THE ESSENTIAL SELF-ADJOINTNESS OF THE WAVE OPERATOR ON RADIATIVE SPACETIMES

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ABSTRACT. We prove the essential self-adjointness of the d'Alembertian $\Box_g$, allowing a larger class of spacetimes than previously considered, including those that arise from perturbing Minkowski spacetime by gravitational radiation. We emphasize the fact, proven by Taira in related settings, that all tempered distributions u satisfying $\Box_g u = \lambda u + f$ for $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and f Schwartz are Schwartz. The proof is fully microlocal and relatively quick given the "de,sc-" machinery recently developed by the third author.

1. INTRODUCTION

Consider an asymptotically flat metric g on spacetime, which for simplicity we take to be diffeomorphic to $\mathbb{R}^{1,d}$, $d \geq 1$. (See below for the precise assumptions.) Then the d'Alembertian $\Box_g$ is symmetric with respect to the $L^2 = L^2(\mathbb{R}^{1,d}, g)$-inner product, in the sense that

$$\langle u, \Box_g v \rangle_{L^2(\mathbb{R}^{1,d},g)} = \langle \Box_g u, v \rangle_{L^2(\mathbb{R}^{1,d},g)} \tag{1}$$

for all $u, v \in \mathcal{S}(\mathbb{R}^{1,d})$, where $\mathcal{S}(\mathbb{R}^{1,d})$ denotes the set of Schwartz functions on spacetime. It is of interest to show that $\Box_g$, or more generally an $L^2(\mathbb{R}^{1,d}, g)$-symmetric operator of the form

$$P = \Box_g + \mathsf{m}^2 + L, \tag{2}$$

$\mathsf{m} \geq 0$ (or even $\mathsf{m} \in i\mathbb{R}$), $L \in \mathrm{Diff}^1(\mathbb{R}^{1,d})$ sufficiently decaying, is, when acting on the domain $\mathcal{S}(\mathbb{R}^{1,d})$ or $C_c^\infty(\mathbb{R}^{1,d})$, essentially self-adjoint. For instance, if one wants to define the Feynman propagator as

$$(P - i0)^{-1} = \lim_{\varepsilon \to 0^+} (P - i\varepsilon)^{-1}, \tag{3}$$

then it is first necessary to make sense of $(P - i\varepsilon)^{-1}$, which, if P is known to be essentially self-adjoint, is an immediate consequence of the functional calculus for self-adjoint operators. This line of inquiry was first followed by Dereziński and Siemssen in [DS18; DS19; DS23], which tackled e.g. the case when g is stationary. Later works by Vasy [Vas20] and Nakamura and Taira [NT21; NT23a; NT23b; Tai21] have tackled some cases of asymptotically Minkowski and asymptotically static g. Relevant to this paper is the fact that Vasy's arguments are microlocal and based on the Parenti–Shubin–Melrose "scattering (sc)-calculus" $\Psi_{sc}(\mathbb{R}^{1,d})$, which consists of certain (but not all) polynomially weighted pseudodifferential operators in Hörmander's standard calculus $\Psi_\infty(\mathbb{R}^{1,d})$. This choice of calculus limits the spacetimes that Vasy can consider.

The arguments of Nakamura–Taira and Vasy begin with the easy consequence of the theory of deficiency indices (see §A) that all one needs to do is prove that

$$u \in L^2, Pu = \pm iu \Rightarrow u = 0. \tag{4}$$

In this way, essential self-adjointness is reduced to a statement regarding the solvability theory of the operator $P \pm i$. The symmetry of P acting on $\mathcal{S}(\mathbb{R}^{1,d})$ yields, via the usual argument that

symmetric operators have only real eigenvalues on their domains, that $u \in \mathcal{S}, Pu = \pm iu \Rightarrow u = 0$. Indeed, if $u \in \mathcal{S} = \mathcal{S}(\mathbb{R}^{1,d})$ and $Pu = \pm iu$, then (with $\|\cdot\|$ denoting the L^2 norm)

$$\pm i \|u\|^2 = \langle Pu, u \rangle_{L^2} = \langle u, Pu \rangle_{L^2} = \mp i \|u\|^2, \tag{5}$$

which is only possible if $u = 0$. So, in order to prove the essential self-adjointness of P, it suffices to prove that

$$u \in L^2, Pu = \pm iu \Rightarrow u \in \mathcal{S}. \tag{6}$$

Taira [Tai21, Thm. 1.6] has proven the stronger result (in some level of generality) that $u \in L^2, f \in \mathcal{S}, Pu = \pm iu + f \Rightarrow u \in \mathcal{S}$. This feature of the solvability theory of P, which we will call the Schwartz-to-Schwartz mapping property of $(P \pm i)^{-1}$, is surprisingly delicate in the variable-coefficient case. (It is rather trivial in the constant-coefficient case — see §B.)

Microlocally, it is straightforward to prove that the Schwartz-to-Schwartz property holds away from the intersection of null infinity and fiber infinity in the appropriate phase space (i.e. infinite frequency). In other words, if $Pu \mp iu \in \mathcal{S}$, and if $A \in \Psi_{\mathrm{sc}}^{0,0}(\mathbb{R}^{1,d})$ is an operator in the Parenti–Shubin calculus whose essential support is disjoint from the portion of fiber infinity over null infinity, then $Au \in \mathcal{S}$. However, it is difficult to remove A. What Vasy does in [Vas20] is prove a weaker and delicate borderline radial point estimate, the conclusion of which is that u has some finite amount of Sobolev regularity. Vasy then proves that this exact amount of Sobolev regularity is sufficient to make the computation eq. (5) go through. In this way, Vasy proves essential self-adjointness without proving the Schwartz-to-Schwartz property of $(P \pm i)^{-1}$.

One goal of this note is to show that, if, instead of the Parenti–Shubin calculus, one uses the double edge-scattering (de,sc-) pseudodifferential calculus $\Psi_{\mathrm{de,sc}}$ recently introduced in [Sus23] by one of the authors in order to analyze the asymptotics of solutions of the Klein–Gordon equation near null infinity, then standard microlocal methods suffice to prove the Schwartz-to-Schwartz mapping property, obviating the need for Vasy's borderline estimate. Moreover, we are able to prove a natural generalization of the mapping property that applies to f with only a finite amount of Sobolev regularity or decay — see Theorem C. This hierarchy of mapping properties is one respect in which our result (and not just the method) is not already contained in [NT21; NT23a; Tai21].

The de,sc-calculus is well-suited to this problem for the same reason it is well-suited to analyze massive waves near null infinity: the oscillations present in solutions of the Klein–Gordon equation lie at finite de,sc-frequency. We explain this perspective in connection with the asymptotics problem in [Sus23, §1]. This finitude of frequency is what allows us to replace Vasy's borderline estimate with *an elliptic* estimate in which the imaginary term in $P \pm i$ provides ellipticity. The Schwartz-to-Schwartz mapping property is then proven by propagating the control provided by ellipticity throughout the characteristic set of P. Heuristically, $P \pm i$ is a Klein–Gordon operator with "complex mass;" it should therefore not be surprising that tools developed to understand the Klein–Gordon equation are useful, even when the original operator P is the massless wave operator $\Box_g$.

A second goal is to extend the essential self-adjointness result to a class of metrics with more singular behavior at null infinity than those considered by Nakamura–Taira and Vasy. The simplest sort of asymptotically Minkowski spacetimes have metrics whose coefficients are well-behaved on the radial compactification[1]

$$\mathbb{M} = \overline{\mathbb{R}^{1,d}} = \mathbb{R}^{1,d} \sqcup \infty \mathbb{S}^d \tag{7}$$

of Minkowski spacetime (at least if we ignore the north/south pole of the compactification). It is to such metrics that Vasy's and Nakamura–Taira's analysis applies. However, on solving the Einstein field equations or other quasilinear wave equations, a typical phenomenon is radiation. Radiation, which is coupled to and therefore has an effect on the metric, tends to null infinity. Consequently,

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[1]This means using the embedding $\mathbb{R}^{1+d} \backslash \mathrm{origin} \ni (t, \mathbf{x}) \mapsto ((t^2 + \|\mathbf{x}\|^2)^{-1/2}, (t, \mathbf{x})/(t^2 + \|\mathbf{x}\|^2)^{1/2}) \in [0, \infty) \times \mathbb{S}^d$. Identifying $\mathbb{R}^{1+d} \backslash \mathrm{origin}$ with the image of this embedding, $\{0\} \times \mathbb{S}^d$ can be thought of as the "sphere at infinity," called $\infty \mathbb{S}^d$ above.

the metric is not expected to be well-behaved on the radial compactification $\mathbb{M}$ but rather on the "octagon"

$$\mathbb{O} = [\overline{\mathbb{R}^{1,d}}; \text{null infinity}; 1/2] \hookleftarrow \mathbb{R}^{1,d} \tag{8}$$

that results from blowing up null infinity.[2] The resultant manifold-with-corners $\mathbb{O}$ has five boundary hypersurfaces (unless $d = 1$, in which case $\mathbb{O}$ is literally an octagon), two of which are the timelike caps, one of which is spacelike infinity, and the other two of which, the front faces of the blowup, correspond to the boundaries of the Penrose diagram of Minkowski spacetime. Indeed, Hintz and Vasy have proven rigorously in [HV20] that solving the Einstein vacuum equations for initial data differing only slightly from the flat case results in a metric polyhomogeneous – i.e. well-behaved, in a precise sense – on $\mathbb{O}$. This compactification is therefore the natural one vis-a-vis the analysis of relativistic wave equations. One attractive feature of $\Psi_{\text{de,sc}}$ is that it permits microlocal analysis directly on $\mathbb{O}$, including PDEs whose coefficients are singular on $\mathbb{M}$. The sc-calculus is not well-suited for this purpose. Another calculus similar to the de,sc-calculus is the edge-b (e,b-) calculus $\Psi_{\text{e,b}}$, developed recently by Hintz–Vasy in [HV23] to study the massless wave equation on $\mathbb{O}$. The massless wave equation was previously analyzed on $\mathbb{O}$ by Baskin–Vasy–Wunsch in [BVW15; BVW18]; however, in these papers, the metric is required to be well-behaved on $\mathbb{M}$. It is the more sophisticated microlocal tools developed in the later work that allowed the handling of radiative spacetimes.

Thus, we prove:

Theorem A. *Suppose that*

- *as discussed in §2, g is non-trapping and well-behaved on $\mathbb{O}$, in the sense that eq. (10) holds,*
- *A is a first-order operator sufficiently decaying in the sense that eq. (12) holds,*

and $\mathsf{m}^2 \geq 0$. Then, the differential operator $P = \Box_g + \mathsf{m}^2 + A$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^{1,d})$ with respect to the $L^2(\mathbb{R}^{1,d}, g)$-inner product.

Remark. Because the essential self-adjointness of P is equivalent to that of $\Box_g + A$, there is no loss of generality in taking $\mathsf{m} = 0$. However, since we make thorough use of the results in [Sus23], in which m must be > 0, we will leave this parameter be.

Example (Vaidya-like metrics). One of the simplest examples of a radiative spacetime with the structure we consider is the *Vaidya spacetime* [Vai43], modified so as to avoid an event horizon. This is constructed by beginning with the Schwarzschild metric

$$g_{\mathsf{m}} = -\left(1 - \frac{2\mathsf{m}}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2\mathsf{m}}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + (\sin\theta)^2 \mathrm{d}\phi^2) \tag{9}$$

on $\mathbb{R}^{1,3}_{r>2\mathsf{m}}$, where θ is the polar angle and ϕ is the azimuthal angle, and then promoting the black hole mass $\mathsf{m} > 0$ to a function $\mathsf{m} \in C^\infty(\mathbb{O}; \mathbb{R})$ such that g_{m} becomes a smooth Lorentzian metric on all of $\mathbb{R}^{1,3} = \mathbb{O}^\circ$. Traditionally, one takes m to just depend on one of the "lightcone coordinates" $t \pm r$ [GP09, §9.5], but this still leads to an event horizon, so allowing general $\mathsf{m} \in C^\infty(\mathbb{O}; \mathbb{R})$ is natural. Vaidya spacetimes are used to model the gravitational fields of spherically symmetric objects which absorb or emit null dust.

See [Bič97; Bič00; Bič02] for other examples of radiating spacetimes. Some, e.g. the pp-wave spacetimes, do not fit into the framework here, since a plane wave is a large perturbation of the metric. Other models, such as the Vaidya example above, only possess the correct structure away from timelike infinity. There exists a large literature regarding the asymptotic structure of generic radiating spacetimes, originating in the works of Bondi et al. [BBM62], Penrose et. al [NP62; Pen63],

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[2] The '1/2' refers to a change of smooth structure at the front face of the blowup, null infinity. This was done in [Sus23] to simplify some aspects of the microlocal analysis. For our purposes here, it suffices to note that this results in a boundary-defining-function being $1/\sqrt{r}$ in the interior of the face instead of $1/r$. Otherwise, this subtlety can be ignored by the reader.

and Sachs [Sac62]. See [CMS95][CGW23] for citations of some of the work that followed. For rigorous mathematical work on small perturbations of Minkowski spacetime, see [CK93][LR10][HV20].

However, $\mathbb{O}$, or closely related constructions, seems to have appeared in the physics literature only recently; indeed, Compère, Gralla, and Wei [CGW23] have introduced what they call the "puzzle piece diagram" of spacetime (essentially compactification by $\mathbb{O}$), about which they claim:

> *It is perhaps surprising that a fully general framework for macroscopic gravitational scattering does not already exist, given the tremendous effort devoted to the study of compact object interactions. Indeed, a framework almost exists, in that various asymptotic formalisms need only be tweaked and stitched together.* [CGW23, p. 3]

"Stitched together" means combining the various asymptotic regimes considered previously into a single compactification. Ashtekar et. al. [AH78; AK24a; AK24b] had previously considered the corner between spacelike and null infinity. Instead of working on $\mathbb{O}$, physicists often work directly on the Penrose diagram $\mathbb{P}$ (see Figure 1), in which timelike and spacelike infinity have been collapsed to various points. A metric is well-behaved (more precisely, symbolic) near points in the interior of null infinity if and only if it is well-behaved near the corresponding points at null infinity in the Penrose diagram. Well-behavedness on $\mathbb{O}$ is a reasonable criterion for what it means for a metric to be well-behaved near the corners of the Penrose diagram. Besides this, there are several reasons for working on $\mathbb{O}$ rather than $\mathbb{P}$:

- unless $d = 1$ (in which case we are working in 1+1D), the Penrose diagram is not a manifold-with-corners locally modeled smoothly on 2^kth-ants of $\mathbb{R}^{1+d}$, which complicates analysis on it. For example, if $d = 2$, the Penrose diagram is, near its timelike corners, a solid cone, not an octant of $\mathbb{R}^3$.
- Even if the metric is well-behaved already on the Penrose diagram, solutions of massive wave equations will not be. For instance, it is necessary to blow up timelike infinity in the Penrose diagram in order to resolve the asymptotics of solutions of the Klein–Gordon equation, as can be seen already in the free case [Hör97]. This is the perspective of [CGW23].

Related to the first point, microlocal tools do not directly apply to the Penrose diagram. Consequently it can be worthwhile to pass to $\mathbb{O}$ even when studying the massless wave equation, in which case the second point above does not apply — see [HV23]. However, if the reader so desires, they may assume that the metric is well-behaved on $\mathbb{P}$ (whatever precise definition one gives for "well-behavedness" near the corners of $\mathbb{P}$, it should imply being symbolic on $\mathbb{O}$) and view the passage to $\mathbb{O}$ as a technical device.

It is worth pointing out that we do not allow perturbations of the Minkowski metric of the form $\eta + \delta\eta$ for $\delta\eta$ static. (In [NT23b], Nakamura and Taira allow time-dependent perturbations of static spacetimes, but only with compact Cauchy hypersurfaces.) From our perspective, asymptotically flat metrics with static perturbations are badly behaved near the north/south pole of the radial compactification of Minkowski spacetime. Fortunately, there exist extensions of the Parenti–Shubin calculus Ψ_{sc} designed to handle such perturbations. For example, Baskin, Doll, and Gell-Redman have used the "3-body-sc"-calculus to analyze the Klein–Gordon equation with an asymptotically static potential [BDGR24]. We expect that it is possible to use their tools to extend the results of this paper to allow static perturbations of the metric. Because the problems we are concerned with here are delicate only near null infinity, and because the north/south pole of the radial compactification of spacetime is disjoint from null infinity, we do not believe that any major obstacles exist to combining the de,sc- and 3-body-sc- analyses.

Let us close this introduction by noting that, in work in progress, Dang, Vasy, and Wrochna [DVW24] apply a second microlocal approach to develop a functional calculus for squares of Dirac operators and study their essential self-adjointness (or lack thereof). Their tools, applied to the problem at hand, yield an independent and arguably simpler proof of the Schwartz-to-Schwartz mapping property of $(P \pm i)^{-1}$ but do not allow more general metrics than those considered already
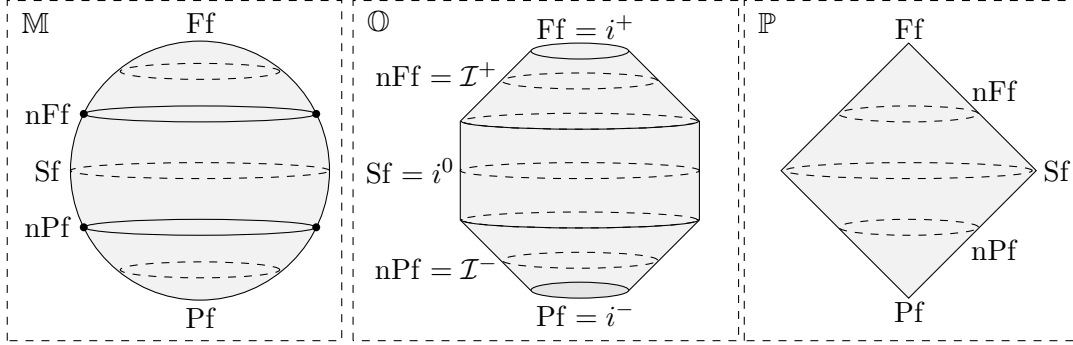
FIGURE 1. (*Left*) The radial compactification $\mathbb{M}$ of Minkowski spacetime, (*middle*) the manifold-with-corners $\mathbb{O}$, and (*right*) the Penrose diagram $\mathbb{P}$ of Minkowski spacetime, all with labeled faces: Ff is future timelike infinity, nFf is future null infinity, Sf is spacelike infinity, nPf is past null infinity, and Pf is past timelike infinity. The compactification $\mathbb{O}$ can be constructed by blowing up the corners of the Penrose diagram or blowing up null infinity in the radial compactification. In this sense, $\mathbb{O}$ is the simplest compactification of Minkowski spacetime refining both the radial and Penrose compactifications. Physicists use the notation $i^\pm, i^0, \mathcal{I}^\pm$ to label the various asymptotic regimes. We have provided the translation in the middle panel. In all three figures, time is increasing vertically and r is increasing moving away from the vertical midline.

by Vasy. This follows previous work by Dang and Wrochna [DW23; DW22] developing a Lorentzian spectral action principle for gravity, for which one needs complex powers of $P \pm i\varepsilon$ to be well-defined, which is simplified if P is essentially self-adjoint. In the context of variational principles for gravity, it is natural to work with classes of metrics which include perturbations of Minkowski space in the sense of solutions to the Einstein equations, and our methods are able to handle these.[3]

Remark (Alternative definition of Feynman propagators)*.* The Schwartz-to-Schwartz mapping property of $(P \pm i)^{-1}$ is proven using propagation estimates for $P \pm i$, where the sign of the imaginary part determines direction of propagation. Concatenating propagation estimates for P itself as in [Sus23] (where now one can choose the direction of propagation independently in the two sheets of the characteristic set, resulting in four possible choices) yields regularity results similar to Theorem C. This leads to four Fredholm realizations of P, following the general approach to Fredholm theory for non-elliptic equations originating in [Vas13]. If we propagate in the same direction as required for $P - i$, the resulting inverse of P is the Feynman propagator — cf. eq. (3). This approach to defining Feynman propagators is taken by Gell-Redman, Haber, and Vasy [GRHV16], working on asymptotically Minkowski spacetimes which are well-behaved on $\mathbb{M}$. For massive waves, the analogous construction on the more general spacetimes considered here is the subject of the work in progress [MV24]. Cf. [GW19; GW20] for other results on Fredholm theory in the massive case and [DG24] for a discussion of the relationship between different notions of Feynman propagators.

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[3]We thank Michał Wrochna for bringing this perspective to our attention.

2. Some setup

By asymptotically Minkowski metric, we mean a Lorentzian metric g on $\mathbb{R}^{1,d}$ such that

$$g - \eta \in \rho^\varepsilon S^0(\mathbb{O}; {}^{\mathrm{de,sc}}\operatorname{Sym}^2 \overline{T}^*\mathbb{O}) \tag{10}$$

for some $\varepsilon > 0$, where η is the exact Minkowski metric and ρ is a total boundary defining function of $\mathbb{O}$: $\rho = \prod_{\mathrm{f}} \rho_{\mathrm{f}}$, where the product is over the boundary hypersurfaces. Here, S^0 is the space of zeroth-order conormal symbols on $\mathbb{O}$, i.e. functions lying in L^∞ after the application of arbitrarily many vector fields tangent to the boundary of $\mathbb{O}$.[4] The assumption eq. (10) is weaker than the corresponding assumption in [Sus23],

$$g - \eta \in \rho^2 C^\infty(\mathbb{O}; {}^{\mathrm{de,sc}}\operatorname{Sym}^2 \overline{T}^*\mathbb{O}). \tag{11}$$

In this previous work, the goal was the production of asymptotic expansions, so stronger assumptions were convenient, but only used in [Sus23, §7]. In forthcoming work by the second author and Vasy [MV24], the germane portion of the analysis in [Sus23, §5, §6] is extended to metrics satisfying only eq. (10). The e,b-analysis by Hintz and Vasy of massless waves near null infinity [HV23] was also carried out in this (and even greater) generality. Solutions of the Einstein vacuum equations with initial data close to those of Minkowski space generally satisfy eq. (10) but not the stronger eq. (11); see [HV20, Theorem 7.1]. In this paper, we use only results in [Sus23, §5], not [Sus23, §7], so we are permitted to assume only eq. (10).

In addition, we assume that g is non-trapping, in the sense that every null geodesic asymptotes to the boundary of $\mathbb{O}$.

Note that $L^2(\mathbb{R}^{1,d}, g) = L^2(\mathbb{R}^{1,d})$ at the level of topological vector spaces. So, unless the specific inner product is relevant, we will just write L^2.

We now consider differential operators P of the form $P = \Box_g + \mathsf{m}^2 + A$, where $\mathsf{m} \geq 0$, $\Box_g$ is the d'Alembertian, and

$$A \in S\operatorname{Diff}_{\mathrm{de,sc}}^{1,-1} \tag{12}$$

is symmetric on $\mathcal{S}(\mathbb{R}^{1,d})$ with respect to the $L^2(\mathbb{R}^{1,d}, g)$-inner product. (See §2.1 for the definition of $S\operatorname{Diff}_{\mathrm{de,sc}}$. The theorem is not less interesting if $A = 0$.) Then P is symmetric on $\mathcal{S}(\mathbb{R}^{1,d})$ with respect to the $L^2(\mathbb{R}^{1,d}, g)$-inner product. Then, our main result is a sharpening of:

Theorem B (Taira's mapping property, for radiating metrics)**.** *Fix* nonreal $\lambda \in \mathbb{C}$. *Then, if* $u \in \mathcal{S}'(\mathbb{R}^{1,d})$ *solves* $Pu + \lambda u = f$ *for* $f \in \mathcal{S}$, *we have* $u \in \mathcal{S}$.

The proof is in the next section, §3. Note that Theorem A follows immediately as a corollary.

Remark. The above also holds if $A \in \rho^\epsilon S\operatorname{Diff}_{\mathrm{de,sc}}^{1,0}$ for some $\epsilon > 0$. This complicates some principal symbols below, but the added terms do not affect the argument.

2.1. **The de,sc-calculus and related objects.** We will use liberally notation introduced in [Sus23, §1, §2]. For example, for $m \in \mathbb{R}, \mathsf{s} \in \mathbb{R}^5$,

$$H_{\mathrm{de,sc}}^{m,\mathsf{s}} = \{u \in \mathcal{S}'(\mathbb{R}^{1,d}) : Lu \in L^2(\mathbb{R}^{1,d}) \text{ for all } L \in \Psi_{\mathrm{de,sc}}^{m,\mathsf{s}}\} \tag{13}$$

is the scale of Sobolev spaces associated to the calculus $\Psi_{\mathrm{de,sc}}$ of de,sc-pseudodifferential operators. We refer to [Sus23] for a description of this pseudodifferential calculus. For the statement of our main theorem, it suffices to consider only $m \in \mathbb{N}$. When $m \in \mathbb{N}$, eq. (13) may be re-expressed in terms of the $C^\infty(\mathbb{O})$-algebra $\operatorname{Diff}_{\mathrm{de,sc}}$ of de,sc- differential operators:

$$H_{\mathrm{de,sc}}^{m,\mathsf{s}} = \{u \in \mathcal{S}'(\mathbb{R}^{1,d}) : Lu \in L^2(\mathbb{R}^{1,d}) \text{ for all } L \in \operatorname{Diff}_{\mathrm{de,sc}}^{m,\mathsf{s}}\}. \tag{14}$$

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[4]See [CMS95] for forceful justification for considering symbolic, or at least fairly general polyhomogeneous, metrics, rather than just those satisfying eq. (11).

Specifically, $\mathrm{Diff}_{\mathrm{de,sc}}^{m,\mathbf{0}}$ is the $C^\infty(\mathbb{O})$-subalgebra of $\mathrm{Diff}^m(\mathbb{R}^{1,d})$ generated by the set $\mathcal{V}_{\mathrm{de,sc}}$ of *de,sc-vector fields*, and

$$\mathrm{Diff}_{\mathrm{de,sc}}^{m,\mathbf{s}} = \rho^{-\mathbf{s}}\,\mathrm{Diff}_{\mathrm{de,sc}}^{m,\mathbf{0}}, \quad \mathbf{s} = (s_{\mathrm{Pf}}, s_{\mathrm{nPf}}, s_{\mathrm{Sf}}, s_{\mathrm{nFf}}, s_{\mathrm{Ff}}) \in \mathbb{R}^5,$$
$$\rho^{-\mathbf{s}} = \rho_{\mathrm{Pf}}^{-s_{\mathrm{Pf}}} \rho_{\mathrm{nPf}}^{-s_{\mathrm{nPf}}} \rho_{\mathrm{Sf}}^{-s_{\mathrm{Sf}}} \rho_{\mathrm{nFf}}^{-s_{\mathrm{nFf}}} \rho_{\mathrm{Ff}}^{-s_{\mathrm{Ff}}}. \tag{15}$$

The set $\mathcal{V}_{\mathrm{de,sc}}$ is the $C^\infty(\mathbb{O})$-submodule of $\mathcal{V}(\mathbb{R}^{1,d})$ generated by vector fields of the form $\chi_1 \partial_t, \chi_1 \partial_{x_j}$ for $\chi_1 \in C^\infty(\mathbb{M})$ vanishing near null infinity and

$$\chi_2 \rho_{\mathrm{nf}}^2 \rho_{\mathrm{Of}} \partial_{\rho_{\mathrm{nf}}}, \chi_2 \rho_{\mathrm{nf}} \rho_{\mathrm{Of}}^2 \partial_{\rho_{\mathrm{Of}}}, \chi_2 \rho_{\mathrm{nf}}^2 \rho_{\mathrm{Of}} \partial_{\theta_j} \tag{16}$$

for $j \in \{1, \ldots, d-1\}$, where

- $\chi_2 \in C^\infty(\mathbb{O})$ is supported near a single corner $\mathrm{nf} \cap \mathrm{Of}$ of $\mathbb{O}$, where $\mathrm{nf} \in \{\mathrm{nPf}, \mathrm{nFf}\}$, $\mathrm{Of} \in \{\mathrm{Pf}, \mathrm{Sf}, \mathrm{Ff}\}$,
- $\rho_{\mathrm{f}} \in C^\infty(\mathbb{O})$ is a boundary-defining-function for the boundary hypersurface f,
- we are defining the partial derivatives using a coordinate system $\rho_{\mathrm{nf}}, \rho_{\mathrm{Of}}, \theta_1, \ldots, \theta_{d-1}$ where $\theta_1, \ldots, \theta_{d-1}$ denotes a local coordinate chart on $\mathbb{S}_{\mathbf{x}/r}^{d-1}$.

(That is, a vector field $V \in \mathcal{V}(\mathbb{R}^{1,d})$ lies in $\mathcal{V}_{\mathrm{de,sc}}$ if and only if it is a scattering vector field at $\mathrm{Pf}, \mathrm{Sf}, \mathrm{Ff}$ and a double edge (de) vector field at $\mathrm{nPf}, \mathrm{nFf}$, with the de- and sc- structures being compatible at the corners in the natural sense.) An equivalent global definition is

$$\mathcal{V}_{\mathrm{de,sc}} = \mathrm{span}_{C^\infty(\mathbb{O})}\{\rho_{\mathrm{nPf}}\rho_{\mathrm{nFf}}\partial_t, \rho_{\mathrm{nPf}}\rho_{\mathrm{nFf}}\partial_{x_j}, \rho_{\mathrm{nPf}}^{-1}\rho_{\mathrm{nFf}}^{-1}\chi_3(\partial_{|t|} + \partial_r), \chi_3 r^{-1}\partial_{\theta_k} : j \le d, k \le d-1\} \tag{17}$$

for $\chi_3 \in C^\infty(\mathbb{M})$ supported away from $\mathrm{cl}_{\mathbb{M}}\{tr = 0\}$ and identically equal to 1 near null infinity (so that $\chi_3(\partial_{|t|} + \partial_r)$, defined using the coordinates $t, r, \theta_1, \ldots, \theta_{d-1}$, is a smooth vector field). In [Sus23], the pseudodifferential calculus $\Psi_{\mathrm{de,sc}}$ is constructed by "quantizing" the Lie algebra $\mathcal{V}_{\mathrm{de,sc}}$ in local coordinates. Near points in the interiors of $\mathrm{nPf}, \mathrm{nFf}$, this gives the double edge calculus of Lauter–Moroianu [LM01]. Near points in the interiors of $\mathrm{Pf}, \mathrm{Sf}, \mathrm{Ff}$, this gives the sc-calculus.

The set $S\,\mathrm{Diff}_{\mathrm{de,sc}}^{m,\mathbf{s}}$ is the $S^0(\mathbb{O})$-module generated by $\mathrm{Diff}_{\mathrm{de,sc}}^{m,\mathbf{s}}$. In other words, it contains de,sc-differential operators with coefficients which are merely symbolic on $\mathbb{O}$ rather than smooth functions times $\rho^{-\mathbf{s}}$.

Note that $H_{\mathrm{de,sc}}^{0,\mathbf{0}} = L^2(\mathbb{R}^{1,d})$, and

$$\bigcap_{m \in \mathbb{N}, \mathbf{s} \in \mathbb{R}^5} H_{\mathrm{de,sc}}^{m,\mathbf{s}} = \mathcal{S}(\mathbb{R}^{1,d}), \qquad \bigcup_{m \in \mathbb{Z}, \mathbf{s} \in \mathbb{R}^5} H_{\mathrm{de,sc}}^{m,\mathbf{s}} = \mathcal{S}'(\mathbb{R}^{1,d}). \tag{18}$$

The order m is the amount of differentiability, and the pentuple $\mathbf{s}$ lists the decay rate, relative to L^2, at each of the five boundary hypersurfaces of $\mathbb{O}$, in the order past timelike infinity, past null infinity, spacelike infinity, future null infinity, future timelike infinity.

The de,sc-vector fields yield a vector bundle $^{\mathrm{de,sc}}T\mathbb{O} \to \mathbb{O}$ whose smooth sections are precisely the elements of $\mathcal{V}_{\mathrm{de,sc}}$. Dualizing yields a new bundle $^{\mathrm{de,sc}}T^*\mathbb{O}$. The (compact) de,sc-phase space

$$^{\mathrm{de,sc}}\overline{T}^*\mathbb{O} \hookleftarrow T^*\mathbb{R}^{1,d} \tag{19}$$

is obtained by radially compactifying the fibers of $^{\mathrm{de,sc}}T^*\mathbb{O}$, making it into a ball bundle over $\mathbb{O}$, so topologically $\mathbb{O} \times \mathbb{B}$. The compactification creates a boundary face df, fiber infinity, which is topologically $\mathbb{O} \times \mathbb{S}^d$. Standard microlocal objects such as wavefront sets, elliptic/characteristic sets, essential supports, etc. are subsets of the boundary of this phase space, and symbols are functions on the interior conormal to the boundary. For the purposes of defining microlocalizers, it suffices to restrict attention to *classical* de,sc-symbols of order zero, which are just smooth functions on $^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$, where it is important to note that smoothness means smoothness all the way up to and including the boundary. (That is, embedding $\mathbb{O}$ in a larger $\mathbb{R}^{1+d}$ as the surface of revolution of an octagon, a function on $\mathbb{O} \times \mathbb{B}$ is smooth if and only if it extends to a smooth function on some open neighborhood in $(\mathbb{R}^{1+d})^2$.)

A classical pseudodifferential operator $A \in \Psi^{0,0}_{\text{de,sc}}$ is elliptic at a point $q \in \partial(^{\text{de,sc}}\overline{T}^*\mathbb{O})$ if and only if its principal symbol

$$\sigma^{0,0}_{\text{de,sc}}(A) \in C^\infty(^{\text{de,sc}}\overline{T}^*\mathbb{O}) \tag{20}$$

is nonvanishing there. (Strictly speaking, $\sigma^{0,0}_{\text{de,sc}}(A)$ is an equivalence class of symbols modulo lower order symbols. A usual abuse of notation is to conflate principal symbols with representatives thereof.)

Like in the sc-calculus, principal symbols in the de,sc-calculus capture operators modulo compact errors.

Let us elaborate on the de,sc- notion of wavefront set, associated with the scale of de,sc-Sobolev spaces. For each $u \in \mathcal{S}'(\mathbb{R}^{1,d})$, its de,sc-wavefront set is a subset

$$\mathrm{WF}^{m,\mathsf{s}}_{\text{de,sc}}(u) \subseteq \partial(^{\text{de,sc}}\overline{T}^*\mathbb{O}). \tag{21}$$

This measures, microlocally, the obstruction to u lying in $H^{m,\mathsf{s}}_{\text{de,sc}}$:

$$q \notin \mathrm{WF}^{m,\mathsf{s}}_{\text{de,sc}}(u) \quad \text{iff} \quad \exists A \in \Psi^{0,0}_{\text{de,sc}} \text{ that is elliptic at } q \text{ and } Au \in H^{m,\mathsf{s}}_{\text{de,sc}}. \tag{22}$$

(And it suffices to restrict to classical A in this definition.) So,

$$u \in H^{m,\mathsf{s}}_{\text{de,sc}} \iff \mathrm{WF}^{m,\mathsf{s}}_{\text{de,sc}}(u) = \varnothing. \tag{23}$$

Over the interior of $\mathbb{O}$, $\mathrm{WF}^{m,\mathsf{s}}_{\text{de,sc}}$ is just the ordinary wavefront set WF^m; over the interiors of $\mathrm{nPf}, \mathrm{nFf}$, $\mathrm{WF}^{m,\mathsf{s}}_{\text{de,sc}}$ is just the de-wavefront set $\mathrm{WF}^{m,s_{\mathrm{nPf}}}_{\text{de}}$, $\mathrm{WF}^{m,s_{\mathrm{nFf}}}_{\text{de}}$ respectively; and over the interiors of $\mathrm{Pf}, \mathrm{Sf}, \mathrm{Ff}$, $\mathrm{WF}^{m,\mathsf{s}}_{\text{de,sc}}$ is just the sc-wavefront set $\mathrm{WF}^{m,s_{\mathrm{Pf}}}_{\text{sc}}$, $\mathrm{WF}^{m,s_{\mathrm{Sf}}}_{\text{sc}}$, $\mathrm{WF}^{m,s_{\mathrm{Ff}}}_{\text{sc}}$ respectively.

3. Proof of main theorem

3.1. Characteristic set and Hamilton flow of P and $P + \lambda$.

We briefly recall the relevant phase space structures; for details see [Sus23, §4], the figures in which we have reproduced here.

The operator P is an element of $\Psi^{2,0}_{\text{de,sc}}(\mathbb{O})$ with principal symbol

$$p(z,\zeta) = -\|\zeta\|_g^2 + \mathsf{m}^2, \tag{24}$$

where $z = (t, \mathbf{x}) \in \mathbb{R}^{1,d}$ and $\zeta = (\tau, \xi)$ is the spacetime frequency. The characteristic set Σ of P consists of the set of limit points on $\partial(^{\text{de,sc}}\overline{T}^*\mathbb{O})$ of the set where $\|\zeta\|_g^2 = \mathsf{m}^2$. At fiber infinity df the m^2 term is negligible, so $\Sigma \cap \mathrm{df}$ is the union of the "points at infinity" of the dual lightcones over every point of $\mathbb{O}$. On the other hand, at finite frequencies over the various faces constituting base infinity, the m^2 term is of the same order as $\|\zeta\|_g^2$, so $\Sigma \backslash \mathrm{df}$ is the union of the mass shells over every point of $\partial\mathbb{O}$. When $\mathsf{m} > 0$, the characteristic set has two connected components: $\Sigma = \Sigma^+ \sqcup \Sigma^-$, where $\Sigma^\pm$ is the component containing the future $(+)$ or past $(-)$ dual light cone. In the case $\mathsf{m} = 0$, the two sheets intersect at the zero section.

Since p is real-valued, as long as λ is nonreal, the principal symbol of $P + \lambda$ never vanishes at finite frequencies. But, at df, λ is lower order. Therefore, the characteristic set of $P + \lambda$ is $\Sigma \cap \mathrm{df}$, this set being independent of m. In particular, note that it has two connected components even for $\mathsf{m} = 0$.

The operators P and $P + \lambda$ give rise to the same Hamiltonian vector field, this being computed at the beginning of [Sus23, §4]. The appropriately rescaled vector field H_p defined in [Sus23, §1] (i.e. H_p minimally weighted so as to make it tangent to all of the boundaries of the compactified de,sc-phase space) is smooth on the compactified phase space and tangent to its boundary and to Σ. Its flow within Σ has a number of vanishing (a.k.a. "radial") sets, which we describe in the $\mathsf{m} > 0$ case:

- $\mathcal{R}^\alpha_\beta$ are the sources/sinks of the flow, located over the timelike caps. Importantly, they lie at finite de,sc-frequency. That is, they do not intersect df. It is these radial sets that are associated with the oscillations present in smooth solutions of the PDE.
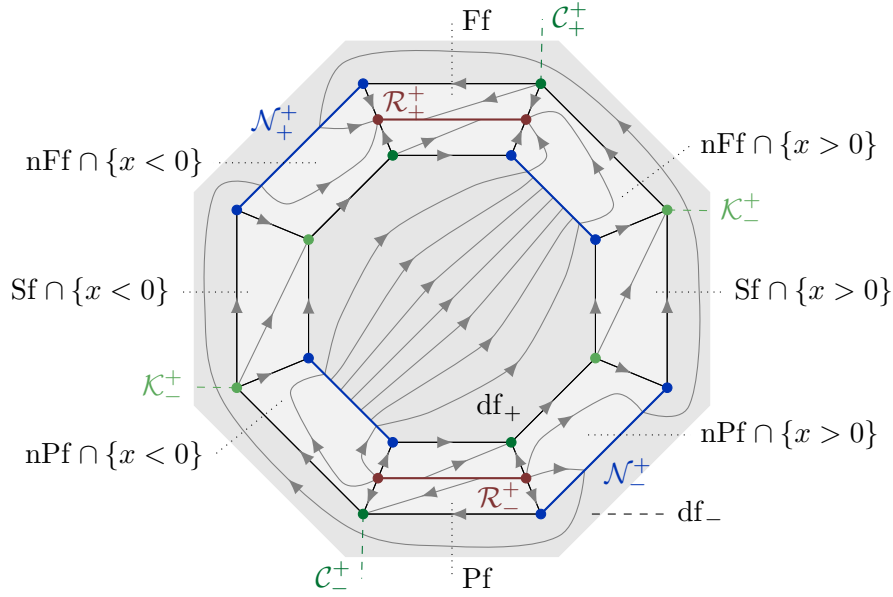
FIGURE 2. The flow of H_p on Σ^+, for $\mathsf{m} > 0$, in the $d = 1$ case when Σ^+ is the boundary of an octagonal prism — see [Sus23] for a full description. Only part of $\mathrm{df} \cap \Sigma^+$ is shown, namely the right moving part, df_+ (central octagon), and a neighborhood in the left moving part df_- of the portion over $\partial \mathbb{O}$ (outermost region). The time axis is vertical and the spatial axis $\mathbb{R}_x$ is horizontal. The labels specify which boundary hypersurface of the compactified phase space the labeled component is a subset of. For $d \geq 2$, the same picture depicts the flow in the portion of the characteristic set corresponding to zero angular momentum if we ignore the angular degrees of freedom. The radial sets $\mathcal{A}$ are located at high angular momentum and are therefore not depicted. See [Sus23, §4] for the computation of the flow. *The key observation on which this paper is based is that the radial sets $\mathcal{R}$ do not intersect* df (the inner and outer regions in the diagram). Contrast with the situation in the sc-cotangent bundle [Sus23, Fig. 1].

- $\mathcal{N}_\beta^\alpha$ are the sources and sinks of the flow in $\Sigma \cap \mathrm{df}$. This flow, over the interior of $\mathbb{O}$, is just the usual null geodesic flow at fiber infinity. So, $\mathcal{N} \subset \mathrm{df}$ contains the endpoints of compactified null geodesics. In Σ, $\mathcal{N}$ is a saddle point of the flow, but in $\Sigma \cap \mathrm{df}$ it is a source/sink.
- $\mathcal{C}_\beta^\alpha, \mathcal{K}_\beta^\alpha, \mathcal{A}_\beta^\alpha$ are additional radial sets over the corners between null infinity and the time-like/spacelike caps. They are all located at fiber infinity and are saddle points for the flow in both $\Sigma, \Sigma \cap \mathrm{df}$.

Here, α, β are signs — there are four copies of each sort of radial set. One sign refers to which of $\Sigma^\pm$ it is in, and the other to whether it is over future or past timelike/null infinity. The structure of the flow is illustrated in Figure 2, Figure 3.

Our sign convention is that, in the $\mathsf{m} > 0$ case, the flow is directed from $\mathcal{R}_-^+$ to $\mathcal{R}_+^+$ (i.e. past to future) in Σ^+ but from $\mathcal{R}_+^-$ to $\mathcal{R}_-^-$ (future to past) in Σ^-.

3.2. **Regularity theory and proof of the main result.** We are interested in the regularity of solutions u to $Pu + \lambda u = f$ given information about the regularity of f. At points away from the characteristic set we can appeal to microlocal elliptic regularity; since the de,sc-characteristic set of

$P + \lambda$ is $\Sigma \cap \mathrm{df}$, we can immediately conclude that

$$\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \backslash \mathrm{WF}_{\mathrm{de,sc}}^{m-2,\mathsf{s}}(Pu + \lambda u) \subseteq \Sigma \cap \mathrm{df}. \tag{25}$$

In particular, if $Pu + \lambda u \in \mathcal{S}$, in which case $\mathrm{WF}_{\mathrm{de,sc}}^{m-2,\mathsf{s}}(Pu + \lambda u) = \varnothing$, we get $\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \subseteq \Sigma \cap \mathrm{df}$.

Within the characteristic set, we need to use propagation estimates. The operator $P + \lambda$ is not of real principal type, but, since the imaginary part of the principal symbol is just the imaginary part $\Im \lambda$ of λ, hence has a definite sign, we can still propagate *along/against the Hamiltonian flow of P within* $\Sigma = \Sigma(\Re \lambda)$, the characteristic set of $P + \Re \lambda$;[5] the imaginary part means simply that we can only propagate regularity in one direction relative to the Hamiltonian flow: forward for $\Im \lambda > 0$ and backward for $\Im \lambda < 0$, according to our sign conventions.

Since, due to eq. (25), u automatically has the desired regularity everywhere except at df, we only need to consider the flow in an arbitrarily small neighborhood of df. At this stage we are working with the characteristic set of $P + \Re \lambda$ (which is of real principal type), not $P + \lambda$, and the flow enters and exits fiber infinity at the "radial points" $\mathcal{N}, \mathcal{C}, \mathcal{K}, \mathcal{A}$. Each of these radial sets is associated with a propagation estimate which says when we can propagate control/singularities through — see [Sus23, §5]. The fact that u is under control (eq. (25)) in the interior of the fibers will be of use here.

First note that for the purpose of concatenating propagation statements, we are free to ignore bicharacteristics which connect radial sets through the fiber interiors, again because we already know eq. (25). Within fiber infinity, the radial sets $\mathcal{N}$ are global sources and sinks of the flow (cf. [HV23]), so they become the initial and final points of propagation. Because they are saddle points from the perspective of the full flow, this propagation does *not* require any Sobolev orders to be above or below an absolute threshold as is true for the radial point estimates at $\mathcal{R}$ in [Sus23, §6] or in the Parenti–Shubin radial point estimates in [Vas20]. In other words, instead of using a high-regularity source/sink estimate to start off the propagation sequence, we can directly propagate regularity into $\mathcal{N}$ *from the fiber interiors*, where we have elliptic regularity. Similarly, instead of using a low-regularity source/sink estimate to complete the sequence, we use a saddle point estimate in [Sus23, §5.3]. While the former has an absolute threshold that limits the amount of regularity one can conclude, the latter does not.

For regularity to propagate through all the saddle points in a component of $\Sigma \cap \mathrm{df}$, the Sobolev orders have to satisfy a system of inequalities. For any threshold, this system has solutions where all orders are above that threshold. This is what allows us to conclude that u is Schwartz if $f = (P + \lambda)u$ is. We state the global propagation result as the following proposition:

Proposition 3.1. *Let* $\pm \Im(\lambda) > 0$ *and* $u \in \mathcal{S}'$.

(1) If $\mathrm{WF}_{\mathrm{de,sc}}^{m-1,\mathsf{s}+1}(Pu + \lambda u) \cap \Sigma^{\pm} \cap \mathrm{df} = \varnothing$ *for a choice of orders satisfying both the inequalities*

$$\begin{cases} m > \max\{1 + s_{\mathrm{nFf}}, \ 1 - s_{\mathrm{nFf}} + 2s_{\mathrm{Ff}}, \ 1/2 + s_{\mathrm{nFf}} - s_{\mathrm{Sf}}, \ 1 - s_{\mathrm{nPf}} + 2s_{\mathrm{Sf}}\}, \\ m < \min\{1 + s_{\mathrm{nPf}}, \ 1 - s_{\mathrm{nPf}} + 2s_{\mathrm{Pf}}, \ 1/2 + s_{\mathrm{nPf}} - s_{\mathrm{Sf}}, \ 1 - s_{\mathrm{nFf}} + 2s_{\mathrm{Sf}}\}, \end{cases} \tag{26}$$

then $\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \cap \Sigma^{\pm} \cap \mathrm{df} = \varnothing$.

(2) If $\mathrm{WF}_{\mathrm{de,sc}}^{m-1,\mathsf{s}+1}(Pu + \lambda u) \cap \Sigma^{\mp} \cap \mathrm{df} = \varnothing$ *for a choice of orders satisfying both the inequalities*

$$\begin{cases} m < \min\{1 + s_{\mathrm{nFf}}, \ 1 - s_{\mathrm{nFf}} + 2s_{\mathrm{Ff}}, \ 1/2 + s_{\mathrm{nFf}} - s_{\mathrm{Sf}}, \ 1 - s_{\mathrm{nPf}} + 2s_{\mathrm{Sf}}\}, \\ m > \max\{1 + s_{\mathrm{nPf}}, \ 1 - s_{\mathrm{nPf}} + 2s_{\mathrm{Pf}}, \ 1/2 + s_{\mathrm{nPf}} - s_{\mathrm{Sf}}, \ 1 - s_{\mathrm{nFf}} + 2s_{\mathrm{Sf}}\}, \end{cases} \tag{27}$$

then $\mathrm{WF}_{\mathrm{de,sc}}^{m,\mathsf{s}}(u) \cap \Sigma^{\mp} \cap \mathrm{df} = \varnothing$.

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[5]When we write Σ below, we mean the characteristic set of $P + \Re \lambda$. If $\Re \lambda + \mathsf{m}^2 > 0$, $\Sigma^{\pm}$ refers to the components of this disconnected set, as before. If $\Re \lambda + \mathsf{m}^2 \le 0$, then Σ has only one component, if $d \ge 2$. However, we can always restrict attention to an arbitrarily small neighborhood of df, where there are two components as before, and these we can refer to as $\Sigma^{\pm}$.

∎

Remark. There should exist exponentially growing solutions u to $Pu + \lambda u = 0$, to which the conclusion of the previous proposition obviously does not apply. The hypothesis that u be tempered is what is excluding such counter-examples to Proposition 3.1.

Proof. By eq. (25) (and the fact that wavefronts are closed), there exists a punctured neighborhood of $\Sigma^{\pm} \cap \mathrm{df}$ (in case one) or $\Sigma^{\mp} \cap \mathrm{df}$ (in case two) disjoint from $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u)$. This only requires that

$$\mathrm{WF}^{m-2,\mathsf{s}}_{\mathrm{de,sc}}(Pu + \lambda u) \cap \Sigma^{\bullet} \cap \mathrm{df} = \varnothing, \tag{28}$$

whereas our assumption here is stronger by one order.

Beyond this starting point, the proposition is a concatenation of [Sus23, Props. 5.6, 5.7, 5.10–5.14], except applied to $P + \lambda$. The propositions were not stated in [Sus23] in this level of generality, but the proofs apply. Indeed, in this former work it was assumed what amounts to $\mathsf{m}^2 + \Re\lambda > 0$, but the structure of Σ and H_p is, sufficiently close to fiber infinity, unchanged if we replace $\mathsf{m}^2 + \Re\lambda$ with another real constant.[6] Moreover, our previous work assumed $\Im\lambda = 0$. The only modification in the $\Im\lambda \neq 0$ case is the already mentioned fact that we can only propagate estimates/singularities in one direction along the flow. When propagating in that one direction, the contributions to the estimates coming from $\Im\lambda$ have the sign which allows those terms to be thrown out. This is by now a standard phenomenon going under the name of *complex absorption* — see [Vas18] for the version of this argument taking place in the Parenti–Shubin calculus.

For the reader not familiar with complex absorption, we provide a sketch.

- First, we recall how propagation estimates work in this context. Consider the half of the propagation of singularities theorem which propagates regularity *forward* along the rescaled Hamilton flow H_p. This is the statement that if u has no de,sc-wavefront set of order (m, s) at a point in Σ, then it also has no wavefront set of order (m, s) at any point downstream, as long as the bicharacteristic segment connecting them does not pass through $\mathrm{WF}^{m-1,\mathsf{s}+1}_{\mathrm{de,sc}}(Pu)$. The proof is a positive-commutator argument based on the construction of a symbol a such that

$$\mathsf{H}_p a = -\delta \rho^{2m-2, 2\mathsf{s}+2} a^2 - b^2 + e, \quad a \in S^{2m-1, 2\mathsf{s}+1}_{\mathrm{de,sc}} \tag{29}$$

where $\delta > 0$ is a constant, $b \in S^{m,\mathsf{s}}_{\mathrm{de,sc}}$ is elliptic at the downstream point, and $e \in S^{2m, 2s}_{\mathrm{de,sc}}$ is supported in a neighborhood of the upstream point. One then quantizes these symbols (in a way preserving essential supports) to get pseudodifferential operators A, B, E satisfying

$$i[A, P] = -\delta(\Lambda A)^*(\Lambda A) - B^* B + E + F, \tag{30}$$

where Λ is a quantization of $\rho^{m-1, \mathsf{s}+1}$, F is an error term which is an order lower than E. Here, one uses that the principal symbol of $i[A, P]$ is $\mathsf{H}_p a$.[7] It can be ensured that $A = \check{A}^2$ for some symmetric $\check{A}$.

For u in a regular enough weighted Sobolev space, eq. (30), combined with the identity

$$-2\Im\langle Pu, Au \rangle = i\langle Pu, Au \rangle - i\langle Au, Pu \rangle = \langle i[A, P]u, u \rangle \tag{31}$$

(assuming that the integration-by-parts here can be justified), yields

$$\|Bu\|^2 = 2\Im\langle Pu, Au \rangle - \delta\|\Lambda Au\|^2 + \langle Eu, u \rangle + \langle Fu, u \rangle. \tag{32}$$

Since we want to assume that $(P + \lambda)u$ is under control, not Pu, we rewrite this as

$$\|Bu\|^2 = 2\Im\langle (P + \lambda)u, Au \rangle - \delta\|\Lambda Au\|^2 + \langle Eu, u \rangle + \langle Fu, u \rangle - 2\Im(\lambda)\|\check{A}u\|^2. \tag{33}$$

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[6]In addition, the constant term does not enter into the calculation of any of the thresholds in any of the estimates.

[7]The sign in the identity $\sigma(i[A, P]) = \pm \mathsf{H}_p a$ depends on sign conventions for Hamiltonian vector fields and the quantization map. We follow the conventions of [Sus23], which differ from those in [Vas20], leading to the direction of propagation (relative to H_p) for a given sign of $\Im\lambda$ being flipped compared to [Vas20].

Since we are interested in bounding $\|Bu\|$ above, as long as $\Im\lambda > 0$ the last term can be ignored, and we have

$$\|Bu\|^2 \leq 2\Im\langle(P+\lambda)u, Au\rangle - \delta\|\Lambda Au\|^2 + \langle Eu, u\rangle + \langle Fu, u\rangle. \tag{34}$$

All the terms on the right-hand side are under control (where controlling $\langle Fu, u\rangle$ requires assuming

$$\mathrm{WF}_{\mathrm{de,sc}}^{m-1/2,\mathsf{s}-1/2}(u) = \varnothing \tag{35}$$

in the propagation region, so in reality an inductive argument is required, gaining one half-order at a time), which leads to control of $\|Bu\|$. By the choice of b, this implies an estimate of the $H_{\mathrm{de,sc}}^{m,\mathsf{s}}$ norm of u microlocalized near the point at which we want to establish control. So, in summary, as long as $\Im\lambda$ has the "right" sign, then the same symbol construction used to prove propagation estimates in the $\lambda \in \mathbb{R}$ case also works here.

If instead $\Im\lambda < 0$, then the discussion above applies, mutatis mutandis, to propagating control in the opposite direction along the flow.

- Now consider the radial point estimates which propagate control into a radial set from the *stable/sink* directions, i.e. forward along the Hamiltonian flow. The proofs are based on constructing a symbol a which instead of eq. (29) satisfies

$$H_p a = -\delta\rho^{2m-2,2\mathsf{s}+2}a^2 - b^2 + e_\mathrm{s}^2 - e_\mathrm{u}^2 + e_\mathrm{e}, \tag{36}$$

where b is elliptic at the radial set, e_s is supported near the stable manifold in Σ of the radial set but away from the unstable manifold (including the radial set itself), e_u is supported near the unstable manifold in Σ but away from the stable manifold, and e_e is supported away from Σ. Quantizing both sides of (36), applying them to u and pairing with u then yields

$$\|Bu\|^2 = 2\Im\langle Pu, Au\rangle - \delta\|\Lambda Au\|^2 + \|E_\mathrm{s}u\|^2 - \|E_\mathrm{u}u\|^2 + \langle E_\mathrm{e}u, u\rangle + \langle Fu, u\rangle. \tag{37}$$

Since the goal is to estimate $\|Bu\|$ from above, the $\|E_\mathrm{u}u\|$ term can be dropped, and every other term on the right-hand side is under control if we assume control on u at the stable manifold. By the same argument as above, the resulting estimate remains valid for $P + \lambda$ when $\Im\lambda > 0$.

To propagate regularity from the unstable manifold, one instead arranges

$$H_p a = \delta\rho^{2m-2,2\mathsf{s}+2}a^2 + b^2 + e_\mathrm{s}^2 - e_\mathrm{u}^2 + e_\mathrm{e}; \tag{38}$$

the corresponding sign changes mean that the estimate remains valid for $P + \lambda$ when $\Im\lambda < 0$.

So, in summary, if $\lambda \in \mathbb{C}\backslash\mathbb{R}$, then the propagation/radial point estimates in [Sus23, §5] apply to $P + \lambda$, as long as we are propagating along the flow if $\Im\lambda > 0$ and against the flow if $\Im\lambda < 0$.

We now return to the manner in which these estimates are concatenated to yield Proposition 3.1. For definiteness, we assume $\Im\lambda > 0$ and consider propagation in Σ^+, i.e. case one; thus, we are propagating control along the arrows in Figure 3. The other cases are analogous. Control on u is propagated in the following order:

- From $(\Sigma^+ \cap \mathrm{nPf})\backslash\mathrm{df}$ (in a neighborhood of df) into $\mathcal{N}_-^+$, as in [Sus23, Prop. 5.10];
- From $\mathcal{N}_-^+$ throughout $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{Pf})\backslash\mathcal{C}_-^+$, $\Sigma^+ \cap \mathrm{df} \cap \pi^{-1}\mathbb{O}^\circ$ (i.e. fiber infinity over the interior), and throughout $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{nPf})\backslash\mathrm{Sf}$ *except* $\mathcal{C}_-^+$ and the bicharacteristics connecting it to $\mathcal{K}_-^+$;
- From $(\Sigma^+ \cap \mathrm{Pf})\backslash\mathcal{C}_-^+$ into $\mathcal{C}_-^+$, using [Sus23, Prop. 5.13], thereby concluding regularity in all of $\Sigma^+ \cap \mathrm{df} \cap \mathrm{Pf}$;
- From $\mathcal{C}_-^+$, along the bicharacteristics connecting it to $\mathcal{K}_-^+$, thereby concluding regularity in all of $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{nPf})\backslash\mathrm{Sf}$;
- From those bicharacteristics into $\mathcal{K}_-^+$, using [Sus23, Prop. 5.11];
- From $\mathcal{K}_-^+$ and $\mathcal{N}_-^+ \cap \mathrm{Sf}$ throughout $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{nPf} \cap \mathrm{Sf})\backslash\mathcal{A}_-^+$;

- From $(\Sigma^+ \cap \mathrm{nPf})\backslash \mathcal{A}^+_-$ into $\mathcal{A}^+_-$, using [Sus23, Prop. 5.7], thereby concluding regularity in all of $\Sigma^+ \cap \mathrm{df} \cap \mathrm{nPf}$;
- From $\Sigma^+ \cap \mathrm{df} \cap \mathrm{nPf}$, throughout $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{Sf})\backslash \mathrm{nFf}$;
- From $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{Sf})\backslash \mathrm{nFf}$ into $\mathcal{A}^+_+$, using [Sus23, Prop. 5.6];
- From $\mathcal{A}^+_+$, throughout $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{nFf} \cap \mathrm{Sf})\backslash(\mathcal{K}^+_+ \cup \mathcal{N}^+_+)$;
- From a neighborhood of $(\Sigma^+ \cap \mathrm{Sf})\backslash \mathcal{K}^+_+$ into $\mathcal{K}^+_+$, using [Sus23, Prop. 5.12], thereby concluding regularity in all of $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{Sf})\backslash \mathcal{N}^+_+$;
- From $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{Sf} \cap \mathrm{nFf})\backslash \mathcal{N}^+_+$, throughout $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{nFf})\backslash(\mathrm{Ff} \cup \mathcal{N}^+_+)$;
- From the bicharacteristics in $\Sigma^+ \cap \mathrm{df} \cap \mathrm{nFf}$ connecting $\mathcal{K}^+_+$ to $\mathcal{C}^+_+$ into $\mathcal{C}^+_+$, using [Sus23, Prop. 5.14];
- From $\mathcal{C}^+_+$, throughout $(\Sigma^+ \cap \mathrm{df} \cap \mathrm{Ff})\backslash \mathcal{N}^+_+$, thereby concluding regularity in all of $(\Sigma^+ \cap \mathrm{df} \cap (\mathrm{nFf} \cup \mathrm{Ff}))\backslash \mathcal{N}^+_+$;
- From $(\Sigma^+ \cap \mathrm{df})\backslash \mathcal{N}^+_+$ into $\mathcal{N}^+_+$, using [Sus23, Prop. 5.10], finally concluding regularity in all of $\Sigma^+ \cap \mathrm{df}$.

(This is the same propagation order used in [HV23]. Beware that this reference uses different notation.)

The inequality eq. (26) is just the conjunction of all of the inequalities in the hypotheses of the cited propositions in [Sus23, §5]. $\qquad\square$

Remark 3.2. The fact that we can use elliptic estimates to establish control in the fiber interiors allows us to sidestep a significant technical difficulty in [Sus23] related to the extended radial set $\mathcal{N}$. In this previous work, there are bicharacteristics in Σ^+ going from $\mathcal{N}^+_+$ to $\mathcal{K}^+_+$ *through the fiber interiors*, then to $\mathcal{C}^+_+$ and then back to (a different point of) $\mathcal{N}^+_+$. Consequently, we had found it required to propagate control through the various radial sets in a different order, using radial point estimates microlocalized to proper subsets of $\mathcal{N}$.

Now we show that Proposition 3.1 implies Theorem B. This amounts to showing that the systems of inequalities in that proposition have solutions with all orders arbitrarily high.

Proof of Theorem B. By elliptic regularity, $\mathrm{WF}_{\mathrm{de,sc}}(u) \subset \Sigma \cap \mathrm{df}$.

For any $N > 1$, let us take

$$s_{\mathrm{Ff}} = s_{\mathrm{nFf}} = N, \qquad s_{\mathrm{Sf}} = 2N, \qquad s_{\mathrm{Pf}} = s_{\mathrm{nPf}} = 4N, \qquad m = 2N. \qquad (39)$$

Then the inequalities in the first part of Proposition 3.1 are satisfied, so from $f \in \mathcal{S}$ we conclude that $\mathrm{WF}^{m,\mathsf{s}}_{\mathrm{de,sc}}(u) \cap \Sigma^\pm \cap \mathrm{df} = \varnothing$. Taking N arbitrarily large, we conclude that $\mathrm{WF}_{\mathrm{de,sc}}(u) \cap \Sigma^\pm \cap \mathrm{df} = \varnothing$.

Taking instead

$$s_{\mathrm{Ff}} = s_{\mathrm{nFf}} = 4N, \qquad s_{\mathrm{Sf}} = 2N, \qquad s_{\mathrm{Pf}} = s_{\mathrm{nPf}} = N, \qquad m = 2N, \qquad (40)$$

by a completely analogous argument we find $\mathrm{WF}_{\mathrm{de,sc}} \cap \Sigma^\mp \cap \mathrm{df} = \varnothing$. Therefore, u has no de,sc-wavefront set anywhere, hence is Schwartz. $\qquad\square$

Proposition 3.1 allows us to make a sharper statement about global Sobolev regularity of solutions to $Pu + \lambda u = f$ assuming that f is in a Sobolev space within the range allowed by the inequalities. However, because we are forced to propagate regularity in the same direction with respect to the Hamiltonian flow (hence in *opposite* directions with respect to time) in both sheets of the characteristic set, the inequalities required for propagation in the two sheets are incompatible. Therefore, to directly translate Proposition 3.1 into a statement on global regularity of solutions, one needs to use *variable-order* Sobolev spaces, whose orders m, s are allowed to vary smoothly on $^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}$. See [Vas18] for precise definitions in the sc- setting, which directly carry over to the de,sc-setting. Propagation of singularities remains valid in the variable-order setting as long as all orders are (not necessarily strictly) decreasing along the direction in which one propagates estimates.
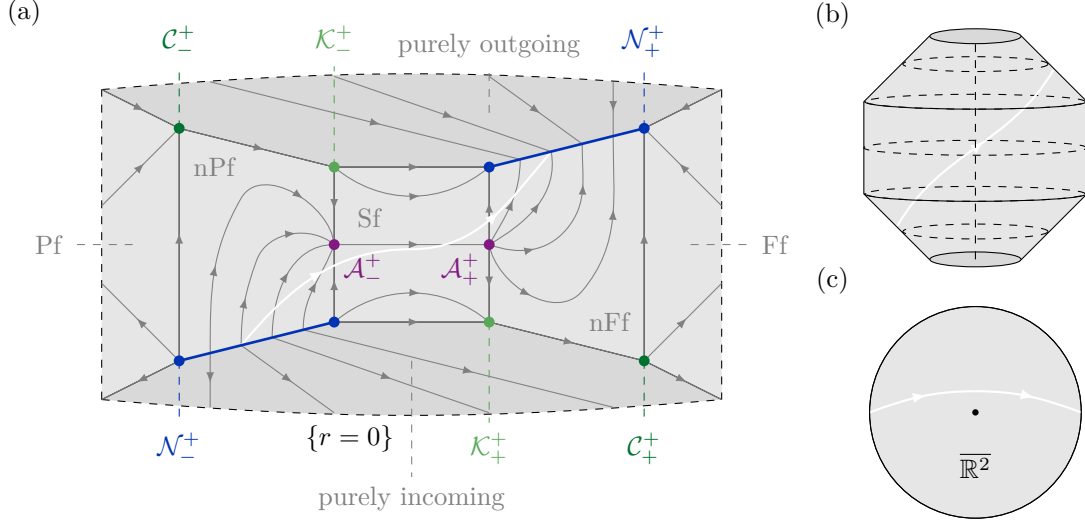
FIGURE 3. (a) The flow, for g the Minkowski metric, in the subset of $\Sigma^+ \cap \mathrm{df}$ with nonnegative angular momentum, in the $d = 2$ case. The top and bottom panels of the figure are the portion of $\Sigma^+ \cap \mathrm{df}$ on which the de,sc-angular momentum vanishes. (The de,sc-angular momentum is a rescaling of angular momentum by a function of the spacetime coordinates and is therefore not conserved.) Such momenta are pointing radially inwards (incoming) or radially outwards (outgoing) The horizontal hyperplane in the vertical middle of the figure, in which $\mathcal{A}$ lies, is the subset of $\Sigma^+ \cap \mathrm{df}$ on which angular momentum is maximized. The vertical panels of the figure are over the faces $\mathrm{Pf}, \dots, \mathrm{Ff}$ of $\mathbb{O}$. The interstitial regions in the figure are the points in $\Sigma^+ \cap \mathrm{df}$ over $\mathbb{O}^\circ$ on which angular momentum is nonvanishing. A typical bicharacteristic in the region (white) snakes from $\mathcal{N}_-^+$ to $\mathcal{N}_+^+$ without hitting the set $\{r = 0\}$ (dashed black). When g is only *asymptotically* flat, the flow over $\mathbb{O}^\circ$ is modified, but the flow over $\partial\mathbb{O}$ is not. (b) The projection of the white bicharacteristic onto spacetime, i.e. ignoring frequency coordinates, using the same conventions as Figure 1(b), and (c) onto the radial compactification $\overline{\mathbb{R}^2}$ of space, i.e. ignoring the time coordinate. Since the angular momentum is nonzero, the curve misses the origin $\{r = 0\}$ (black). The purely outgoing/incoming bicharacteristics are those that hit the origin.

(That is, one cannot propagate weak control somewhere to strong control elsewhere, but the reverse is fine.) The result is the following:

Theorem C. *Fix* nonreal $\lambda \in \mathbb{C}$, $m \in C^\infty(^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}; \mathbb{R})$, $\mathsf{s} \in C^\infty(^{\mathrm{de,sc}}\overline{T}^*\mathbb{O}; \mathbb{R}^5)$. *Assume that* $\pm\mathrm{Im}(\lambda) > 0$ *and that* $\pm m$ *as well as each component of* $\pm\mathsf{s}$ *is monotone non-increasing along the Hamiltonian flow in a neighborhood of* $\Sigma \cap \mathrm{df}$. *Suppose that the following inequalities are satisfied:*

- $m < 1 + s_{\mathrm{nPf}}$ *near* $\mathcal{N}_-^\pm$ *and* $m > 1 + s_{\mathrm{nPf}}$ *near* $\mathcal{N}_-^\mp$;
- $m < 1 + 2s_{\mathrm{Pf}} - s_{\mathrm{nPf}}$ *near* $\mathcal{C}_-^\pm$ *and* $m > 1 + 2s_{\mathrm{Pf}} - s_{\mathrm{nPf}}$ *near* $\mathcal{C}_-^\mp$;
- $m > 1 + 2s_{\mathrm{Sf}} - s_{\mathrm{nPf}}$ *near* $\mathcal{K}_-^\pm$ *and* $m < 1 + 2s_{\mathrm{Sf}} - s_{\mathrm{nPf}}$ *near* $\mathcal{K}_-^\mp$;
- $m < 1/2 + s_{\mathrm{nPf}} - s_{\mathrm{Sf}}$ *near* $\mathcal{A}_-^\pm$ *and* $m > 1/2 + s_{\mathrm{nPf}} - s_{\mathrm{Sf}}$ *near* $\mathcal{A}_-^\mp$;
- $m > 1/2 + s_{\mathrm{nFf}} - s_{\mathrm{Sf}}$ *near* $\mathcal{A}_+^\pm$ *and* $m < 1/2 + s_{\mathrm{nFf}} - s_{\mathrm{Sf}}$ *near* $\mathcal{A}_+^\mp$;
- $m < 1 + 2s_{\mathrm{Sf}} - s_{\mathrm{nFf}}$ *near* $\mathcal{K}_+^\pm$ *and* $m > 1 + 2s_{\mathrm{Sf}} - s_{\mathrm{nFf}}$ *near* $\mathcal{K}_+^\mp$;
- $m > 1 + 2s_{\mathrm{Ff}} - s_{\mathrm{nFf}}$ *near* $\mathcal{C}_+^\pm$ *and* $m < 1 + 2s_{\mathrm{Ff}} - s_{\mathrm{nFf}}$ *near* $\mathcal{C}_+^\mp$;
- $m > 1 + s_{\mathrm{nFf}}$ *near* $\mathcal{N}_+^\pm$ *and* $m < 1 + s_{\mathrm{nFf}}$ *near* $\mathcal{N}_+^\mp$.

Then, if $u \in \mathcal{S}'(\mathbb{R}^{1,d})$ solves $Pu + \lambda u = f$ for $f \in H^{m,\mathsf{s}}_{\mathrm{de,sc}}$, we have $u \in H^{m+1,\mathsf{s}-1}_{\mathrm{de,sc}}$.

Remark 3.3. The hypotheses are satisfied if the orders are taken constant near each sheet of $\Sigma \cap \mathrm{df}$, satisfying the inequalities of Proposition 3.1.(1), on $\Sigma^{\pm} \cap \mathrm{df}$ and those of Proposition 3.1.(2), on $\Sigma^{\mp} \cap \mathrm{df}$, where the sign is that of the imaginary part of λ. Strictly speaking, only this special case follows directly from Proposition 3.1 because the latter uses orders which are constant on each sheet of Σ; but the full statement of Theorem C is proven by exactly the same argument using the variable-order versions of propagation estimates.

Remark 3.4. Variable-order s is required to make the sharpest possible statements; however, since the relevant systems of inequalities have solutions with all orders arbitrarily high, there are refinements of Theorem B that only assume that f lies in some constant order de,sc-Sobolev space. For example, consider what regularity of f is sufficient to guarantee $u \in H^{m,\mathsf{s}}_{\mathrm{de,sc}}$. We can find constant orders (m_1, s_1) and (m_2, s_2) which satisfy the propagation inequalities of Proposition 3.1.(1), Proposition 3.1.(2) respectively and such that $m_1, m_2 \geq m$ and $\mathsf{s}_1, \mathsf{s}_2 \geq \mathsf{s}$. Then for any constant orders $m_0 \geqslant \max\{m_1, m_2\}$ and $\mathsf{s}_0 \geq \max\{\mathsf{s}_1, \mathsf{s}_2\}$, it follows from Proposition 3.1 that $Pu + \lambda u \in H^{m_0-1,\mathsf{s}_0+1}_{\mathrm{de,sc}}$ implies $u \in H^{m,\mathsf{s}}_{\mathrm{de,sc}}$.

Remark 3.5. One can reformulate Theorem C as an invertibility statement for $(P + \lambda) : \mathcal{X}^{m,\mathsf{s}} \to \mathcal{Y}^{m,\mathsf{s}}$, where

$$\mathcal{X}^{m,\mathsf{s}} = \{u \in H^{m+1,\mathsf{s}-1}_{\mathrm{de,sc}} : (P + \lambda)u \in H^{m,\mathsf{s}}_{\mathrm{de,sc}}\}, \qquad \mathcal{Y}^{m,\mathsf{s}} = H^{m,\mathsf{s}}_{\mathrm{de,sc}} \tag{41}$$

with orders satisfying the inequalities above; cf. [Vas20, Cor. 4]. This provides a precise meaning to $(P \pm i\varepsilon)^{-1}$ independently of the functional calculus for self-adjoint operators. In addition, the signs in $\pm i\varepsilon$ distinguishing the Feynman and anti-Feynman propagators correspond to opposite monotonicity and threshold conditions of m, s in Theorem C.

APPENDIX A. DEFICIENCY INDEX THEORY AS APPLIED TO $\Box_g$

To complete our exposition, we include the proof of the following proposition:

Proposition A.1. *Let g, P be as in §2. Then, P is essentially self-adjoint on $C^{\infty}_{\mathrm{c}}(\mathbb{R}^{1,d})$ if and only if the only $u \in L^2$ satisfying $Pu = \pm iu$ is $u = 0$, for both choices of sign.* ∎

Proof. By [RS72, Chp. VIII §2], essential self-adjointness is equivalent to $\mathrm{Ran}_{C^{\infty}_{\mathrm{c}}}(P \pm i)$ being dense in L^2 for both choices of sign, which is equivalent to the subspace $\mathrm{Ran}_{C^{\infty}_{\mathrm{c}}}(P \pm i)^{\perp} \subset L^2$ being $\{0\}$.

For any $w \in \mathcal{D}'(\mathbb{R}^{1,d})$ and $v \in C^{\infty}_{\mathrm{c}}(\mathbb{R}^{1,d})$, we can interpret $\langle w, v \rangle_{L^2}$ as a distributional pairing, jointly continuous in both slots, which implies (by symmetry on Schwartz functions) that $\langle Pw, v \rangle_{L^2} = \langle w, Pv \rangle_{L^2}$ for all $w \in \mathcal{D}'(\mathbb{R}^{1,d})$ and $v \in C^{\infty}_{\mathrm{c}}(\mathbb{R}^{1,d})$. Then, we can compute

$$
\begin{aligned}
u \in \mathrm{Ran}_{C^{\infty}_{\mathrm{c}}}(P \pm i)^{\perp} &\iff \langle u, (P \pm i)v \rangle_{L^2} = 0 \text{ for all } v \in C^{\infty}_{\mathrm{c}}(\mathbb{R}^{1,d}) \\
&\iff \langle (P \mp i)u, v \rangle_{L^2} = 0 \text{ for all } v \in C^{\infty}_{\mathrm{c}}(\mathbb{R}^{1,d}) \\
&\iff (P \mp i)u = 0.
\end{aligned}
\tag{42}
$$

So, $\mathrm{Ran}_{C^{\infty}_{\mathrm{c}}}(P \pm i)^{\perp} = \{0\}$ if and only if the only $u \in L^2$ satisfying $Pu = \pm iu$ is $u = 0$, for both choices of sign. □

APPENDIX B. PROOF OF TAIRA'S MAPPING PROPERTY FOR THE EXACT MINKOWSKI CASE

For the exact Minkowski D'Alembertian $\Box = \partial_t^2 - \sum_{j=1}^d \partial_{x_j}^2 \in \mathrm{Diff}^2(\mathbb{R}^{1,d}_{t,x})$, Taira's mapping property says that, if $\lambda \in \mathbb{C} \backslash \mathbb{R}$, then, if $f \in \mathcal{S}(\mathbb{R}^{1,d})$,

$$u \in \mathcal{S}'(\mathbb{R}^{1,d}), \Box u = \lambda u + f \Rightarrow u \in \mathcal{S}(\mathbb{R}^{1,d}). \tag{43}$$

Proof. Letting $\mathcal{F}u(\tau, \xi)$ denote the Fourier transform of u, where $\tau \in \mathbb{R}$ is the coordinate dual to t and $\xi \in \mathbb{R}^d$ is dual to $x = (x_1, \ldots, x_d)$, $\Box u = \lambda u + f$ is equivalent to $-(\tau^2 - \xi^2 + \lambda)\mathcal{F}u = \mathcal{F}f(\tau, \xi)$. So,

$$u(t, x) = -\mathcal{F}^{-1}_{\tau \to t, \xi \to x}\left(\frac{\mathcal{F}f(\tau, \xi)}{\tau^2 - \xi^2 + \lambda}\right); \tag{44}$$

note that the division by $\tau^2 - \xi^2 + \lambda$ is well-defined because $\lambda \notin \mathbb{R}$. Because $f \in \mathcal{S}(\mathbb{R}^{1,d})$, the same is true for $\mathcal{F}f$. It follows that $(\tau^2 - \xi^2 + \lambda)^{-1}\mathcal{F}f$ is Schwartz. Indeed, hitting this with a constant-coefficient differential operator, the result is a linear combination of functions of the form

$$\frac{p(\tau, \xi)\partial_\tau^j \partial_\xi^\alpha \mathcal{F}f}{(\tau^2 - \xi^2 + \lambda)^k} \tag{45}$$

for some $j, k \in \mathbb{N}$, multi-index α, and polynomial p. Since $\lambda \notin \mathbb{R}$, we have, for any $N \in \mathbb{R}$,

$$\left\|(1 + \tau^2 + \xi^2)^N \frac{p(\tau, \xi)\partial_\tau^j \partial_\xi^\alpha \mathcal{F}f}{(\tau^2 - \xi^2 + \lambda)^k}\right\|_{L^\infty} \leq \frac{1}{|\Im\lambda^k|}\|(1 + \tau^2 + \xi^2)^{N_0}\partial_\tau^j \partial_\xi^\alpha \mathcal{F}f\|_{L^\infty} < \infty, \tag{46}$$

for some N_0. So, $(\tau^2 - \xi^2 + \lambda)^{-1}\mathcal{F}f$ is indeed Schwartz.

So, if follows from eq. (44) that u is Schwartz. $\square$

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