over the corresponding hemisphere. We can therefore apply sc-propagation results [Vas18] (noting that the wavefront sets $\mathrm{WF}_{\mathrm{sc}}(\delta'(t) f_1(\mathbf{x}))$, $\mathrm{WF}_{\mathrm{sc}}(\delta(t) f_2(\mathbf{x}))$ do not interrupt the propagation) to conclude that the portion of $\mathrm{WF}_{\mathrm{sc}}(u^\pm)$ inside the sc-characteristic set is a subset of the radial sets of the sc-Hamiltonian flow. The same therefore applies to $u = u^- + u^+$. But, by elliptic regularity in the sc-calculus (using that f is Schwartz), $\mathrm{WF}_{\mathrm{sc}}(u)$ is a subset of the sc-characteristic set of P. So, $\mathrm{WF}_{\mathrm{sc}}(u)$ is a subset of the radial set of the sc-Hamiltonian flow. This certainly implies that u is Schwartz in a neighborhood of $\mathrm{cl}_{\mathbb{M}}\{t = 0\}$ in $\mathbb{M}$.

This implies that

$$\mathrm{WF}_{\mathrm{de,sc}}(u) \subseteq \mathcal{R} \cup {}^{\mathrm{de,sc}}\pi^{-1}(\mathrm{nPf} \cup \mathrm{nFf}). \tag{495}$$

By Theorem 2, we can strengthen this to

$$\mathrm{WF}_{\mathrm{de,sc}}(u) \subseteq \mathcal{R}. \tag{496}$$

Indeed, given any $m \in \mathbb{R}$ and $\mathsf{s} \in \mathbb{R}^5$, we can find some $m_0 > m$ and $\mathsf{s}_0 > \mathsf{s}$ such that the pair (m_0, s_0) satisfies the hypotheses of that theorem. The theorem then tells us that

$$\mathrm{WF}^{m_0, \mathsf{s}_0}_{\mathrm{de,sc}}(u) \subseteq \mathcal{R}. \tag{497}$$

Equation (496) then follows from $\mathrm{WF}_{\mathrm{de,sc}}(u) = \mathrm{cl} \bigcup_{m_0, \mathsf{s}_0} \mathrm{WF}^{m_0, \mathsf{s}_0}_{\mathrm{de,sc}}(u)$, since $\mathcal{R}$ is closed.

Now, we can find $Q_\pm \in \Psi^{0,\mathbf{0}}_{\mathrm{de,sc}}$ such that

- $1 = Q_- + Q_+$,
- $\mathrm{WF}'_{\mathrm{de,sc}}(Q_\pm) \cap \Sigma_{\mathsf{m},\mp} = \varnothing$.

Let $\chi_0 \in C^\infty(\mathbb{M})$ be identically equal to 0 in some neighborhood of the past cap and identically equal to 1 in some neighborhood of the future cap. Then, we can define

$$Q^\pm_- = (1 - \chi_0) Q_\pm, \qquad Q^\pm_+ = \chi_0 Q_\pm. \tag{498}$$

For signs $\varsigma, \sigma \in \{-, +\}$, let $u^\varsigma_\sigma = Q^\varsigma_\sigma u$. Observe that $Pu^\varsigma_\sigma = Q^\varsigma_\sigma f + [P, Q^\varsigma_\sigma]u$. Since $\mathrm{WF}'_{\mathrm{de,sc}}([P, Q^\varsigma_\sigma]) \cap \mathcal{R} = \varnothing$, we have

$$\mathrm{WF}'_{\mathrm{de,sc}}([P, Q^\varsigma_\sigma]) \cap \mathrm{WF}_{\mathrm{de,sc}}(u) = \varnothing, \tag{499}$$

which implies that $[P, Q^{\varsigma}_{\sigma}]u$ is Schwartz. So, $f^{\varsigma}_{\sigma} = Q^{\varsigma}_{\sigma} f + [P, Q^{\varsigma}_{\sigma}]u$ is Schwartz. Moreover, by construction,

$$\mathrm{WF}_{\mathrm{de,sc}}(u^{\varsigma}_{\sigma}) \subseteq \mathcal{R}^{\varsigma}_{\sigma}. \tag{500}$$

For s with $s_{\mathrm{Pf}}, s_{\mathrm{Ff}} < -1/2$, we can apply Theorem 5 (for each possible pair of signs) to the u^{ς}_{σ}, the hypothesis of which is trivially satisfied as a consequence of eq. (496), eq. (500). The conclusion is that

$$\begin{aligned} u^{\varsigma}_{-} &\in H^{\infty,(s_{\mathrm{Pf}},\infty,\infty,\infty,\infty);\infty,\infty}_{\mathrm{de,sc};\varsigma,-}, \\ u^{\varsigma}_{+} &\in H^{\infty,(\infty,\infty,\infty,\infty,s_{\mathrm{Ff}});\infty,\infty}_{\mathrm{de,sc};\varsigma,-}. \end{aligned} \tag{501}$$

Taking $s_{\mathrm{Pf}}, s_{\mathrm{Ff}} \in (-3/2, -1/2)$, we can cite Proposition 3.19 to conclude that $u = u^{-}_{-} + u^{+}_{-} + u^{-}_{+} + u^{+}_{+}$ has the form specified in the theorem □

If $f \in \mathcal{S}(\mathbb{R}^d)$, then f, viewed initially as a function on $\Sigma_0 = \{(t, \mathbf{x}) \in \mathbb{R}^{1,d} : t = 0\}$, can be extended to a Schwartz function F on $\mathbb{R}^{1,d}$. This implies that, for any $m, s \in \mathbb{R}$,

$$\mathrm{WF}^{m,s}_{\mathrm{sc}}(\delta'(t) f(\mathbf{x})) \subseteq \mathrm{WF}^{m,s_0}_{\mathrm{sc}}(\delta'(t)) \tag{502}$$

for any $s_0 \in \mathbb{R}$, because $\delta'(t) f(\mathbf{x}) = M_F \delta'(t)$ and $M_F \in \mathrm{Diff}^{0,-\infty}_{\mathrm{sc}}(\mathbb{M})$. Similarly, $\mathrm{WF}^{m,s}_{\mathrm{sc}}(\delta(t) f(\mathbf{x})) \subseteq \mathrm{WF}^{m,s_0}_{\mathrm{sc}}(\delta(t))$. Consequently, in order to verify eq. (494), it suffices to prove that

$$\mathrm{WF}^{m,s_0}_{\mathrm{sc}}(\delta'(t)), \mathrm{WF}^{m,s_0}_{\mathrm{sc}}(\delta(t)) \subseteq {}^{\mathrm{sc}}N^* \mathrm{cl}_{\mathbb{M}}\{t = 0\} \cap {}^{\mathrm{sc}}\mathbb{S}^* \mathbb{M} \tag{503}$$

for some $s_0 = s_0(m) \in \mathbb{R}$. Moreover, since $\partial_t \in \mathrm{Diff}^{1,0}_{\mathrm{sc}}(\mathbb{M})$, we know that $\mathrm{WF}^{m,s_0}_{\mathrm{sc}}(\delta'(t)) \subseteq \mathrm{WF}^{m+1,s_0}_{\mathrm{sc}}(\delta(t))$, so it suffices to prove the above for just $\delta(t)$. In order to do this, we use that $\mathrm{WF}^{m,s}_{\mathrm{sc}}(w) = \mathcal{F}^{-1}_{*} \circ \mathrm{WF}^{s,m}_{\mathrm{sc}}(\mathcal{F} w)$ for every $w \in \mathcal{S}'(\mathbb{R}^{1,d})$, where $\mathcal{F}$ is the spacetime Fourier transform and $\mathcal{F}^{-1}_{*}$ is the involution of ${}^{\mathrm{sc}}\overline{T}^* \mathbb{M}$ switching frequency and position (choosing sign conventions appropriately). Thus,

$$\mathrm{WF}^{m,s_0}_{\mathrm{sc}}(\delta(t)) = \mathcal{F}_{*} \mathrm{WF}^{s_0,m}_{\mathrm{sc}}(\delta(\mathbf{x})). \tag{504}$$

Recalling that the portion of $\mathrm{WF}^{s_0,m}_{\mathrm{sc}}(\delta(\mathbf{x}))$ over the interior is just $\mathrm{WF}^{s_0}(\delta(\mathbf{x}))$, if s_0 is sufficiently negative then $\mathrm{WF}^{s_0,m}_{\mathrm{sc}}(\delta(\mathbf{x}))$ is contained entirely over the boundary, which says that $\mathcal{F}_{*} \mathrm{WF}^{s_0,m}_{\mathrm{sc}}(\delta(\mathbf{x}))$ is contained entirely at fiber infinity. Thus, $\mathrm{WF}^{m,s_0}_{\mathrm{sc}}(\delta(t)) \subseteq {}^{\mathrm{sc}}\mathbb{S}^* \mathbb{M}$. In order to see that $\mathrm{WF}^{m,s_0}_{\mathrm{sc}}(\delta(t)) \subseteq {}^{\mathrm{sc}}N^* \mathrm{cl}_{\mathbb{M}}\{t = 0\}$, note that $t\delta = 0$ and $\triangle \delta = 0$, where $\triangle = -(\partial^2_{x_1} + \cdots + \partial^2_{x_d})$ is the spatial Laplacian. The former implies that $\mathrm{WF}^{m,s_0}_{\mathrm{sc}}(\delta(t))$ is contained over $\mathrm{cl}_{\mathbb{M}}\{t = 0\}$, and the latter implies that

$$\mathrm{WF}_{\mathrm{sc}}(\delta(t)) \subseteq \mathrm{Char}^{2,0}_{\mathrm{sc}}(\triangle). \tag{505}$$

As $\mathrm{Char}^{2,0}_{\mathrm{sc}}(\triangle) \cap {}^{\mathrm{sc}}\pi^{-1} \mathrm{cl}_{\mathbb{M}}\{t = 0\} = N^* \mathrm{cl}_{\mathbb{M}}\{t = 0\}$, this completes the verification.

In order to see that a solution to the IVP eq. (5) with Schwartz initial data (and indeed, much worse initial data) is automatically tempered, a basic energy estimate suffices; this is proved in §7.3. Thus, the temperedness hypothesis of the previous proposition can be removed, yielding finally Theorem 1.

7.2. **Scattering problems.** Say that the forward problem for P is well-posed if, for any $f \in \mathcal{S}(\mathbb{R}^{1,d})$, there exists a unique $u \in C^\infty(\mathbb{R}^{1,d}) \cap \mathcal{S}'(\mathbb{R}^{1,d})$ such that

- $Pu = f$, and
- $\chi_0 u \in \mathcal{S}(\mathbb{R}^{1,d})$ whenever $\chi_0 \in C^\infty(\mathbb{M})$ is identically 0 near the future timelike cap.

There exist criteria in the literature that suffice for this. One can prove this for the P considered above using the energy estimate in §7.3 in conjunction with a duality argument, but we do not present the details here, so the next proposition is stated with well-posedness of the forward problem as an assumption.

Proposition 7.2. *Let $v_\pm$ denote Schwartz functions on the past timelike cap of $\mathbb{M}$. Then, assuming that the forward problem for P is well-posed, there exists a unique function $u \in C^\infty(\mathbb{R}^{1,d})$ such that $Pu = 0$ and*

$$u = u_0 + \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{-i\mathsf{m}\sqrt{t^2-r^2}} u_- + \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{+i\mathsf{m}\sqrt{t^2-r^2}} u_+ \tag{506}$$

for some Schwartz $u_0 \in \mathcal{S}(\mathbb{R}^{1,d})$ and $u_\pm \in \varrho_{\mathrm{nPf}}^\infty \varrho_{\mathrm{Sf}}^\infty \varrho_{\mathrm{nFf}}^\infty C^\infty(\mathbb{O})$ such that, restricted to the past timelike cap, $u_\pm$ agree with $v_\pm$. ■

Proof. By Proposition 3.20, there exists functions $u_{-,\mathrm{pre}}, u_{+,\mathrm{pre}} \in \varrho_{\mathrm{nPf}}^\infty \varrho_{\mathrm{Sf}}^\infty \varrho_{\mathrm{nFf}}^\infty \varrho_{\mathrm{Ff}}^\infty \in C^\infty(\mathbb{O})$ such that $u_{\pm,\mathrm{pre}}$, when restricted to the past timelike cap, agree with $v_\pm$, and such that the function u_{pre} defined by

$$u_{\mathrm{pre}} = \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{-i\mathsf{m}\sqrt{t^2-r^2}} u_{-,\mathrm{pre}} + \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{+i\mathsf{m}\sqrt{t^2-r^2}} u_{+,\mathrm{pre}} \tag{507}$$

satisfies $Pu_{\mathrm{pre}} \in \mathcal{S}(\mathbb{R}^{1,d})$. Let $f = Pu_{\mathrm{pre}}$. By the existence clause of the well-posedness of the forward problem, there exists a function $w \in \mathcal{S}'(\mathbb{R}^{1,d})$ such that $Pw = -f$ and $\chi_0 w$ is Schwartz whenever χ_0 is identically 0 near the future timelike cap. In particular, w solves the IVP

$$\begin{cases} Pw = -f, \\ w|_{t=0} = w^{(0)}, \\ \partial_t w|_{t=0} = w^{(1)}, \end{cases} \tag{508}$$

for some $w^{(0)}, w^{(1)} \in \mathcal{S}(\mathbb{R}^d)$. By Proposition 7.1, w has the form

$$w = w_0 + \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{-i\mathsf{m}\sqrt{t^2-r^2}} w_- + \chi \varrho_{\mathrm{Pf}}^{d/2} \varrho_{\mathrm{Ff}}^{d/2} e^{+i\mathsf{m}\sqrt{t^2-r^2}} w_+ \tag{509}$$

for some Schwartz $w_0 \in \mathcal{S}(\mathbb{R}^{1,d})$ and $w_\pm \in C^\infty(\mathbb{O})$. Moreover, $w_\pm$ can be chosen to be supported near $\mathrm{nFf} \cup \mathrm{Ff}$ (or even just near Ff). Set $u = u_{\mathrm{pre}} + w$. This solves $Pu = 0$ and has the form eq. (507) for $u_\pm = u_{\pm,\mathrm{pre}} + w_\pm$ and $u_0 = w_0$. By the support condition on $w_\pm$, the restrictions of $u_\pm$ to the past timelike caps are the same as the restrictions of $u_{\pm,\mathrm{pre}}$.

Conversely, suppose that we are given u of the form eq. (506) with $u_0 \in \mathcal{S}(\mathbb{R}^{1,d})$, $u_\pm \in \varrho_{\mathrm{nPf}}^\infty \varrho_{\mathrm{Sf}}^\infty \varrho_{\mathrm{nFf}}^\infty C^\infty(\mathbb{O})$ restricting to $v_\pm$ at the past timelike caps. Define $w_? = u - u_{\mathrm{pre}}$. This satisfies $Pw_? = -f$. Choosing $\chi_0 \in C^\infty(\mathbb{M})$ to be both identically 0 near the future timelike cap *and* identically 1 near the past timelike cap, we have

$$P(\chi_0 w_?) = -\chi_0 f + [P, \chi_0] w_? \in \mathcal{S}(\mathbb{R}^{1,d}). \tag{510}$$

Any function of the form eq. (506) lies in $H_{\mathrm{de,sc}}^{\infty,(-1/2-,\infty,\infty,\infty,\infty)}(\mathbb{O})$. Since the leading order terms in the asymptotic expansions of u, u_{pre} at the past timelike cap agree,

$$\chi_0 w_? \in H_{\mathrm{de,sc}}^{\infty,(-1/2+\varepsilon,\infty,\infty,\infty,\infty)}(\mathbb{O}) \tag{511}$$

for any $\varepsilon < 1$. By Theorem 4, we can actually conclude that $\mathrm{WF}_{\mathrm{de,sc}}(\chi_0 w_?) \cap \mathcal{R}_- = \varnothing$. Thus, by Theorem 3, $\chi_0 w_? \in \mathcal{S}(\mathbb{R}^{1,d})$. This implies that $\chi_1 w_? \in \mathcal{S}(\mathbb{R}^{1,d})$ whenever χ_1 is identically 0 near the future timelike cap. So, $w_?$ solves the same forward problem that w does. By the uniqueness clause of the well-posedness of the forward problem, $w_? = w$. This shows that u is unique. □

7.3. **Temperedness.** Here, we give a self-contained proof that the solutions u to the initial value problem are tempered. The argument below is, unsurprisingly, of a standard sort via an energy estimate. The point is just that the specific assumptions under which the main theorem is stated suffice for the argument to go through. The operators considered in the body of the paper, as well as their formal $L^2(\mathbb{R}^{1,d})$-based adjoints, have the form

$$L = (1 + a_{00}) \frac{\partial^2}{\partial t^2} + \sum_{j=1}^d a_{0j} \frac{\partial}{\partial t} \frac{\partial}{\partial x_j} - \sum_{j,k=1}^d (1 - a_{jk}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} + \sum_{j=0}^d b_j \frac{\partial}{\partial x_j} + V + \mathsf{m}^2 \tag{512}$$

for some $a_{jk} = a_{kj} \in C^\infty(\mathbb{R}^{1,d};\mathbb{R})$, $b_j \in C^\infty(\mathbb{R}^{1,d})$, and $V \in C^\infty(\mathbb{R}^{1,d})$, all of which are decaying symbols on $\mathbb{O}$. In particular, on each Cauchy hypersurface $\Sigma_T = \{(t,\mathbf{x}) \in \mathbb{R}^{1,d} : t = T\}$, which stays away from null infinity, $\partial_t a_{j,k}(t,\mathbf{x}), \partial_{x_\ell} a_{j,k}(t,\mathbf{x}) \in \langle \mathbf{x}\rangle^{-2} L^\infty(\mathbb{R}^d_{\mathbf{x}})$, $b_j(t,\mathbf{x}), V(t,\mathbf{x}) \in \langle \mathbf{x}\rangle^{-1} L^\infty(\mathbb{R}^d_{\mathbf{x}})$, and likewise for higher derivatives. These suffice to prove the most basic estimates. Proving estimates that are uniform as $t \to \pm\infty$ will require taking into account temporal decay.

Consider the H^1-energy

$$E[u](t) = \int_{\mathbb{R}^d} \Big(\Big|\frac{\partial u}{\partial t}\Big|^2 + \sum_{j=1}^d \Big|\frac{\partial u}{\partial x_j}\Big|^2 + |u|^2 \Big) \,\mathrm{d}^d x. \tag{513}$$

Because $1 + a_{00} > 0$, owing to the assumption that ∇t is timelike, and because the matrix $\{1 - a_{jk}\}_{j,k=1}^d$ is strictly positive definite, owing to the assumption that the hypersurface Σ_T is spacelike for each T, $E[u](t)$ can be bounded above by some multiple of

$$E_0[u](t) = \int_{\mathbb{R}^d} \Big((1+a_{00}) \Big|\frac{\partial u}{\partial t}\Big|^2 + \sum_{j,k=1}^d (1-a_{jk}) \frac{\partial u^*}{\partial x_j}\frac{\partial u}{\partial x_k} + \mathsf{m}^2 |u|^2 \Big) \,\mathrm{d}^d x. \tag{514}$$

Indeed, the assumptions imply that $\inf_{(t,\mathbf{x})\in\mathbb{R}^{1,d}} (1+a_{00}) > 0$, as well as a similar uniform lower bound on the matrix $\{1-a_{jk}\}_{j,k=1}^d$. Thus, $E[u](t) \leq C E_0[u](t)$ for some $C > 0$. Conversely, $E_0[u](t) \leq C_0 E[u](t)$ for some other $C_0 > 0$.

If $u \in C^\infty(\mathbb{R}_t; C^\infty_{\mathrm{c}}(\mathbb{R}^d_{\mathbf{x}}))$, then

$$\begin{aligned} \frac{\mathrm{d}E_0[u]}{\mathrm{d}t} &= \int_{\mathbb{R}^d} 2\Re\Big[\frac{\partial u^*}{\partial t} \Big((1+a_{00}) \frac{\partial^2 u}{\partial t^2} - \sum_{j,k=1}^d (1-a_{jk}) \frac{\partial^2 u}{\partial x_j \partial x_k} + \mathsf{m}^2 u \Big) \Big] \,\mathrm{d}^d x \\ &\quad + \int_{\mathbb{R}^d} \Big(\frac{\partial a_{00}}{\partial t} \Big|\frac{\partial u}{\partial t}\Big|^2 + 2 \sum_{j,k=1}^d \frac{\partial a_{jk}}{\partial x_j} \Re\Big[\frac{\partial u^*}{\partial t}\frac{\partial u}{\partial x_k}\Big] - \sum_{j,k=1}^d \frac{\partial a_{jk}}{\partial t} \frac{\partial u^*}{\partial x_j}\frac{\partial u}{\partial x_k} \Big) \,\mathrm{d}^d x. \end{aligned} \tag{515}$$

The integral on the first line is

$$2\int_{\mathbb{R}^d} \Re\Big[\frac{\partial u^*}{\partial t} P^* u\Big] \,\mathrm{d}^d x - 2\int_{\mathbb{R}^d} \Re\Big[\frac{\partial u^*}{\partial t} \Big(\sum_{j=1}^d a_{0j} \frac{\partial^2 u}{\partial t \partial x_j} + \sum_{j=0}^d b_j \frac{\partial u}{\partial x_j} + Vu \Big) \Big] \,\mathrm{d}^d x. \tag{516}$$

Using Cauchy–Schwarz and AM-GM, the first term here is bounded as follows:

$$2\Big| \int_{\mathbb{R}^d} \Re\Big[\frac{\partial u^*}{\partial t} Lu\Big]\Big| \leq \frac{C}{\langle t\rangle^2} E_0[u](t) + \langle t\rangle^2 \|Lu(t,-)\|_{L^2(\mathbb{R}^d)}. \tag{517}$$

Likewise,

$$2\Big| \int_{\mathbb{R}^d} \Re\Big[\frac{\partial u^*}{\partial t} \Big(Vu + \sum_{j=0}^d b_j \frac{\partial u}{\partial x_j} \Big)\Big]\Big| \leq C\Big(\sum_{j=0}^d \sup_{\mathbf{x}\in\mathbb{R}^d} |b_j(t,\mathbf{x})| + \sup_{\mathbf{x}\in\mathbb{R}^d} |V(t,\mathbf{x})| \Big) E_0[u](t). \tag{518}$$

Finally, since $2\Re[\partial_t u^* \partial_t \partial_{x_j} u] = \partial_{x_j} |\partial_t u|^2$, we can write

$$2\int_{\mathbb{R}^d} \Re\Big[\frac{\partial u^*}{\partial t} \sum_{j=1}^d a_{0j} \frac{\partial^2 u}{\partial t \partial x_j}\Big] \,\mathrm{d}^d x = -2\int_{\mathbb{R}^d} \Big|\frac{\partial u}{\partial t}\Big|^2 \sum_{j=1}^d \frac{\partial a_{0j}}{\partial x_j} \,\mathrm{d}^d x, \tag{519}$$

integrating by parts. This satisfies $2|\int_{\mathbb{R}^d} |\partial_t u|^2 \operatorname{div} a_{0\bullet} \,\mathrm{d}^d x| \leq 2C \sup_{\mathbf{x}\in\mathbb{R}^d} |\operatorname{div} a_{0\bullet}(t,\mathbf{x})| E_0[u](t)$. Turning to the second line of eq. (515),

$$\Big| \int_{\mathbb{R}^d} \frac{\partial a_{00}}{\partial t} \Big|\frac{\partial u}{\partial t}\Big|^2 \,\mathrm{d}^d x\Big| \leq C \sup_{\mathbf{x}\in\mathbb{R}^d} \Big|\frac{a_{00}}{\partial t}\Big| E_0[u](t), \tag{520}$$

and

$$\Big|\int_{\mathbb{R}^d} 2\sum_{j,k=1}^d \frac{\partial a_{jk}}{\partial x_k}\Re\Big[\frac{\partial u^*}{\partial t}\frac{\partial u}{\partial x_j}\Big] - \sum_{j,k=1}^d \frac{\partial a_{jk}}{\partial t}\frac{\partial u^*}{\partial x_j}\frac{\partial u}{\partial x_k}\,\mathrm{d}^d x\Big| \leq \frac{3C}{2}\sup_{j,k}\sup_{\mathbf{x}\in\mathbb{R}^d}\|\nabla a_{j,k}\| E_0[u](t). \tag{521}$$

Let

$$c(t) = C\Big(\langle t\rangle^{-2} + \sum_{j=0}^d \sup_{\mathbf{x}\in\mathbb{R}^d}|b_j(t,\mathbf{x})| + \sup_{\mathbf{x}\in\mathbb{R}^d}|V(t,\mathbf{x})| + 2\sup_{\mathbf{x}\in\mathbb{R}^d}|\operatorname{div} a_{0\bullet}(t,\mathbf{x})| + \frac{3}{2}\sup_{\mathbf{x}\in\mathbb{R}^d}\sup_{j,k}\|\nabla a_{j,k}\|\Big). \tag{522}$$

The above shows that

$$\frac{\mathrm{d}E_0[u]}{\mathrm{d}t} \leq c(t)E_0[u] + \langle t\rangle^2\|Lu(t,-)\|_{L^2(\mathbb{R}^d)}. \tag{523}$$

Because each of the $b_j, V, \operatorname{div} a_{0\bullet}, \nabla a_{j,k}$ is a decaying symbol on $\mathbb{M}$, their supremums over Σ_T depend continuously on T. Thus, $c\in C^0(\mathbb{R};\mathbb{R}^+)$. Grönwall's inequality then says that $E_0[u](t)\leq \exp(\int_0^t c(s)\,\mathrm{d}s)(E_0[u](0) + \int_0^t\langle s\rangle^2\|Lu(s,-)\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}s)$, which implies

$$E[u](t) \leq C\exp\Big(\int_0^t c(s)\,\mathrm{d}s\Big)\Big(C_0E[u](0) + \int_0^t\langle s\rangle^2\|Lu(s,-)\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}s\Big). \tag{524}$$

This was proven under the assumption that $u(t,-)$ be compactly supported, but using e.g. finite speed of propagation this assumption can be removed. Consequently, if $u\in C^\infty(\mathbb{R}^{1,d})$ solves $Lu=0$, then

$$E[u](t)\leq CC_0\exp\Big(\int_0^t c(s)\,\mathrm{d}s\Big)E[u](0), \tag{525}$$

where part of the conclusion is that, if $\partial_t u|_{t=0}\in L^2(\mathbb{R}^d_{\mathbf{x}})$ and $u|_{t=0}\in H^1(\mathbb{R}^d_{\mathbf{x}})$, then $u(t,\mathbf{x})\in H^1(\mathbb{R}^d_{\mathbf{x}})$ for each $t\in\mathbb{R}$.

Now we use that being a decaying symbol on $\mathbb{O}$ implies improved decay as $t\to\infty$. Indeed, we are assuming that b_j, V are symbols of order -2 on $\mathbb{O}$, so that $b_j, V\in(1+t^2+x^2)^{-1/2}L^\infty(\mathbb{R}^{1,d})$. We are also assuming this of the $a_{j,k}$, and eq. (67), eq. (68) imply that $\partial_t a_{j,k}, \partial_{x_\ell}a_{j,k}$ are then also symbols of the same order, so

$$\partial_t a_{j,k}, \partial_{x_\ell}a_{j,k} \in \frac{1}{(1+t^2+x^2)^{1/2}}L^\infty(\mathbb{R}^{1,d}) \tag{526}$$

as well. This all implies that $c\in\langle t\rangle^{-1}L^\infty(\mathbb{R}_t)$. Thus, $E[u](t)\leq C_1\langle t\rangle^{C_2}E[u](0)$ for some $C_1, C_2>0$. So, $u\in\langle t\rangle^{C_2}L^\infty(\mathbb{R}_t;H^1(\mathbb{R}^d))\subseteq\mathcal{S}'(\mathbb{R}^{1,d})$.

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